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WEAK CONTINUITY PROPERTIES OF TOPOLOGIZED GROUPS

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Abstract. We explore (weak) continuity properties of group operations. For this purpose, the Novak number and developability number are applied. It is shown that if $(G, \cdot, \tau)$ is a regular right (left) semitopological group with $\operatorname{dev}(G) < \operatorname{Nov}(G)$ such that all left (right) translations are feebly continuous, then $(G, \cdot, \tau)$ is a topological group. This extends several results in literature.

Keywords: developability number, feebly continuous, nearly continuous, Novak number, paratopological group, semitopological group, topological group

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1. Introduction

A *topological group* is a group with a topology such that both its multiplication and inversion are continuous. The problem of finding weaker and less restrictive conditions that can be used to characterize a group topology has many precedents in literature. The first result in this line was given by Montgomery [23] for groups that are Polish and the multiplication is separately continuous. It suggests a question as to when a separately continuous multiplication is continuous and further when the operation of taking the inverse is continuous. In the past seventy years, a lot of papers have appeared in connection with these problems. Recently, Ferri, Hernández and Wu [13] have considered a group equipped with a Baire metrizable topology and used weaker conditions on left and right translations to characterize a group topology. They used Frolík's *feeble continuity* (or *somewhat continuity*) under the name of

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"almost continuity", see the definition below. It turns out that the study of the latter question involves topological games, and it has a great impact on topological dynamics and the theory of Banach spaces, see [9], [20] and [22] for more information. In the meantime, Arhangel'skii and Reznichenko [3] also considered the problem as to when a paratopological group is a topological group. As seen in [3], these types of studies involve more and more of various types of almost continuous functions. In fact, the role that weak continuity properties play here is not surprising. It can be tracked back to the early papers on separate vs. joint continuity, see [27] and [28] for historical remarks. All these fore-mentioned facts motivate us to conduct further investigation on weak continuity properties of group operations. This is the main purpose of the present paper.

To proceed further, we need to introduce some notation and terminology. In the sequel, a *topologized group* is a triple $(G,\cdot,\tau)$, where $(G,\cdot)$ is a group and (G,τ) is a topological space. The neutral element of $(G,\cdot)$ is denoted by e. We shall use $\mathfrak{m}$ to denote the *multiplication* of $(G,\cdot)$, i.e., $\mathfrak{m}(g,h) = g\cdot h$ for $g,h\in G$. For any $h\in G$, let $\lambda_h\colon G\to G$ be the *left translation*, i.e., $\lambda_h(g)=\mathfrak{m}(h,g)$ for all $g\in G$. Similarly, we use $\varrho_h\colon G\to G$ to denote the *right translation* defined by $\varrho_h(g)=\mathfrak{m}(g,h)$. If τ makes all left (right) translations continuous, then we call $(G,\cdot,\tau)$ a *left* (*right*) *semitopological group*. Further, if τ makes all λ_h and ϱ_h continuous, $(G,\cdot,\tau)$ is called a *semitopological group*. In the case that τ makes $\mathfrak{m}$ (jointly) continuous, $(G,\cdot,\tau)$ is called a *paratopological group*. The inversion of (G,τ) is denoted by $\mathfrak{i}$. We should note that $(G,\cdot,\tau)$ is semitopological if and only if τ is invariant to translations. Similarly, $(G,\cdot,\tau)$ is left (right) semitopological if and only if τ is invariant to left (right) translations. The *Novák number* [6] of a topological space (G,τ), denoted by $\mathrm{Nov}(G)$, is defined by

$$\mathrm{Nov}(G)=\min\{\kappa\colon G \text{ can be covered by } \leqslant\kappa \text{ nowhere dense sets}\}.$$

Evidently, (G,τ) is of second category if and only if $\mathrm{Nov}(G)\geqslant\aleph_1$; and (G,τ) is Baire if and only if $\mathrm{Nov}(O)\geqslant\aleph_1$ for every nonempty open set $O\subseteq G$. The *developability number* of (G,τ), denoted by $\mathrm{dev}(G)$, is defined by

$$\begin{aligned}\mathrm{dev}(G)=\min\{\kappa\colon\ &\text{there is a family } \{\mathscr{U}_\alpha\colon \alpha<\kappa\} \text{ of open covers}\\ &\text{such that } \{\mathrm{st}(x,\mathscr{U}_\alpha)\colon \alpha<\kappa\} \text{ is a local base for } x\in G\},\end{aligned}$$

where $\mathrm{st}(x,\mathscr{U}_\alpha) := \bigcup\{U\colon x\in U\in\mathscr{U}_\alpha\}$. A regular space (G,τ) is a *Moore space* if and only if $\mathrm{dev}(G)=\aleph_0$, and ω_1+1 with the order topology is a space with $\mathrm{dev}(\omega_1+1)=\aleph_1$. The weight and character of a topological space (G,τ) are denoted by $w(G)$ and $\chi(G)$ respectively. It is easy to see that $\chi(G)\leqslant\mathrm{dev}(G)$ for any space (G,τ).

This paper is organized as follows. In Section 2, we discuss various weak continuity properties of semitopological and paratopological groups. It is pointed out that the weakest condition that makes a paratopological group $(G, \cdot, \tau)$ a topological group is the semi-precontinuity of i. Moreover, it is also pointed out that conditions in Lemma 1.2 and Lemma 1.3 in [3] are indeed equivalent, and cardinal functions $\mathrm{dev}(G)$ and $\mathrm{Nov}(G)$ can be used to derive some automatic continuity property of left and right semitopological groups. Then, these discoveries are used in Section 3 to find weaker conditions on the left and right translations to characterize a group topology. Our main theorem in Section 3 extends several results in [3], [29], etc, and several examples are given to show the subtle difference between weak continuity properties of group operations. In the last section, a number of open questions are posed.

