Pratulananda Das; Santanu Bhunia

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TWO VALUED MEASURE AND SUMMABILITY OF DOUBLE SEQUENCES

Pratulananda Das and Santanu Bhunia, West Bengal

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Abstract. In this paper, following the methods of Connor [2], we extend the idea of statistical convergence of a double sequence (studied by Muresaleen and Edely [12]) to μ-statistical convergence and convergence in μ-density using a two valued measure μ. We also apply the same methods to extend the ideas of divergence and Cauchy criteria for double sequences. We then introduce a property of the measure μ called the (APO_2) condition, inspired by the (APO) condition of Connor [3]. We mainly investigate the interrelationships between the two types of convergence, divergence and Cauchy criteria and ultimately show that they become equivalent if and only if the measure μ has the condition (APO_2).

Keywords: double sequences, μ-statistical convergence, divergence and Cauchy criteria, convergence, divergence and Cauchy criteria in μ-density, condition (APO_2)

MSC 2010: 40A30, 40A05

1. Introduction

The usual notion of convergence does not always capture in fine details the properties of the vast class of sequences that are not convergent. One way of including more sequences under preview is to consider those sequences that are convergent when restricted to some 'big' set of natural numbers. By a 'big' set one understands a set $K \subset \mathbb{N}$ having asymptotic density equal to 1. Investigations in this line was initiated by Fast [8] and independently by Schoenberg [17] who introduced the idea of statistical convergence. Since then this concept was studied by *Šalát* [16], Fridy [9], Connor ([2], [3]) and many others (see [5], [6], [10], [12], [13]) where more references can be found about related works). In particular, in [2] and [3] Connor proposed two very interesting extensions of the concept of statistical convergence using a complete

$\{0,1\}$ valued measure μ defined on an algebra of subsets of $\mathbb{N}$ which form the basis of many more recent works ([4] where more references can be found).

The notion of statistical convergence was introduced for double sequences by Muresaleen and Edely [12] (also by Móricz [11] who introduced it for multiple sequences). More results on double sequences can be found in [1], [5], [6], [7]. In Section 3 of the paper we introduce the notions of μ-statistical convergence and convergence in μ-density (following the line of Connor [2]) using a two valued measure μ defined on an algebra of subsets of $\mathbb{N}\times\mathbb{N}$ and mainly investigate the inter-relationship between these two concepts.

In Section 4 of the paper we focus on the Cauchy criteria and introduce the Cauchy conditions associated with the two types of convergence defined in Section 3. Though one of them, namely the 'μ-statistical Cauchy condition' appeared in [3], the other 'Cauchy condition in μ-density' and in particular the relation between these two concepts was never explored before. We do precisely this in this section and as the underlying structure we take a metric space (X,ϱ).

Finally, in Section 5 we explore another relatively unexplored concept, namely, the divergence of double sequences of real numbers corresponding to the measure μ. We also introduce a new property of the measure μ called (APO_2) which plays the most important role throughout the paper, and show by an example that this condition is strictly weaker than the condition (APO) of Connor [3].

2. Definitions and notation

Throughout the paper $\mathbb{N}$ denotes the set of all natural numbers, χ_A represents the characteristic function of $A\subseteq\mathbb{N}$ and $\mathbb{R}$ represents the set of all real numbers. Recall that a set $A\subseteq\mathbb{N}$ is said to have the asymptotic density $d(A)$ if $d(A)=\lim\limits_{n\to\infty} n^{-1}\sum\limits_{j=1}^{n}\chi_A(j)$.

Definition 1 ([8]). A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be statistically convergent to $\xi\in\mathbb{R}$ if for any $\varepsilon>0$ we have $d(A(\varepsilon))=0$, where $A(\varepsilon)=\{n\in\mathbb{N}\colon |x_n-\xi|\geqslant\varepsilon\}$.

By the convergence of a double sequence we mean the convergence in Pringsheim's sense (see [14]):

A double sequence $x=\{x_{ij}\}_{i,j\in\mathbb{N}}$ of real numbers is said to be convergent to $\xi\in\mathbb{R}$ if for any $\varepsilon>0$, there exists $N_\varepsilon\in\mathbb{N}$ such that $|x_{ij}-\xi|<\varepsilon$ whenever $i,j\geqslant N_\varepsilon$. In this case we write $\lim\limits_{i,j\to\infty} x_{ij}=\xi$.

A double sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ of real numbers is said to be bounded if there exists a positive real number M such that $|x_{ij}| < M$ for all $i, j \in \mathbb{N}$. That is, $||x||_{(\infty,2)} = \sup_{i,j\in\mathbb{N}} |x_{ij}| < \infty$.

Let $K \subseteq \mathbb{N} \times \mathbb{N}$ and let $K(i,j) = |\{(m,n) \in K \colon m \leqslant i, n \leqslant j\}|$. If the sequence $\{K(i,j)/(i\cdot j)\}_{i,j\in\mathbb{N}}$ has a limit in Pringsheim's sense then we say that K has double natural density and is denoted by $d_2(K) = \lim_{i,j\to\infty} K(i,j)/(i\cdot j)$.

Definition 2 ([12]). A double sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ of real numbers is said to be statistically convergent to $\xi \in \mathbb{R}$ if for any $\varepsilon > 0$ we have $d_2(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{(i,j) \in \mathbb{N}\times\mathbb{N} \colon |x_{ij} - \xi| \geqslant \varepsilon\}$.

A statistically convergent double sequence of elements of a metric space (X, ϱ) is defined essentially in the same way (with $\varrho(x_{ij}, \xi) \geqslant \varepsilon$ instead of $|x_{ij} - \xi| \geqslant \varepsilon$).

Throughout the paper μ will denote a complete $\{0,1\}$ valued finite additive measure defined on an algebra Γ of subsets of $\mathbb{N}\times\mathbb{N}$ that contains all subsets of $\mathbb{N}\times\mathbb{N}$ that are contained in the union of a finite number of rows and columns of $\mathbb{N}\times\mathbb{N}$ and $\mu(A) = 0$ if A is contained in the union of a finite number of rows and columns of $\mathbb{N}\times\mathbb{N}$.

3. μ-statistical convergence and convergence in μ-density

We first introduce the following two definitions.

Definition 3. A double sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ of real numbers is said to be μ-statistically convergent to $L \in \mathbb{R}$ if and only if for any $\varepsilon > 0$, $\mu(\{(i,j) \in \mathbb{N}\times\mathbb{N} \colon |x_{ij} - L| \geqslant \varepsilon\}) = 0$.

Definition 4. A double sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ of real numbers is said to be convergent to $L \in \mathbb{R}$ in μ-density if there exists an $A \in \Gamma$ with $\mu(A) = 1$ such that $\{x_{ij}\}_{(i,j)\in A}$ is convergent to L.