2. Weak continuity in topologized groups

Let us begin this section with some definitions. Given a topological space X and a subset $A \subseteq X$, in the sequel, $\overline{A}$ and $\mathrm{int}\, A$ stand for the closure and interior of A in X respectively. Recall that a function $f : X \to Y$ from a space X into another space Y is said to be

$$\left\{\begin{array}{c} \text{continuous} \\ \text{nearly continuous [30]} \\ \text{quasi-continuous [24]} \\ \text{semi-precontinuous} \\ \text{feebly continuous [14], [15]} \end{array}\right\} \text{ at } x \in X \text{ if } \left\{\begin{array}{c} x \in \mathrm{int}\, f^{-1}(V) \\ x \in \mathrm{int}\overline{f^{-1}(V)} \\ x \in \overline{\mathrm{int}\, f^{-1}(V)} \\ x \in \overline{\mathrm{int}\, \overline{f^{-1}(V)}} \\ \mathrm{int}\, f^{-1}(V) \neq \emptyset \end{array}\right\}$$

for each neighborhood V of $f(x)$. Then f is *continuous*, *nearly continuous*, *quasi-continuous*, *semi-precontinuous*, *feebly continuous* if it has the respective property at each point, that is, if and only if for each open set $V \subseteq Y$ we have respectively $f^{-1}(V) \subseteq \mathrm{int}\, f^{-1}(V)$, $f^{-1}(V) \subseteq \mathrm{int}\,\overline{f^{-1}(V)}$, $f^{-1}(V) \subseteq \overline{\mathrm{int}\, f^{-1}(V)}$, $f^{-1}(V) \subseteq \overline{\mathrm{int}\, \overline{f^{-1}(V)}}$, $\mathrm{int}\, f^{-1}(V) \neq \emptyset$ if $f^{-1}(V) \neq \emptyset$. The concept of quasi-continuous functions first appeared in [18] for real functions of several real variables. By definition, quasi-continuous functions are feebly continuous, but in general the converse does not hold. There are nearly continuous functions which are not even feebly continuous, and there are quasi-continuous functions which are not nearly continuous. The notion of semi-precontinuous functions is based on a concept in [1]. Quasi-continuous and nearly continuous functions are semi-precontinuous.

Lemma 2.1. *Let $(G, \cdot, \tau)$ be a semitopological group.*

(a) $\mathfrak{i}$ *is quasi-(nearly, semi-pre) continuous if and only if it is quasi- (nearly, semi-pre) continuous at a point x_0.*

(b) $\mathfrak{m}$ *is quasi-(nearly, semi-pre) continuous if and only if it is quasi- (nearly, semi-pre) continuous at a point (x_0, y_0).*

Quasi-continuity was used by Kenderov et al to study when a paratopological group is topological, and they showed in [19] that a paratopological group is a topological group if and only if its inversion is quasi-continuous. To obtain conditions which make a paratopological group $(G, \cdot, \tau)$ a topological group, Arhangel'skii and Reznichenko [3] proved the following lemma.

Lemma 2.2 ([3])**.** *Suppose that $(G, \cdot, \tau)$ is a paratopological group such that $e \in \operatorname{int}\overline{U^{-1}}$ for each open neighborhood U of the neutral element e of G. Then $(G, \cdot, \tau)$ is a topological group.*

In our terminology, we can say that each paratopological group $(G, \cdot, \tau)$ whose inversion is semi-precontinuous at e is a topological group. Indeed, this fact is equivalent to Lemma 1.2 of [3], which asserts that *if a paratopological group $(G, \cdot, \tau)$ is not topological, then there is a neighborhood U of e such that $U \cap U^{-1}$ is nowhere dense.* Our next theorem summarizes all these facts where weak continuity properties play roles.

Theorem 2.3 ([3], [19])**.** *For a paratopological group $(G, \cdot, \tau)$, the following statements are equivalent:*

(a) *The inversion $\mathfrak{i}$ of $(G, \cdot, \tau)$ is semi-precontinuous at e.*
(b) *$(G, \cdot, \tau)$ is a topological group.*
(c) *For every open neighbourhood U of e, $e \in \operatorname{int}\overline{U \cap U^{-1}}$.*
(d) *For every open neighbourhood U of e, $\operatorname{int}\overline{U \cap U^{-1}} \neq \emptyset$.*
(e) *The inversion $\mathfrak{i}$ of $(G, \cdot, \tau)$ is quasi-continuous at e.*

Corollary 2.4. *A paratopological group $(G, \cdot, \tau)$ is a topological group if and only if its inversion $\mathfrak{i}$ is nearly continuous at e.*

Remark 2.5. It is important to note that the condition (e) of Theorem 2.3 cannot be replaced by the weaker condition that the inversion $\mathfrak{i}$ is feebly continuous. Indeed, there are paratopological but not topological groups whose inversions are feebly continuous, for example, the Sorgenfrey line, *totally bounded* paratopological groups. A paratopological group with feebly continuous inversion is called *saturated* by Guran in [16]. In many aspects, saturated paratopological groups behave like topological groups, we refer to [4] and [5]. Guran [16] also asked if a Baire regular paratopological group must be saturated. The general case of this question was answered negatively by Ravsky [31]. However, Cao and Greenwwood [11] showed that its answer is affirmative with the appearance of a countable π-network.