If C_μ and C^*_μ denote respectively the sets of all double sequences which are μ-statistically convergent and convergent in μ-density then as in [2] (see also [6]) it is easy to prove that C^*_μ is a dense subset of C_μ which again is closed in l^∞_2 (the set of all bounded double sequences of real numbers endowed with the sup metric). Further, following the methods of [2] one can easily verify that there exists a measure μ (the details of μ are given in the next section) such that it is always possible to construct a double sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ which is μ-statistically convergent but does not converge to any point in μ-density.

This brings us to the most important question: for which measure μ we have $C_\mu = C^*_\mu$. In [3] it was proved that μ-statistical convergence and convergence in

μ-density of ordinary sequences of real numbers are equivalent if and only if the measure μ defined on an algebra of subsets of $\mathbb{N}$ satisfies the following condition (APO):

A measure μ satisfies the condition (APO) if for every sequence $\{A_n\}_{n\in\mathbb{N}}$ of mutually disjoint μ-null sets there exists a countable family of μ-null sets $\{B_n\}_{n\in\mathbb{N}}$ such that $A_n \Delta B_n$ is finite for all $n \in \mathbb{N}$ and $B = \bigcup\limits_{n\in\mathbb{N}} B_n \in \Gamma$ with $\mu(B) = 0$.

If a measure μ satisfies the condition (APO) (the definition of (APO) being the same as in the case of ordinary sequences) then as in Theorem 1 [3] we can easily prove that $C_\mu = C^*_\mu$. However, unlike single sequences, the condition (APO) is not necessary in the case of double sequences. For example, consider the algebra Γ_0 consisting of only those subsets of $\mathbb{N}\times\mathbb{N}$ that are contained in the union of a finite number of rows and columns of $\mathbb{N}\times\mathbb{N}$ and their complements and the corresponding measure μ_0 (which corresponds to Pringsheim's convergence). Obviously $C_{\mu_0} = C^*_{\mu_0}$ for this measure μ_0. However, note that the sets $A_i = \{i\}\times\mathbb{N} \in \Gamma_0$ for all $i \in \mathbb{N}$ with $\mu_0(A_i) = 0$ for all $i \in \mathbb{N}$ and $A_i \cap A_j = \emptyset$ if $i \neq j$. If we omit from $\mathbb{N}\times\mathbb{N}$ only finitely many elements of each A_i (or some A_i's), the resulting set cannot be a μ_0-null set, which shows that μ_0 does not satisfy the condition (APO).

From the above we can come to the conclusion that the situation is different for double sequences and we now introduce the following condition:

(APO$_2$) (Additive property of null sets)

The measure μ is said to satisfy the condition (APO$_2$) if for every sequence $\{A_i\}_{i\in\mathbb{N}}$ of mutually disjoint μ-null sets (i.e. $\mu(A_i) = 0$ for all $i \in \mathbb{N}$) there exists a countable family of sets $\{B_i\}_{i\in\mathbb{N}}$ such that $A_i \Delta B_i$ is included in the union of a finite number of rows and columns of $\mathbb{N}\times\mathbb{N}$ for every $i \in \mathbb{N}$ and $\mu(B) = 0$ where $B = \bigcup\limits_{i\in\mathbb{N}} B_i$ (hence $\mu(B_i) = 0$ for every $i \in \mathbb{N}$).

Theorem 1. *$C_\mu = C^*_\mu$ if μ satisfies the condition* (APO$_2$).

P r o o f. Suppose μ satisfies the condition (APO$_2$). We shall prove that $C_\mu = C^*_\mu$. To prove this it is sufficient to show that $C_\mu \subseteq C^*_\mu$. Let $x = \{x_{ij}\}_{i,j\in\mathbb{N}} \in C_\mu$ and let x be μ-statistically convergent to l. Then for any $\varepsilon > 0$, $\mu(A(\varepsilon)) = 0$ where $A(\varepsilon) = \{(i,j) \in \mathbb{N}\times\mathbb{N}\colon\ |x_{ij} - l| \geqslant \varepsilon\}$. Put $A_1 = \{(i,j) \in \mathbb{N}\times\mathbb{N}\colon\ |x_{ij} - l| \geqslant 1\}$ and $A_k = \{(i,j) \in \mathbb{N}\times\mathbb{N}\colon\ 1/k \leqslant |x_{ij} - l| < 1/(k-1)\}$ for $k \geqslant 2$. Thus we get a collection $\{A_i\}_{i\in\mathbb{N}}$ of subsets of $\mathbb{N}\times\mathbb{N}$ such that $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\mu(A_i) = 0$ for each $i \in \mathbb{N}$. By virtue of the condition (APO$_2$) there exists a sequence $\{B_i\}_{i\in\mathbb{N}}$ of sets such that $A_i \Delta B_i$ is included in the union of a finite number of rows and columns of $\mathbb{N}\times\mathbb{N}$ for each $i \in \mathbb{N}$, and $\mu(B) = 0$ where $B = \bigcup\limits_{i\in\mathbb{N}} B_i$.

Let $\varepsilon > 0$ be given. Choose $k \in \mathbb{N}$ such that $1/k < \varepsilon$. Then $\{(i,j) \in \mathbb{N} \times \mathbb{N}\colon |x_{ij} - l| \geqslant \varepsilon\} \subseteq \bigcup_{i=1}^{k} A_i$. As $A_i \Delta B_i$ $(i = 1, 2, \ldots, k)$ are included in the union of a finite number of rows and columns of $\mathbb{N} \times \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that $\Big(\bigcup_{m=1}^{k} B_m\Big) \cap \{(i,j) \in \mathbb{N} \times \mathbb{N}\colon i \geqslant n_0 \wedge j \geqslant n_0\} = \Big(\bigcup_{m=1}^{k} A_m\Big) \cap \{(i,j) \in \mathbb{N} \times \mathbb{N}\colon i \geqslant n_0 \wedge j \geqslant n_0\}$.

If $i, j \geqslant n_0$ and $(i,j) \notin B$ then $(i,j) \notin \bigcup_{m=1}^{k} B_m$ and so $(i,j) \notin \bigcup_{m=1}^{k} A_m$. This implies that $|x_{ij} - l| < 1/k < \varepsilon$. Hence $\{x_{ij}\}_{(i,j) \in \mathbb{N} \times \mathbb{N} \setminus B}$ is convergent to l where $\mu(\mathbb{N} \times \mathbb{N} \setminus B) = 1$. Therefore $x \in C_\mu^*$, that is $C_\mu \subseteq C_\mu^*$. Hence $C_\mu = C_\mu^*$.