In the light of Theorem 2.3, one may consider the following "dual" problem: *Let $(G, \cdot, \tau)$ be a topologized group such that $\mathfrak{i}$ is continuous. Must semi-precontinuity and continuity of $\mathfrak{m}$ be equivalent?* Unfortunately, the answer is negative as shown by the following examples.

Example 2.6. (a) Let $(\mathbb{Z}_2, +)$ be the group of integers modulo 2 with the usual operation of addition, i.e., $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$. Let $\tau_s = \{\emptyset, \{\bar{0}\}, \mathbb{Z}_2\}$ be the Sierpiński topology on $\mathbb{Z}_2$. It is shown in [7] that $\mathfrak{m}$ is quasi-continuous. Furthermore, one can also prove that $\mathfrak{m}$ is nearly continuous and $\mathfrak{i}$ is continuous. Since $(\mathbb{Z}_2, +)$ is abelian and $\lambda_{\bar{1}}$ is not continuous, $(\mathbb{Z}_2, \cdot, \tau_s)$ is neither a left nor a right semitopological group.

(b) Let τ_{cf} be the co-finite topology on $\mathbb{R}$. It is well-known that τ_{cf} is a T_1-topology. We consider the topologized group $(\mathbb{R}, +, \tau_{\text{cf}})$, where $+$ is the usual addition on $\mathbb{R}$. It is easy to see that $\mathfrak{i}$ is continuous. For any nonempty open subset V of $(\mathbb{R}, \tau_{\text{cf}})$, $\mathfrak{m}^{-1}(V)$ has empty interior in $(\mathbb{R} \times \mathbb{R}, \tau_{\text{cf}} \times \tau_{\text{cf}})$ but $\overline{\mathfrak{m}^{-1}(V)} = \mathbb{R} \times \mathbb{R}$. It follows that $\mathfrak{m}^{-1}(V) \subseteq \operatorname{int} \overline{\mathfrak{m}^{-1}(V)}$, and thus $\mathfrak{m}$ is nearly continuous but not quasi-continuous (not even feebly continuous). Furthermore, one can check easily that all left and right translations are continuous. Therefore, $(\mathbb{R}, +, \tau_{\text{cf}})$ is a semitopological (but not paratopological) group.

Example 2.7. *There exists a topologized abelian group $(G, +, \tau)$ which is a Tychonoff space such that $\mathfrak{i}$ is continuous, $\mathfrak{m}$ is quasi-continuous but neither nearly nor separately continuous.* Let $(G, +)$ be the set of reals with the usual addition. Equip $(G, +)$ with the following topology τ: rational numbers are isolated points and neighborhoods of irrational numbers are standard Euclidean ones. Then (G, τ) is a Tychonoff space, see [12, Example 5.1.22]. Clearly, $\mathfrak{i}$ is continuous, as the inverse of a rational number is rational and the inverse of an irrational number is irrational. Let x be any irrational number and y any rational number. Then the translation by x is discontinuous at $y - x$. Thus, $(G, +, \tau)$ is not semitopological. Finally, we claim that $\mathfrak{m}$ is quasi-continuous but not nearly continuous. Let $x, y, z \in G$ and $z = x + y$. If z is irrational, then clearly $\mathfrak{m}$ is continuous at (x, y). Suppose that z is rational. If x is rational, then y is also rational, so (x, y) is isolated in $G \times G$ and $\mathfrak{m}$ is continuous at (x, y). In case that x is irrational, then $\mathfrak{m}^{-1}(\{z\}) = \{(u, z - u) \colon u \in G\}$ is closed in $G \times G$ and $\operatorname{int} \mathfrak{m}^{-1}(\{z\}) = \{(u, z - u) \colon u \in \mathbb{Q}\}$. Since $\operatorname{int} \mathfrak{m}^{-1}(\{z\})$ is dense in $\mathfrak{m}^{-1}(\{z\})$, $\mathfrak{m}$ is quasi-continuous but not nearly continuous at (x, y).

Remark 2.8. In summary, we see that there are examples of topologized abelian groups:

(a) T_0-group which is not semitopological, $\mathfrak{i}$ is continuous, $\mathfrak{m}$ is both nearly continuous and quasi-continuous.

(b) T_1-semitopological group, $\mathfrak{i}$ is continuous, $\mathfrak{m}$ is nearly continuous, but not feebly continuous.

(c) Tychonoff group which is not semitopological, $\mathfrak{i}$ is continuous, $\mathfrak{m}$ is quasi-continuous, but not nearly continuous. These facts raise the question whether there exist some better examples. Note that a Hausdorff paratopological group can fail to be regular. Consider $(\mathbb{R}^2, +)$ equipped with the "semidisc topology" τ_B, where the family of the sets

$$B_\varepsilon = \{(x, y) \in \mathbb{R}^2 \colon\ x^2 + y^2 < \varepsilon,\ y > 0\} \cup \{(0, 0)\} \text{ with } \varepsilon > 0$$

is a local base at the element $(0, 0)$ of $\mathbb{R}^2$ for the topology τ_B, and the neighborhoods of other points are translations of neighborhoods of $(0, 0)$. It was pointed out by Tkachenko in [36] that $(\mathbb{R}^2, +, \tau_B)$ is a Hausdorff paratopological group, but it is not regular. Furthermore, he also mentioned in [36] that it is still an open problem whether every regular paratopological group must be a Tychonoff space.

Next, we give an example of a topologized group with a completely metrizable topology and continuous inversion, such that the feeble continuity and near (quasi-) continuity of its multiplication are not equivalent.

Example 2.9. *There exists a topologized group such that* $\mathfrak{i}$ *is continuous and* $\mathfrak{m}$ *is feebly continuous, but* $\mathfrak{m}$ *is not semi-precontinuous.* Let $(\mathbb{R}, +)$ be the group of reals with the usual addition. We equip $\mathbb{R}$ with the metric d defined by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y; \\ \max\{|x|, |y|\}, & x \neq y. \end{cases}$$

It is shown in [32] that $(\mathbb{R}, d)$ is a complete metric space. Let τ be the topology generated by d on $\mathbb{R}$. The neighborhoods of 0 in $(\mathbb{R}, \tau)$ are the same as those in the Euclidean topology. It can be checked easily that $\mathfrak{i}$ is continuous. To see that $\mathfrak{m}$ is feebly continuous, for every nonempty open set V of $(\mathbb{R}, d)$ pick an arbitrary point $a \in V$. Since every point in $(\mathbb{R}, \tau)$ except 0 is isolated, the set $\mathfrak{m}^{-1}(\{a\}) = \{(x, y) \in \mathbb{R} \times \mathbb{R} \colon\ x + y = a\}$ has nonempty interior, and so does $\mathfrak{m}^{-1}(V)$. This shows that $\mathfrak{m}$ is feebly continuous. Note that $\mathfrak{m}^{-1}(\{1\}) = \{(x, y) \in \mathbb{R} \times \mathbb{R} \colon\ x + y = 1\}$ is closed in $\mathbb{R} \times \mathbb{R}$, and that

$$\operatorname{int} \overline{\mathfrak{m}^{-1}(\{1\})} = \{(x, y) \in \mathbb{R} \times \mathbb{R} \colon\ x + y = 1\} \setminus \{(0, 1), (1, 0)\}$$

is also closed in $\mathbb{R} \times \mathbb{R}$. It follows that $\mathfrak{m}^{-1}(\{1\}) \not\subseteq \overline{\operatorname{int} \overline{\mathfrak{m}^{-1}(\{1\})}}$. Thus, $\mathfrak{m}$ is not semi-precontinuous.

Theorem 2.10. *Let $(G,\cdot,\tau)$ be a left or right semitopological group such that* $\operatorname{dev}(G) < \operatorname{Nov}(G)$. *Then* $\mathfrak{i}$ *is nearly continuous at* e.

Proof. Suppose that $(G,\cdot,\tau)$ is a left semitopological group with $\operatorname{dev}(G) < \operatorname{Nov}(G)$. Let V be an open neighborhood of e. We shall show that $\overline{V^{-1}}$ is also a neighborhood of e. Assume that $\operatorname{dev}(G) = \kappa$, and let $\{\mathscr{U}_\alpha\colon \alpha < \kappa\}$ be a family of open covers of G such that for any $x \in G$, $\{\operatorname{st}(x,\mathscr{U}_\alpha)\colon \alpha < \kappa\}$ is a neighborhood base of x. For each $\alpha < \kappa$ we define

$$A_\alpha := \{x \in G\colon \operatorname{st}(x,\mathscr{U}_\alpha) \subseteq x \cdot V\}.$$

Since (G,τ) is a left semitopological group, $x \cdot V$ is an open set for each $x \in X$. Thus, for each $x \in X$ there is an $\alpha < \kappa$ such that $\operatorname{st}(x,\mathscr{U}_\alpha) \subseteq x \cdot V$. This implies that $G = \bigcup\{A_\alpha\colon \alpha < \kappa\}$. Moreover, as $\operatorname{dev}(G) < \operatorname{Nov}(G)$, $\operatorname{int}(\overline{A_\alpha}) \neq \emptyset$ for some $\alpha < \kappa$. Thus, there are a point $x_0 \in G$ and $\beta < \kappa$ such that $\operatorname{st}(x_0,\mathscr{U}_\beta) \subseteq \overline{A_\alpha}$. Let $\mathscr{V}$ be the canonical common refinement of $\mathscr{U}_\beta$ and $\mathscr{U}_\alpha$. Set $U = \operatorname{st}(x_0,\mathscr{V})$. Then for each $z \in U \cap A_\alpha$ we have

$$x_0 \in \operatorname{st}(z,\mathscr{V}) \subseteq \operatorname{st}(z,\mathscr{U}_\alpha) \subseteq z \cdot V,$$

therefore $z^{-1}x_0 \in V$, so $x_0^{-1}z \in V^{-1}$, hence $x_0^{-1} \cdot (U \cap A_\alpha) \subseteq V^{-1}$. On the other hand, $U \subseteq \overline{A_\alpha}$ implies that $U \subseteq \overline{U \cap A_\alpha}$. As a consequence,

$$e \in x_0^{-1} \cdot U \subseteq x_0^{-1} \cdot \overline{U \cap A_\alpha} \subseteq \overline{x_0^{-1} \cdot (U \cap A_\alpha)} \subseteq \overline{V^{-1}}.$$

Since U is an open neighborhood of x_0, $x_0^{-1} \cdot U$ is an open neighborhood of e. We have shown that $\overline{V^{-1}}$ is also a neighborhood of e, and therefore $\mathfrak{i}$ is nearly continuous at e. □

Corollary 2.11. *Let $(G,\cdot,\tau)$ be a left or right semitopological group such that (G,τ) is a Baire and Moore space. Then* $\mathfrak{i}$ *is nearly continuous at* e.