Theorem 2. *If $C_\mu = C_\mu^*$ for a measure μ, then μ has the condition* (APO$_2$).

Proof. Suppose $C_\mu = C_\mu^*$ for a measure μ, and $\xi \in \mathbb{R}$. Choose a monotonic sequence $\{z_n\}_{n \in \mathbb{N}}$ of distinct elements of $\mathbb{R}$ such that $z_n \neq \xi$ for any $n \in \mathbb{N}$ and $\lim_{n \to \infty} z_n = \xi$. Then the sequence $\{|z_n - \xi|\}_{n \in \mathbb{N}}$ is a decreasing sequence converging to zero. Let $\varepsilon_n = |z_n - \xi|$, $n \in \mathbb{N}$. Let $\{A_j\}_{j \in \mathbb{N}}$ be a family of mutually disjoint sets with $\mu(A_j) = 0$ for each $j \in \mathbb{N}$. Define a double sequence $\{x_{mn}\}_{m,n \in \mathbb{N}}$ in the following way: $x_{mn} = z_j$ if $(m,n) \in A_j$ and $x_{mn} = \xi$ if $(m,n) \notin A_j$ for any $j \in \mathbb{N}$.

Let $\eta > 0$ be given. Choose $k \in \mathbb{N}$ such that $\varepsilon_k < \eta$. Then $A(\eta) = \{(m,n) \in \mathbb{N} \times \mathbb{N}\colon |x_{mn} - \xi| \geqslant \eta\} \subseteq A_1 \cup A_2 \cup \ldots \cup A_k$. Since μ is finitely additive so $\mu(A(\eta)) = 0$. Hence x is μ-statistically convergent to ξ, i.e. $x \in C_\mu$. Then by our assumption $x \in C_\mu^*$. So there exists $M \subseteq \mathbb{N} \times \mathbb{N}$ with $\mu(M) = 1$ such that $\{x_{mn}\}_{(m,n) \in M}$ is convergent to ξ. Let $B = \mathbb{N} \times \mathbb{N} \setminus M$. Then $\mu(B) = 0$. Let $B_j = A_j \cap B$. Then $\bigcup_{j=1}^{\infty} B_j = \bigcup_{j=1}^{\infty} (A_j \cap B) = B \cap \Big(\bigcup_{j=1}^{\infty} A_j\Big) \subseteq B$. Therefore $\mu\Big(\bigcup_{j=1}^{\infty} B_j\Big) = 0$.

Now we claim that $A_j \cap M$ is included in the union of a finite number of rows and columns of $\mathbb{N} \times \mathbb{N}$. If not, then M must contain an infinite sequence of elements $\{(m_k, n_k)\}_{k \in \mathbb{N}}$ where both $m_k, n_k \to \infty$ and $x_{m_k n_k} = z_j \neq \xi$ for all $k \in \mathbb{N}$, which contradicts that $\{x_{mn}\}_{(m,n) \in M}$ is convergent to ξ. Hence each $A_j \cap M$ must be contained in the union of a finite number of rows and columns of $\mathbb{N} \times \mathbb{N}$. Thus $A_i \Delta B_i = A_i \setminus B_i = A_i \cap M$ is also included in the union of a finite number of rows and columns of $\mathbb{N} \times \mathbb{N}$. This proves that the measure μ satisfies the condition (APO$_2$). $\square$

Remark 1. A similar condition like condition (APO$_2$) was used in [6] recently while studying the convergence of double sequences with respect to ideals. The importance of the condition (APO$_2$) will be more clear in the next sections when we discuss the Cauchy criteria and then the divergence of double sequences with respect to the measure μ.

4. Cauchy criteria for the measure μ

Following the idea of classical Cauchy condition Fridy [9] formulated the statistical Cauchy condition for sequences of real numbers which was further extended to μ-statistical Cauchy condition by Connor [3] (extensions of Cauchy conditions with respect to ideals were also studied by Dems [7]). We modify it to define a Cauchy type condition associated with μ-statistical convergence of double sequences in a metric space (X, ϱ).

We first introduce the following definition.

Definition 5. A double sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ in a metric space (X, ϱ) is said to be a μ- statistically Cauchy sequence if and only if for every $\varepsilon > 0$ there exists an $A \subset \mathbb{N} \times \mathbb{N}$ with $\mu(A) = 0$ such that $(i, j), (i_1, j_1) \notin A$ implies that $\varrho(x_{ij}, x_{i_1 j_1}) < \varepsilon$.

Theorem 3. *If a double sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ is a μ-statistically convergent then it is μ-statistically Cauchy sequence.*

P r o o f. The proof is straightforward. □

Remark 2. The converse is not generally true as can be readily seen by taking the double sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$ in $\wp[a, b]$ (the metric space of all polynomials on [a,b]) where $x_{1j}(t) = 1$, $j \in \mathbb{N}$, $x_{2j}(t) = 1 + t$, $j \in \mathbb{N}$, $x_{3j}(t) = 1 + t + t^2/2!$, $j \in \mathbb{N}$, …, $x_{nj}(t) = 1 + t + t^2/2! + \ldots + t^{n-1}/(n-1)!$, $j \in \mathbb{N}$, etc.

Then $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a usual Cauchy double sequence and so it is μ-statistically Cauchy for any measure μ. But $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is not μ-statistically convergent in $\wp[a, b]$.

Theorem 4. *In a metric space (X, ϱ), for any double sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ the following statements are equivalent.*

(i) *x is μ-statistically Cauchy;*

(ii) *for every $\varepsilon > 0$ there exists $(m_0, n_0) \in \mathbb{N} \times \mathbb{N}$ such that $\mu(\{(i, j) \in \mathbb{N} \times \mathbb{N}\colon \varrho(x_{ij}, x_{m_0 n_0}) < \varepsilon\}) = 1$.*

P r o o f. (i) ⇒ (ii)

Suppose that $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ is a μ-statistically Cauchy sequence. Then for every $\varepsilon > 0$ there exists an $A \subset \mathbb{N} \times \mathbb{N}$ with $\mu(A) = 0$ such that $(i, j), (i_1, j_1) \notin A$ implies that $\varrho(x_{ij}, x_{i_1 j_1}) < \varepsilon$. This implies that if $\varrho(x_{ij}, x_{i_1 j_1}) \geqslant \varepsilon$ for $(i, j), (i_1, j_1) \in \mathbb{N} \times \mathbb{N}$, then at least one of $(i, j), (i_1, j_1)$ must be in A. Since $A^c \neq \emptyset$, choose $(m_0, n_0) \in A^c$. Then $\varrho(x_{ij}, x_{m_0 n_0}) \geqslant \varepsilon$ implies that $(i, j) \in A$. Hence $\{(i, j) \in \mathbb{N}\times\mathbb{N}\colon \varrho(x_{ij}, x_{m_0 n_0}) \geqslant \varepsilon\} \subseteq A$, which implies that $\mu(\{(i, j) \in \mathbb{N} \times \mathbb{N}\colon \varrho(x_{ij}, x_{m_0 n_0}) \geqslant \varepsilon\}) = 0$ and so $\mu(\{(i, j) \in \mathbb{N} \times \mathbb{N}\colon \varrho(x_{ij}, x_{m_0 n_0}) < \varepsilon\}) = 1$. Thus (ii) holds.