Lemma 2.12 ([17], p. 95). *Let X be any infinite dimensional linear topological space of the second category. Then X contains a maximal proper linear subspace Y that is of the second category in X. Moreover,*

(a) *Y does not have the Baire property;*

(b) *if X is an infinite dimensional complete metric (or normed) space it must contain a subspace that is infinite dimensional, of the second category and metric (or normed) but not complete.*

Next, we use Lemma 2.12 to construct a left semitopological group whose inversion is nearly continuous but not quasi-continuous.

Example 2.13. *There exists a Baire and metric left semitopological group which is not a right semitopological group, and whose inversion is nearly continuous but not feebly continuous.* Let us consider the Banach space c_0 of all sequences convergent to zero with the sup-norm. It is known that c_0 is infinite dimensional. By Lemma 2.12, it contains a maximal proper subspace G_0 which is of the second category and does not have the Baire property. Since G_0 is a topological group, it is a Baire space as it is of the second category. Let us fix $\beta \in c_0 \setminus G_0$ and define $G := G_0 \cup (\beta + G_0)$. Note that G with the topology τ inherited from c_0 is a metric space. The subspace G_0 is dense and Baire in G, thus (G, τ) is a metric Baire space. Since $\beta \notin G_0$, we have $G_0 \cap (\beta + G_0) = \emptyset$. Consider the operation "$\star$" defined by

$$a \star b = \begin{cases} a + b, & \text{if } a \in G_0; \\ a - b, & \text{if } a \in \beta + G_0. \end{cases}$$

One can check that the operation "$\star$" is well-defined on G and $(G, \star)$ in fact forms a group. Furthermore, it is not difficult to see that the inversion $\mathfrak{i}\colon G \to G$ is defined as

$$\mathfrak{i}(a) = \begin{cases} -a, & \text{if } a \in G_0; \\ a, & \text{if } a \in \beta + G_0. \end{cases}$$

First, we claim that $(G, \star, \tau)$ is a left semitopological group but not a right semitopological group. Indeed, if $a \in G_0$, $b \in G$ and $\{b_n\colon n \in \mathbb{N}\}$ is an arbitrary sequence in G with $\lim_{n\to\infty} b_n = b$, then we have

$$\lim_{n\to\infty} (a \star b_n) = \lim_{n\to\infty} (a + b_n) = a + b = a \star b.$$

Similarly, if we take $a \in \beta + G_0$, $b \in G$ and $\{b_n\colon n \in \mathbb{N}\}$ is an arbitrary sequence in G with $\lim_{n\to\infty} b_n = b$, then

$$\lim_{n\to\infty} (a \star b_n) = \lim_{n\to\infty} (a - b_n) = a - b = a \star b.$$

This verifies that G is a left semitopological group. Now, take $b \in G$, $b \neq 0$ and a sequence $\{a_n\colon n \in \mathbb{N}\} \subseteq \beta + G_0$ such that $a_n \to 0$. Then $0 \star b = b$ and $a_n \star b = a_n - b \to -b$. Thus, $(G, \star, \tau)$ is not a right semitopological group.

Now, we show that $\mathfrak{i}$ is nearly continuous. Let V be a neighborhood of $\mathfrak{i}(a)$, where $a \in G$, i.e., V is of the form $W \cap G$, where W is open in c_0. We consider two cases. Suppose that $a \in \beta + G_0$. In this case, since $\beta + G_0$ is dense in c_0, we have

$$\overline{V^{-1}} \supseteq \overline{W \cap (\beta + G_0)} \supseteq W \cap G = V.$$

This shows that $\mathfrak{i}$ is nearly continuous at each $a \in \beta + G_0$. On the other hand, if $a \in G_0$ then $\mathfrak{i}(a) = -a$. In this case, G_0 is dense in c_0, thus

$$\overline{V^{-1}} \supseteq \overline{(-W) \cap (-G_0)} = \overline{(-W) \cap G_0} \supseteq \overline{-W}^{c_0} \cap G \supseteq -W \cap G,$$

where $\overline{-W}^{c_0}$ is the closure of $-W$ in c_0. It follows that $\overline{V^{-1}}$ contains an open neighborhood of a in G, i.e., $\mathfrak{i}$ is nearly continuous at each $a \in G_0$. Finally, we show that $\mathfrak{i}$ is not feebly continuous. Take a point $x_0 \in G$ such that $\|x_0\| = 1$, and consider the open ball $W = B(x_0, 1/2)$ in c_0 centered at x_0 with radius $1/2$. Let $V = W \cap G$. Then,

$$\mathfrak{i}^{-1}(V) = V^{-1} = (-W \cap G_0) \cup (W \cap (\beta + G_0)).$$

Since $(-W) \cap W = \emptyset$, G_0 and $\beta + G_0$ are disjoint dense subsets of G, the interior of $\mathfrak{i}^{-1}(V)$ in G is empty. Hence, $\mathfrak{i}$ is not feebly continuous.

Note that in Example 2.13, (G, τ) is a metric and Baire space, but $\mathfrak{i}$ is not feebly continuous. We will see in Section 3 that the reason why this happens is that $(G, \star, \tau)$ fails to be a semitopological group (i.e., not a right semitopological group, as it is a left semitopological group).

3. The main result

In this section we investigate when a topologized group $(G, \cdot, \tau)$ is a topological group. Let X, Y, Z be three topological spaces, and let $f\colon X \times Y \to Z$ be a function from $X \times Y$ into Z. Recall that f is called *quasi-continuous with respect to y* [26] at (x, y) if for every open neighborhood W of $f(x, y)$ and every open neighborhood $U \times V$ of (x, y), there are an open neighborhood V' of y and a nonempty open set $U' \subseteq U$ such that $f(U' \times V') \subseteq W$. Quasi-continuity with respect to the second variable plays an important role in the theory of separate vs. joint continuity. It is called strong quasi-continuity in [10] and [19], where it is applied to the study of the problem when a semitopological group is a topological group. In general, quasi-continuity and quasi-continuity with respect to the second variable are two distinct notions for the multiplication operation $\mathfrak{m}$ of a topologized group $(G, \cdot, \tau)$. For example, let $(\mathbb{Z}_2, \cdot, \tau)$ be the topologized group given in Example 2.6. We have seen that $\mathfrak{m}$ is quasi-continuous. However, it can be checked readily that $\mathfrak{m}$ is not quasi-continuous with respect to the second variable at the point $(\bar{1}, \bar{1}) \in \mathbb{Z}_2 \times \mathbb{Z}_2$.

The next lemma is essentially proved in [26, Theorem 1], where X and Y are assumed to be Baire and first countable respectively, and $D = Y$. Hence, its proof is omitted. Results similar to this one can be found in [10, Theorem 1] and [19, Lemma 1], where some properties defined by topological games are used.

Lemma 3.1 ([26]). *Let $f\colon X\times Y\to Z$ be a function from the product space $X\times Y$ into a regular space Z such that $f(x,\cdot)\colon Y\to Z$ is continuous for all $x\in X$, and there is a dense subset $D\subseteq Y$ such that $f(\cdot,y)\colon X\to Z$ is quasi-continuous for all $y\in D$. If $\mathrm{Nov}(O)>\chi(Y)$ for every nonempty open subset $O\subseteq X$, then f is quasi-continuous with respect to y at every point $(x,y)\in X\times D$.*

Lemma 3.2. *Let $(G,\cdot,\tau)$ be a left or right semitopological group. Then $\mathrm{Nov}(G)=\mathrm{Nov}(O)$ for every nonempty open set $O\subseteq G$.*

Proof. It is clear that $\mathrm{Nov}(O)\leqslant\mathrm{Nov}(G)$ for any nonempty open subset $O\subseteq G$. Now, for any fixed nonempty open subset $O\subseteq G$, suppose $\mathrm{Nov}(O)=\kappa$. Pick up any point $x\in O$. For any point $y\in G$, since $\lambda_{y\cdot x^{-1}}$ is continuous, $(y\cdot x^{-1})\cdot O$ is an open neighborhood of y with $\mathrm{Nov}((y\cdot x^{-1})\cdot O)=\kappa$. Thus, G has an open cover $\mathscr{U}=\{U_\alpha\colon\ \alpha\in I\}$ such that $\mathrm{Nov}(U_\alpha)=\kappa$ for all $\alpha\in I$. Let $\mathscr{V}=\{V_\beta\colon\ \beta\in J\}$ be a maximal disjoint open family (not necessarily a cover) which refines $\mathscr{U}$. Then $G\setminus\bigcup\{V_\beta\colon\ \beta\in J\}$ is nowhere dense in G and $\mathrm{Nov}(V_\beta)\leqslant\kappa$ for all $\beta\in J$. Suppose $V_\beta=\bigcup\{A_{\beta,\gamma}\colon\ \gamma<\kappa\}$, where each $A_{\beta,\gamma}$ is nowhere dense in G. Since $\{V_\beta\colon\ \beta\in J\}$ is a pairwise disjoint open family, each $A_\gamma=\bigcup\{A_{\beta,\gamma}\colon\ \beta\in J\}$ is nowhere dense in G. It follows that

$$G=\Bigl(\bigcup\{A_\gamma\colon\ \gamma<\kappa\}\Bigr)\cup\Bigl(G\setminus\bigcup\{V_\beta\colon\ \beta\in J\}\Bigr).$$

Thus, we have shown that $\mathrm{Nov}(G)=\kappa$. □

Theorem 3.3. *Let $(G,\cdot,\tau)$ be a topologized group endowed with a regular topology τ such that $\mathrm{dev}(G)<\mathrm{Nov}(G)$. If all left (right) translations are feebly continuous and all right (left) translations are continuous, then $(G,\cdot,\tau)$ is a topological group.*

Proof. Suppose that all left translations are feebly continuous and all right translations are continuous. By Theorem 2.10, the inversion $\mathfrak{i}$ of $(G,\cdot,\tau)$ must be nearly continuous at e. Thus, by Corollary 2.4, we are done if we can show that $(G,\cdot,\tau)$ is a paratopological group. Assume that $\mathrm{dev}(G)=\kappa$. Then there exists an ordered family $\{\mathscr{U}_\alpha\colon\ \alpha<\kappa\}$ of open covers of (G,τ) such that for any $x\in G$, $\{\mathrm{st}(x,\mathscr{U}_\alpha)\colon\ \alpha<\kappa\}$ is a neighborhood base of x.