(ii) $\Rightarrow$ (i)

Let (ii) hold and $\varepsilon > 0$ be given. Then there exists $(m_0, n_0) \in \mathbb{N} \times \mathbb{N}$ such that $\mu(\{(i,j) \in \mathbb{N} \times \mathbb{N}\colon \varrho(x_{ij}, x_{m_0n_0}) \geqslant \frac{1}{2}\varepsilon\}) = 0$. Let $A = \{(i,j) \in \mathbb{N} \times \mathbb{N}\colon \varrho(x_{ij}, x_{m_0n_0}) \geqslant \frac{1}{2}\varepsilon\}$. Then $\mu(A) = 0$. Let $(i,j), (i_1, j_1) \notin A$. Then $\varrho(x_{ij}, x_{m_0n_0}) < \frac{1}{2}\varepsilon$ and $\varrho(x_{i_1j_1}, x_{m_0n_0}) < \frac{1}{2}\varepsilon$ and consequently $\varrho(x_{ij}, x_{i_1j_1}) < \varepsilon$. Thus x is a μ-statistically Cauchy sequence. Hence (i) holds.

Just as μ-statistical Cauchy criterion comes from the concept of μ-statistical convergence, it seems natural to consider a Cauchy like condition associated with the convergence in μ-density and examine its relationship with μ-statistical Cauchy condition. It appears from literature that no such study has been done not only for double sequences but not for sequences (viz. $\{x_n\}_{n\in\mathbb{N}}$) either. We now intend to do precisely this. The whole analysis is done for double sequences only in metric spaces. One can easily extend the results to sequences by necessary modifications.

The following definition is now introduced.

Definition 6. A double sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ in a metric space (X, ϱ) is said to be a Cauchy double sequence in μ-density if and only if there exists a set $A \subseteq \mathbb{N}\times\mathbb{N}$ with $\mu(A) = 1$ such that $\{x_{ij}\}_{(i,j)\in A}$ is a usual Cauchy double sequence.

Theorem 5. If $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ is convergent in μ-density then it is also a Cauchy double sequence in μ-density.

Proof. The proof is straightforward. $\square$

Remark 3. The converse is not generally true as can be seen by taking the example mentioned in Remark 2.

Theorem 6. *Every Cauchy double sequence in μ-density is also μ-statistically Cauchy.*

Proof. Let $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ be a Cauchy double sequence in μ-density. Then there exists $A \subseteq \mathbb{N} \times \mathbb{N}$ with $\mu(A) = 1$ such that $\{x_{ij}\}_{(i,j)\in A}$ is a Cauchy double sequence. Then for every $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $\varrho(x_{ij}, x_{mn}) < \varepsilon$ for all $i, j, m, n \geqslant k$ and $(i,j), (m,n) \in A$. Choose $(m_0, n_0) \in A$ with $m_0, n_0 \geqslant k$. Clearly $\varrho(x_{ij}, x_{m_0n_0}) < \varepsilon$ for all $i, j \geqslant k$ and $(i,j) \in A$. Hence $\{(i,j) \in \mathbb{N} \times \mathbb{N}\colon \varrho(x_{ij}, x_{m_0n_0}) \geqslant \varepsilon\} \subseteq A^c \cup F$ where F is the union of the first k rows and first k columns of $\mathbb{N} \times \mathbb{N}$ and so $\mu(\{(i,j) \in \mathbb{N} \times \mathbb{N}\colon \varrho(x_{ij}, x_{m_0n_0}) \geqslant \varepsilon\}) = 0$. Therefore x is also μ-statistically Cauchy. $\square$

The following example shows that the converse of the above theorem is not always true.

Example 1. Let (X,ϱ) be a metric space which has at least one Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ (say) of distinct points, i.e. $x_i \neq x_j$ for all $i,j\in\mathbb{N}$ with $i\neq j$. Let $\mathbb{N}=\bigcup\limits_{j\in\mathbb{N}}\Delta_j$ be a decomposition of $\mathbb{N}$ such that each Δ_j is infinite and $\Delta_i\cap\Delta_j=\emptyset$ for $i\neq j$. Let $B_j=\Delta_j\times\mathbb{N}$. Then $\mathbb{N}\times\mathbb{N}=\bigcup\limits_{j\in\mathbb{N}}B_j$. Let Γ' be the class of all those sets $A\subset\mathbb{N}\times\mathbb{N}$ that intersect only a finite number of B_i's. Let $\Gamma=\Gamma'\cup(\Gamma')^c$. Then Γ is an algebra of subsets of $\mathbb{N}\times\mathbb{N}$.

We define a measure μ on Γ by

$$\begin{aligned}\mu(A)&=0\quad\text{if } A\in\Gamma',\\&=1\quad\text{if } A\notin\Gamma'.\end{aligned}$$

Let us now define a double sequence $y=\{y_{ij}\}_{i,j\in\mathbb{N}}$ by $y_{ij}=x_n$ if $(i,j)\in B_n$. Let $\eta>0$ be given. Since $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, so there exists $k\in\mathbb{N}$ such that $\varrho(x_n,x_m)<\eta$ for all $m,n\geqslant k$. Now $B=B_1\cup B_2\cup\ldots\cup B_k$ and so $\mu(A(\frac{1}{2}\eta))=0$. It is evident that $(i,j),(i_1,j_1)\notin A(\frac{1}{2}\eta)$ implies that $\varrho(y_{ij},y_{i_1j_1})<\eta$. Hence y is μ-statistically Cauchy.