We first prove that $(G,\cdot,\tau)$ is a semitopological group. It suffices to show that for any fixed $h\in G$, λ_h is continuous at an arbitrary point $g\in G$. For this purpose, for each $\alpha<\kappa$ we define

$$O(\lambda_h,\alpha):=\bigcup\{V\colon\ V\subseteq G \text{ is open, and } \lambda_h(V)\subseteq P \text{ for some } P\in\mathscr{U}_\alpha\}.$$

Clearly, $O(\lambda_h,\alpha)$ is open in (G,τ). Further, $O(\lambda_h,\alpha)$ is also dense in (G,τ). To see this, let $U\subseteq G$ be a nonempty open set. Since $\lambda_{h^{-1}}$ is feebly continuous, $\mathrm{int}(\lambda_h(U))$

is nonempty. Thus, we can choose a nonempty open set W and $P \in \mathscr{U}_\alpha$ such that $W \subseteq h \cdot U$ and $W \subseteq P$. On the other hand, λ_h is also feebly continuous. Thus, we can choose a nonempty open set V such that $V \subseteq \lambda_{h^{-1}}(W)$. Then $V \subseteq U \cap O(\lambda_h, \alpha)$. Since $\mathrm{Nov}(G) > \kappa$, we have

$$\bigcap\{O(\lambda_h, \alpha)\colon\ \alpha < \kappa\} \neq \emptyset.$$

Also, it is clear that λ_h is continuous at every point of $\bigcap\{O(\lambda_h, \alpha)\colon\ \alpha < \kappa\}$. To show that λ_h is continuous at g, choose a point $s \in G$ such that

$$g \cdot s \in \bigcap\{O(\lambda_h, \alpha)\colon\ \alpha < \kappa\}.$$

Then λ_h is continuous at $g \cdot s$. Note that for any $s \in G$, $\varrho_{s^{-1}}$ and ϱ_s are continuous. Further, we have $\lambda_h = \varrho_{s^{-1}} \circ \lambda_h \circ \varrho_s$. This implies that λ_h is continuous at the point $g \in G$.

Next, we show that $(G, \cdot, \tau)$ is a paratopological group. By Lemma 3.2 and Lemma 3.1, $\mathfrak{m}$ is quasi-continuous with respect to y at every point $(x, y) \in G \times G$. Now, for any given point $y \in G$ and for every $\alpha < \kappa$ define

$$O(\mathfrak{m}, \alpha) := \bigcup\{U\colon\ U \subseteq G \text{ is an open subset, and } \mathfrak{m}(U \times V) \subseteq P \text{ for some open neighborhood } V \text{ of } y \text{ and } P \in \mathscr{U}_\alpha\}.$$

Using quasi-continuity of $\mathfrak{m}$ with respect to the second variable, it can be shown that for each $\alpha < \kappa$, $O(\mathfrak{m}, \alpha)$ is a dense and open set in (G, τ). Moreover, since $\mathrm{Nov}(G) > \kappa$, we have $\bigcap\{O(\mathfrak{m}, \alpha)\colon\ \alpha < \kappa\} \neq \emptyset$. It is clear that $\mathfrak{m}$ is (jointly) continuous at (x, y) for any point $x \in \bigcap\{O(\mathfrak{m}, \alpha)\colon\ \alpha < \kappa\}$. Hence, $(G, \cdot, \tau)$ is a paratopological group. □

Corollary 3.4. *Let $(G, \cdot, \tau)$ be a topologized group such that (G, τ) is a Baire Moore space. If all left (right) translations are feebly continuous and all right (left) translations are continuous, then $(G, \cdot, \tau)$ is a metrizable topological group.*

Theorem 3.3 and Corollary 3.4 extends several results in literature.

Corollary 3.5 ([29])**.** *Every Baire and Moore semitopological group is a paratopological group.*

According to [34, p. 66], the next result is due to an unpublished work of Reznichenko.

Corollary 3.6. *Every semitopological group which is a Baire metrizable space is a topological group.*

The class of *strongly Baire* spaces was introduced in [19]. It was shown that every strongly Baire semitopological group is a topological group. It is easy to show that every Baire Moore space is strongly Baire. Thus, Corollary 3.5 and Corollary 3.6 can also be derived from the results in [19]. Recall that a topological space is *symmetrizable* [2] if its topology is generated by a symmetric, i.e., a distance function satisfying all the usual axioms for a metric, except for the triangle inequality.

Corollary 3.7 ([3]). *Every symmetrizable Hausdorff Baire paratopological group is a metrizable topological group.*

Proof. It was shown in [21] that every symmetrizable paratopological group is a Moore space. Therefore, the conclusion follows from Corollary 3.4. □

Example 2.9 shows that in general the conclusion of Theorem 3.3 does not hold for topologized groups when both left and right translations are feebly continuous, under the same assumption on the topology. Our next example shows that in general the conclusion of Theorem 3.3 does not hold for topologized groups even when both left and right translations are quasi-continuous, and the topology is separable, metrizable and Baire.

Example 3.8. *There exists a topologized group $(G,\cdot,\tau)$ with a separable, metrizable and Baire topology such that $\mathfrak{m}$ is separately quasi-continuous, but $(G,\cdot,\tau)$ is not a paratopological group.* Let $G=[0,1)$ be the half-open and half-closed unit interval of $\mathbb{R}$ equipped with the multiplication

$$x\cdot y=\begin{cases} x+y, & \text{if } x+y<1;\\ x+y-1, & \text{if } x+y\geqslant 1.\end{cases}$$

Let τ be the Euclidean topology on G. Then it is clear that (G,τ) is a separable metric Baire space, and $(G,\cdot)$ is a group with $e=0$.