Next we shall show that y is not a Cauchy double sequence in μ-density. On the contrary, assume that y is a Cauchy double sequence in μ-density. Then there exists $A\subseteq\mathbb{N}\times\mathbb{N}$ with $\mu(A)=1$ such that $\{y_{ij}\}_{(i,j)\in A}$ is a Cauchy double sequence in the usual sense. Then $\mu(\mathbb{N}\times\mathbb{N}\setminus A)=0$ and so $\mathbb{N}\times\mathbb{N}\setminus A\in\Gamma'$ and there exists an $l\in\mathbb{N}$ such that $\mathbb{N}\times\mathbb{N}\setminus A\subseteq B_1\cup B_2\cup\ldots\cup B_l$. But then $B_i\subseteq A$ for $i>l$. In particular, B_{l+1}, $B_{l+2}\subseteq A$. From the construction of B_j's it clearly follows that given any $k\in\mathbb{N}$, there are $(m_1,n_1)\in B_{l+1}$ and $(m_2,n_2)\in B_{l+2}$ such that $m_1,n_1\geqslant k$ as well as $m_2,n_2\geqslant k$. Hence there is no $k\in\mathbb{N}$ such that whenever $(i,j),(i_1,j_1)\in A$ with $i,j,i_1,j_1\geqslant k$, then $\varrho(x_{ij},x_{i_1j_1})<\varepsilon_0$ where $\varepsilon_0=\frac{1}{2}\varrho(x_{l+1},x_{l+2})>0$. This contradicts the fact that $\{y_{ij}\}_{(i,j)\in A}$ is Cauchy. Thus y is not a Cauchy double sequence in μ-density.

In the following we will study the equivalence of the μ-statistically Cauchy condition and Cauchy condition in μ-density under the certain assumption (namely condition (APO_2)) which becomes necessary as well as sufficient on certain restrictions of the space.

Theorem 7. *Let (X,ϱ) be an arbitrary metric space. Then a μ-statistically Cauchy double sequence is a Cauchy double sequence in μ-density if μ satisfies the condition* (APO_2).

Proof. Let $x=\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a μ-statistically Cauchy double sequence and let μ satisfies the condition (APO_2). Then for each $n\in\mathbb{N}$, there exists an A'_n

with $\mu(A'_n) = 0$ such that $(i,j),(i_1,j_1) \notin A'_n$ implies that $\varrho(x_{ij}, x_{i_1j_1}) < 1/n$. Let $A_1 = A'_1, A_2 = A'_2 \setminus A_1, A_3 = A'_3 \setminus A_1 \cup A_2, \ldots, A_j = A'_j \setminus A_1 \cup A_2 \cup \ldots \cup A_{j-1}$ and so on. Then $\mu(A_j) = 0$ for all $j \in \mathbb{N}$ and obviously $A_i \cap A_j = \emptyset$ if $i \neq j$. Then by the condition (APO_2) there exists a family $\{B_j\}_{j\in\mathbb{N}}$ of sets such that $A_j \Delta B_j$ is included in the union of a finite number of rows and columns of $\mathbb{N} \times \mathbb{N}$ for each $j \in \mathbb{N}$, and $\mu(B) = 0$ where $B = \bigcup\limits_{j\in\mathbb{N}} B_j$. Let $M = \mathbb{N} \times \mathbb{N} \setminus B$. Then $\mu(M) = 1$. We shall show that $\{x_{ij}\}_{(i,j)\in M}$ is a Cauchy double sequence.

Let $\eta > 0$ be given. Choose $l \in \mathbb{N}$ such that $1/l < \eta$. Now $A'_l \setminus B = \bigcup\limits_{i=1}^{l} (A_i \setminus B) \subseteq \bigcup\limits_{i=1}^{l} (A_i \setminus B_i) \subseteq \bigcup\limits_{i=1}^{l} F_{r_i}$ for some $r_1, r_2, \ldots, r_l \in \mathbb{N}$ (by the condition (APO_2)) where F_{r_i} is the union of the first r_i rows and first r_i columns of $\mathbb{N} \times \mathbb{N}$. Choose $n_0 \in \mathbb{N}$ such that $n_0 \geqslant r_1, r_2, \ldots, r_l$. Then $A'_l \setminus B \subseteq \bigcup\limits_{i=1}^{l} F_{r_i} \subseteq F_{n_0}$. Clearly $\mu(F_{n_0}) = 0$ and so $\mu(\mathbb{N} \times \mathbb{N} \setminus F_{n_0}) = 1$. Hence $\mu(M \cap (\mathbb{N} \times \mathbb{N} \setminus F_{n_0})) = 1$. This shows that there is an infinite number of elements (i,j) in M with $i,j > n_0$. It now easily follows that if $(i,j),(i_1,j_1) \in M$ with $i,j,i_1,j_1 > n_0$ then $(i,j),(i_1,j_1) \notin A'_l$ and so $\varrho(x_{ij}, x_{i_1j_1}) < 1/l < \eta$. This completes the proof of the theorem. $\square$

Theorem 8. *If (X,ϱ) is a metric space containing at least one limit point and every μ-statistically Cauchy double sequence is a Cauchy double sequence in μ-density, then μ satisfies the condition* (APO_2).

P r o o f. Let ξ be a limit point of X. Then there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ of distinct points in X such that $\lim\limits_{n\to\infty} x_n = \xi$ and $x_n \neq \xi$ for all $n \in \mathbb{N}$. Suppose $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of mutually disjoint non-empty sets with $\mu(A_n) = 0$ for all $n \in \mathbb{N}$. Define a double sequence $y = \{y_{ij}\}_{i,j\in\mathbb{N}}$ by $y_{ij} = x_n$ if $(i,j) \in A_n$ and $y_{ij} = \xi$ if $(i,j) \notin A_n$ for any $n \in \mathbb{N}$. Let $\eta > 0$ be given. Then there exists $m \in \mathbb{N}$ such that $\varrho(x_n, \xi) < \frac{1}{2}\eta$ for all $n \geqslant m$. Then $A(\eta) = \{(i,j) \in \mathbb{N} \times \mathbb{N} \colon \varrho(y_{ij}, \xi) \geqslant \frac{1}{2}\eta\} \subseteq A_1 \cup A_2 \cup \ldots \cup A_m$ and so $\mu(A(\eta)) = 0$. Now clearly $(i,j),(i_1,j_1) \notin A(\eta)$ implies that $\varrho(y_{ij}, \xi) < \frac{1}{2}\eta$ and $\varrho(y_{i_1j_1}, \xi) < \frac{1}{2}\eta$. So $\varrho(y_{ij}, y_{i_1j_1}) < \eta$. This shows that y is μ-statistically Cauchy. Therefore by our assumption y is also Cauchy in μ-density. Then there exists $M \subseteq \mathbb{N} \times \mathbb{N}$ with $\mu(M) = 1$ such that $\{y_{ij}\}_{(i,j)\in M}$ is a Cauchy double sequence.

Let $B = \mathbb{N} \times \mathbb{N} \setminus M$. Then $\mu(B) = 0$. First put $B_j = A_j \cap B$ for $j \in \mathbb{N}$. Then $\mu(B_j) = 0$ for all j. Further, $\bigcup\limits_{j\in\mathbb{N}} B_j = B \cap \Big(\bigcup\limits_{j\in\mathbb{N}} A_j\Big) \subset B$. Therefore $\mu\Big(\bigcup\limits_{j\in\mathbb{N}} B_j\Big) = 0$. Now for the sets $A_i \cap M$, $i \in \mathbb{N}$ the following three cases may arise:

C a s e I: Each $A_i \cap M$ is included in the union of a finite number of rows and columns of $\mathbb{N} \times \mathbb{N}$.