We first verify that $\mathfrak{m}$ is quasi-continuous with respect to y at every point $(x,y)\in G\times G$. At any point (x,y) with $x+y\neq 1$, $\mathfrak{m}$ is continuous. Suppose that (x,y) is a point with $x+y=1$. Then $\mathfrak{m}(x,y)=0$. For any $0<\varepsilon<1$ and every neighborhood $U\times V$ of (x,y) we can find $\delta_1,\delta_2>0$ such that $(x-\delta_1,x+\delta_1)\subseteq U$, $(y-\delta_1,y+\delta_2)\subseteq V$ and $0<\delta_1-\delta_2<\varepsilon$. Let $U'=(x+\frac12(\delta_1+\delta_2),x+\delta_1)$ and $V'=(y-\delta_2,y+\delta_2)$. Then it can be checked that $\mathfrak{m}(U'\times V')\subseteq[0,\varepsilon)$. This verifies the claim. However, $\mathfrak{m}$ is not continuous at $(\frac12,\frac12)$. To see this, for each $n\in\mathbb{N}$ let $x_n=y_n=\frac12-1/(n+1)$. Then $(x_n,y_n)\to(\frac12,\frac12)$ but $\mathfrak{m}(x_n,y_n)=1-2/(n+1)$ does not converge to $\mathfrak{m}(\frac12,\frac12)$. In fact, $\mathfrak{m}$ is not continuous at any point on the line segment $\{(x,y)\in G\times G\colon\ x+y=1\}$. Therefore, $(G,\cdot,\tau)$ is not a paratopological group.

Further, we can show that $\mathfrak{i}$ is not continuous at e either. In fact, let $x_n = 1/(n+1)$. Then $x_n \to e$ and $\mathfrak{i}(x_n) = 1 - 1/(n+1)$. However, $\{\mathfrak{i}(x_n)\colon n \in \mathbb{N}\}$ does not converge to e.

4. Open questions

In this section we give some additional remarks which are related to the results in the previous two sections, and pose several open questions.

Remark 4.1. First, Corollary 3.4 should be compared with Theorem 1.1 in [34], which says that *if a group G with a metric, separable and Baire topology such that ϱ_h is Baire measurable for densely many $h \in G$ and λ_g is continuous for all $g \in G$, then G is a topological group.* As mentioned in [34], there is an example of a group and a compact metric topology on it such that λ_g is continuous for all g but the multiplication is discontinuous. This shows that the assumption about feeble continuity in Theorem 3.3 cannot be dropped. Furthermore, Theorem 3.3 should also be compared with Theorem 1 in [13], which claims the following: *For a Baire metrizable group G, if there is a dense set S of the second category in G such that the right translations ϱ_s and $\varrho_{s^{-1}}$ are continuous for all $s \in S$ and also for each $s \in G$, there is a residual set R_s of G such that the left translation λ_s is feebly continuous on R_s, $\lambda_s(R_s)$ is residual and $\lambda_{s^{-1}}$ is feebly continuous on $\lambda_s(R_s)$, then G is a topological group.*

Question 4.2. Can we relax hypothesis: "all right translations are continuous" in Theorem 3.3 to the condition: "there is a dense subset S of the second category in G such that the right translations ϱ_s and $\varrho_{s^{-1}}$ are continuous for all $s \in S$"?

It is unclear to the authors whether the conclusion of Corollary 3.7 is still valid for semitopological groups. In addition, it is unclear whether every symmetrizable semitopological group must be a Moore space either. These motivate us to pose the following natural question.

Question 4.3. Must every symmetrizable Hausdorff Baire semitopological group be a topological group?

Remark 4.4. Note that in Example 3.8, $\mathfrak{m}$ is continuous at any point (x, y) with $x + y \neq 1$. This outcome is somehow not surprising. If $(G, \cdot, \tau)$ is a topologized group with a separable, metrizable and Baire topology τ such that all left and right translations are quasi-continuous, then by Theorem 2 of [24], $\mathfrak{m}$ is quasi-continuous. Now, $G \times G$ is a separable, metrizable and Baire space, hence by a classical result, the set of points of continuity of $\mathfrak{m}$ is a dense G_δ-set in $G \times G$. Furthermore, by a result of Neubrunn in [25], if $(G, \cdot, \tau)$ is a topologized group with a topology τ such that $w(G) < \mathrm{Nov}(G)$, all left (right) translations are feebly continuous and all right (left) translations are quasi-continuous, then $\mathfrak{m}$ is feebly continuous. As a consequence, under the same assumption on τ, if all left and right translations are quasi-continuous, then $\mathfrak{m}$ is quasi-continuous. In [24], a separately feebly continuous function $f\colon [-1,1] \times [-1,1] \to \mathbb{R}$ which is not feebly continuous is constructed.

Question 4.5. Let $(G, \cdot, \tau)$ be a topologized group with a separable, metrizable and Baire topology τ such that all left and right translations are feebly continuous. Must $\mathfrak{m}$ be feebly continuous?

If the answer to Question 4.5 is affirmative, then we conclude that the set $C(\mathfrak{m})$ of points of continuity of $\mathfrak{m}$ is somewhere dense, and thus $C(\mathfrak{m}) \neq \emptyset$. Hence, we can view Question 4.5 as an analog of Talagrand's question in [35]. Finally, we conclude the paper with one more question.

Question 4.6. Let $(G, \cdot, \tau)$ be a topologized group with a Polish topology τ such that all left and right translations are quasi-continuous. Must $(G, \cdot, \tau)$ be a topological group?

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