C a s e I I: Only one of $A_i \cap M$'s, namely $A_k \cap M$ (say), is not included in the union of a finite number of rows and columns of $\mathbb{N} \times \mathbb{N}$.

C a s e I I I: More than one of $A_i \cap M$'s are not included in the union of a finite number of rows and columns of $\mathbb{N} \times \mathbb{N}$.

If (I) holds, then $A_j \Delta B_j = A_j \setminus B_j = A_j \setminus B = A_j \cap M$ is included in the union of a finite number of rows and columns of $\mathbb{N} \times \mathbb{N}$ and this implies that μ satisfies the (APO_2) condition.

If (II) holds, then we redefine $B_k = A_k$ and $B_j = A_j \cap B$ for $j \neq k$. Then

$$\bigcup_{j \in \mathbb{N}} B_j = \Big[B \cap \Big(\bigcup_{j \neq k} A_j\Big)\Big] \cup A_k \subset B \cup A_k$$

and so $\mu(\bigcup_{j \in \mathbb{N}} B_j) = 0$. Also, since $A_j \Delta B_j = A_j \cap M$ for $j \neq k$ and $A_k \Delta B_k = \emptyset$, so as in case I the criterion for (APO_2) condition is satisfied.

If (III) holds, then there exist $k, l \in \mathbb{N}$ with $k \neq l$ such that $A_k \cap M$ and $A_l \cap M$ are not included in the union of a finite number of rows and columns of $\mathbb{N} \times \mathbb{N}$. Let $\varepsilon_0 = \frac{1}{2}\varrho(x_k, x_l) > 0$. As $\{y_{ij}\}_{(i,j) \in M}$ is a Cauchy double sequence, so for the above $\varepsilon_0 > 0$ there exists $k_0 \in \mathbb{N}$ such that $\varrho(y_{ij}, y_{i_1 j_1}) < \varepsilon_0$ for all $i, j, i_1, j_1 \geqslant k_0$ and $(i,j), (i_1, j_1) \in M$. Now since $A_k \cap M, A_l \cap M$ are not included in the union of a finite number of rows and columns of $\mathbb{N} \times \mathbb{N}$, we can choose $(i,j) \in A_k \cap M$ and $(i_1, j_1) \in A_l \cap M$ with $i, j, i_1, j_1 > k_0$. But then $y_{ij} = x_k$ and $y_{i_1 j_1} = x_l$ and so $\varrho(y_{ij}, y_{i_1 j_1}) = \varrho(x_k, x_l) > \varepsilon_0$ (in fact there is an infinite number of indices (i,j), (i_1, j_1) in M with this property). This contradicts the fact that $\{y_{ij}\}_{(i,j) \in M}$ is Cauchy. Therefore Case III cannot arise. And in view of Case I and Case II μ satisfies the (APO_2) condition.

5. μ-statistical divergence and divergence in μ-density

Just as the notion of convergence of double sequences can be extended using a two valued measure μ, it seems very natural to investigate whether this can also be done for divergent double sequences of real numbers. It appears from literature that so far no such study has been done, not only for double sequences, but not for sequences either. In this section we do precisely this and introduce the ideas of μ-statistical divergence and divergence in μ-density and mainly investigate their interrelationship where again surprisingly the condition (APO_2) plays a very prominent role. We first introduce the following two definitions.

Definition 7. A double sequence $x = \{x_{ij}\}_{i,j \in \mathbb{N}}$ of real numbers is said to be μ-statistically divergent to $+\infty$ if $\mu(\{(i,j) \in \mathbb{N} \times \mathbb{N} \colon x_{ij} \leqslant G\}) = 0$ for any positive real number G.

Definition 8. A double sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ is said to be divergent in μ-density to $+\infty$ if there exists $A \subseteq \mathbb{N}\times\mathbb{N}$ with $\mu(A) = 1$ such that $\{x_{ij}\}_{(i,j)\in A}$ is divergent to $+\infty$.

Throughout we shall denote by S_2 the set of all double sequences of real numbers, and by D_μ the set of all real double sequences which are μ-statistically divergent to $+\infty$ and by D^*_μ the set of all real double sequences which are divergent in μ-density to $+\infty$.

Theorem 9. $D^*_\mu \subseteq D_\mu$.

Proof. Let $x = \{x_{ij}\}_{i,j\in\mathbb{N}} \in D^*_\mu$. Then there exists $A \subseteq \mathbb{N}\times\mathbb{N}$ with $\mu(A) = 1$ such that $\{x_{ij}\}_{(i,j)\in A}$ is divergent to $+\infty$. Therefore for any real number $G > 0$ there exists $m_0 \in \mathbb{N}$ such that $x_{ij} > G$ for all $i \geqslant m_0, j \geqslant m_0$ and $(i,j) \in A$. Let $A(G) = \{(i,j) \in \mathbb{N}\times\mathbb{N}\colon\ x_{ij} \leqslant G\}$. Then clearly $A(G) \subseteq (\mathbb{N}\times\mathbb{N}\setminus A) \cup [(\{1,2,3,\dots,m_0\}\times\mathbb{N}) \cup (\mathbb{N}\times\{1,2,3,\dots,m_0\})]$ and so $\mu(A(G)) = 0$. Thus $x \in D_\mu$ and $D^*_\mu \subseteq D_\mu$.

The following example shows that the converse of the above theorem is not generally true.

Example 2. Let $\mathbb{N} = \bigcup\limits_{j\in\mathbb{N}} \Delta_j$ be a decomposition of $\mathbb{N}$ such that each Δ_j is infinite and $\Delta_i \cap \Delta_j = \emptyset$ for $i \neq j$. Let $B_j = \Delta_j \times \mathbb{N}$. Then $\mathbb{N}\times\mathbb{N} = \bigcup\limits_{j\in\mathbb{N}} B_j$. Let Γ' be the class of all those sets $A \subset \mathbb{N}\times\mathbb{N}$ that intersect only a finite number of B_i's. Let $\Gamma = \Gamma' \cup (\Gamma')^c$. Then Γ is an algebra of subsets of $\mathbb{N}\times\mathbb{N}$.

We define a measure μ on Γ by

$$\begin{aligned} \mu(A) &= 0 \quad \text{if } A \in \Gamma', \\ &= 1 \quad \text{if } A \notin \Gamma'. \end{aligned}$$

Construct a double sequence $y = \{y_{ij}\}_{i,j\in\mathbb{N}}$ of real numbers by $y_{ij} = n$ if $(i,j) \in B_n$ for all $n \in \mathbb{N}$. Let $G > 0$ be a real number. Then there exists $m \in \mathbb{N}$ such that $G < m$. Now $A(G) = \{(i,j) \in \mathbb{N}\times\mathbb{N}\colon\ y_{ij} \leqslant G\} \subseteq B_1 \cup B_2 \cup \ldots \cup B_m$. Clearly then $\mu(A(G)) = 0$ and so $y \in D_\mu$.

Now let y be divergent in μ-density to $+\infty$. Then there exists $H \in \Gamma'$ (i.e. $\mu(H) = 0$) such that $(y_{ij})_{(i,j)\in M}$ is divergent to $+\infty$ where $M = \mathbb{N}\times\mathbb{N}\setminus H$. Since $H \in \Gamma'$, so there exists $l \in \mathbb{N}$ such that $H \subseteq B_1 \cup B_2 \cup \ldots \cup B_l$ and $B_i \subseteq M$ for all $i > l$, and so in particular $B_{l+1} \subseteq M$. But this implies that $\{y_{ij}\}_{(i,j)\in B_{l+1}}$ is a subsequence of $\{y_{ij}\}_{(i,j)\in M}$ which is convergent to $l+1$. This contradicts the fact that $\{y_{ij}\}_{(i,j)\in M}$ is divergent to $+\infty$. Therefore y is not divergent in μ-density to $+\infty$, i.e. $y \notin D^*_\mu$.

In the following we study the equivalence of the μ-statistical divergence and divergence in the μ-density of double sequences of real numbers and show that just like convergence also the condition (APO$_2$) plays a very prominent role here.

Theorem 10. *$D_\mu = D^*_\mu$ if and only if μ satisfies the (APO$_2$) condition.*

Proof. First suppose that μ satisfies the (APO$_2$) condition. To prove that $D_\mu = D^*_\mu$ it is sufficient to prove that $D_\mu \subseteq D^*_\mu$. Let $x = \{x_{ij}\}_{i,j\in\mathbb{N}} \in D_\mu$. Then for any real number $G > 0$, $\mu(\{(i,j) \in \mathbb{N}\times\mathbb{N}\colon x_{ij} \leqslant G\}) = 0$. Now we put $A_1 = \{(i,j) \in \mathbb{N}\times\mathbb{N}\colon x_{ij} \leqslant 1\}$, $A_2 = \{(i,j) \in \mathbb{N}\times\mathbb{N}\colon 1 < x_{ij} \leqslant 2\}$, $\ldots$, $A_k = \{(i,j) \in \mathbb{N}\times\mathbb{N}\colon k-1 < x_{ij} \leqslant k\}$ for all $k \geqslant 2$. Thus we get a collection of mutually disjoint sets $\{A_i\}_{i\in\mathbb{N}}$ with $\mu(A_i) = 0$ for all $i \in \mathbb{N}$. By the (APO$_2$) condition there exists a family of sets $\{B_i\}_{i\in\mathbb{N}}$ such that $A_i \Delta B_i$ is included in the union of a finite number of rows and columns of $\mathbb{N}\times\mathbb{N}$ and $\mu(B) = 0$ where $B = \bigcup\limits_{i\in\mathbb{N}} B_i$. Let $M = \mathbb{N}\times\mathbb{N}\setminus B$. Then $\mu(M) = 1$. Suppose $G > 0$ is any given real number. Choose $k \in \mathbb{N}$ such that $G < k$. Then $\{(i,j) \in \mathbb{N}\times\mathbb{N}\colon x_{ij} \leqslant G\} \subseteq A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_k$. Since $A_i \Delta B_i$ is included in the union of a finite number of rows and columns of $\mathbb{N}\times\mathbb{N}$ for all $i = 1,2,3,\ldots,k$, we can choose an $n_0 \in \mathbb{N}$ such that $\Bigl(\bigcup\limits_{i=1}^{k} B_i\Bigr) \cap \{(i,j) \in \mathbb{N}\times\mathbb{N}\colon i \geqslant n_0 \wedge j \geqslant n_0\} = \Bigl(\bigcup\limits_{i=1}^{k} A_i\Bigr) \cap \{(i,j) \in \mathbb{N}\times\mathbb{N}\colon i \geqslant n_0 \wedge j \geqslant n_0\}$.

Clearly if $i,j \geqslant n_0$ and $(i,j) \in M$ then $(i,j) \notin \bigcup\limits_{i=1}^{k} B_i$ and so $(i,j) \notin \bigcup\limits_{i=1}^{k} A_i$. Therefore $x_{ij} > k \geqslant G$. Thus $\{x_{ij}\}_{(i,j)\in M}$ is divergent to $+\infty$.

Conversely, assume that $D_\mu = D^*_\mu$. Let $\{A_i\}_{i\in\mathbb{N}}$ be a family of mutually disjoint sets with $\mu(A_i) = 0$ for all $i \in \mathbb{N}$. Define a double sequence $x = \{x_{mn}\}$ in the following way:

$$x_{mn} = \begin{cases} j & \text{if } (m,n) \in A_j, \\ m+n & \text{if } (m,n) \notin A_j, \quad \text{for all } j \in \mathbb{N}. \end{cases}$$

Let $G > 0$ be any real number. Choose $k \in \mathbb{N}$ such that $G < k$. Then $A(G) = \{(m,n) \in \mathbb{N}\times\mathbb{N}\colon x_{ij} \leqslant G\} \subseteq A_1 \cup A_2 \cup \ldots \cup A_k \cup F$, where F is the union of the first k rows and the first k columns of $\mathbb{N}\times\mathbb{N}$. Then clearly $\mu(A(G)) = 0$ and so $x \in D_\mu$. By our assumption $x \in D^*_\mu$, so there exists $M \subseteq \mathbb{N}\times\mathbb{N}$ such that $\mu(M) = 1$ and $\{x_{ij}\}_{(i,j)\in M}$ is divergent to $+\infty$. Let $B = \mathbb{N}\times\mathbb{N}\setminus M$. Then $\mu(B) = 0$. Put $B_j = A_j \cap B$ for all $j \in \mathbb{N}$. Since $\bigcup\limits_{j\in\mathbb{N}} B_j \subseteq B$, so $\mu\Bigl(\bigcup\limits_{j\in\mathbb{N}} B_j\Bigr) = 0$.

Let $j \in \mathbb{N}$. We claim that $A_j \cap M$ is included in the union of a finite number of rows and columns of $\mathbb{N}\times\mathbb{N}$. If not then M must contain an infinite sequence of elements $\{(m_k,n_k)\}_{k\in\mathbb{N}}$ where both $m_k, n_k \to \infty$ and $x_{m_k n_k} = j$ for all $k \in \mathbb{N}$.

But this contradicts the fact that $\{x_{ij}\}_{(i,j)\in M}$ is divergent to $+\infty$. Hence $A_j\Delta B_j = A_j\setminus B_j = A_j\setminus B = A_j\cap M$ is included in the union of a finite number of rows and columns of $\mathbb{N}\times\mathbb{N}$. Since this is true for each $j\in\mathbb{N}$, this proves that μ satisfies the (APO_2) condition.

Remark 4. One question that comes naturally is what is the relation between the conditions (APO) [3] and (APO_2). Obviously (APO) implies (APO_2). But the converse is not true. As an example we consider the algebra $\Gamma=\Gamma'\cup(\Gamma')^c$ of $\mathbb{N}\times\mathbb{N}$ where $\Gamma'=\{A\subset\mathbb{N}\times\mathbb{N}\colon\ d_2(A)=0\}$ and the measure μ where $\mu(A)=0$ if $A\in\Gamma'$ and $\mu(A)=1$ if $A\in(\Gamma')^c$. In view of Theorem 2.1 [12], $C_\mu=C_\mu^*$ and so μ satisfies the condition (APO_2). We can show that μ does not satisfy (APO) (see [6]). We produce the proof below for the sake of completeness.

First let $\{E_p\}_{p\in\mathbb{N}}$ be a sequence of subsets of $\mathbb{N}$ with density zero such that $\bigcup\limits_{p=1}^{\infty}E_p=\mathbb{N}$. Put $A_p=E_p\times\mathbb{N}$ for $p\in\mathbb{N}$. Then $d_2(A_p)=0$ for $p\in\mathbb{N}$. Let $\{B_p\}_{p\in\mathbb{N}}$ be an arbitrary sequence of subsets of $\mathbb{N}\times\mathbb{N}$ such that $\operatorname{card}(A_p\Delta B_p)$ is finite. Then there exists a sequence of finite sets $\{F_p\}_{p\in\mathbb{N}}$ such that $B_p\supseteq A_p\setminus F_p$. We shall show that $d_2(\bigcup\limits_{p=1}^{\infty}B_p)\neq 0$.

Let $m\in\mathbb{N}$. We shall show that for each $\eta>0$ there exists $n\in\mathbb{N}$ such that $n\geqslant m$ and

$$\frac{1}{m\cdot n}\operatorname{card}\Big\{(j,k)\colon\ j\leqslant m\wedge k\leqslant n\wedge(j,k)\in\bigcup_{p=1}^{\infty}(A_p\setminus F_p)\Big\}>1-\eta.$$

To this end we first choose $p_0\in\mathbb{N}$ such that $\bigcup\limits_{i=1}^{p_0}E_i\supset\{1,2,3,\dots,m\}$, since $\bigcup\limits_{p=1}^{\infty}E_p=\mathbb{N}$. So $\bigcup\limits_{i=1}^{p_0}A_i\supset\{1,2,\dots,m\}\times\mathbb{N}$.

Hence $\bigcup\limits_{i=1}^{p}(A_i\setminus F_i)\supset(\{1,2,\dots,m\}\times\mathbb{N})\setminus F$, where F is a finite set. So for each $n\in\mathbb{N}$ we have $(\{1,2,\dots,m\}\times\{1,2,\dots,n\})\cap\bigcup\limits_{i=1}^{\infty}(A_i\setminus F_i)\supset(\{1,2,\dots,m\}\times\{1,2,\dots,n\})\cap\bigcup\limits_{i=1}^{p_0}(A_i\setminus F_i)\supset(\{1,2,\dots,m\}\times\{1,2,\dots,n\})\setminus F$ (where F does not depend on n) and for sufficiently big $n\in\mathbb{N}$ ($n\geqslant m$ also) we have the inequality $(m\cdot n)^{-1}\operatorname{card}\Big((\{1,2,\dots,m\}\times\{1,2,\dots,n\})\cap\bigcup\limits_{i=1}^{\infty}(A_i\setminus F_i)\Big)>1-\eta$. This shows that $\bar{d}_2\Big(\bigcup\limits_{p=1}^{\infty}B_p\Big)=1$ which implies that μ does not fulfil (APO).

For the next result we extend the class D_μ to include also those real double sequences which are μ-statistically divergent to $-\infty$.

Theorem 11. *$S_2 = C_\mu \cup D_\mu$ if and only if μ is defined on the whole of $\wp(\mathbb{N} \times \mathbb{N})$.*

Proof. Suppose first that μ is defined on the whole of $\wp(\mathbb{N} \times \mathbb{N})$. Let $x = \{x_{ij}\}_{i,j\in\mathbb{N}} \in S_2$ and let $G > 0$ be any real number. Consider the set $A(G) = \{(i,j) \in \mathbb{N} \times \mathbb{N} \colon x_{ij} \leqslant G\}$. Then either $\mu(A(G)) = 0$ or $\mu(A(G)) = 1$ as μ is defined on the whole of $\wp(\mathbb{N} \times \mathbb{N})$. If $\mu(A(G)) = 0$ for all $G > 0$, then $x \in D_\mu$. Otherwise $\mu(A(G)) \neq 0$ for some $G > 0$. Let $G' > 0$ be such that $\mu(A(G)) = 1$. Now we have two possibilities:

i) for any real number $g > 0$, $\mu(\{(i,j) \in \mathbb{N} \times \mathbb{N} \colon x_{ij} < -g\}) = 1$.

ii) there exists a $g' > 0$ such that $\mu(\{(i,j) \in \mathbb{N} \times \mathbb{N} \colon -g' \leqslant x_{ij} < G'\}) = 1$.

In the first case x is μ-statistically divergent to $-\infty$ and so $x \in D_\mu$. If (ii) holds then let $T = \{(i,j) \in \mathbb{N} \times \mathbb{N} \colon a \leqslant x_{ij} \leqslant b\}$ where $a = -g'$, $b = G'$. Note that $\mu(T) = 1$. Let us construct the sets $A_1 = \{(i,j) \in T \colon a \leqslant x_{ij} \leqslant \frac{1}{2}((a+b)\}$ and $B_1 = \{(i,j) \in T \colon \frac{1}{2}(a+b) \leqslant x_{ij} \leqslant b\}$. Since $\mu(T) = 1$, $T = A_1 \cup B_1$ and it is clear that both $\mu(A_1)$ and $\mu(B_1)$ cannot be 0 simultaneously (since μ is finitely additive). Thus one of $\mu(A_1)$ and $\mu(B_1)$ is equal to 1. Denote it by D_1 and the interval corresponding to it by J_1. Thus $D_1 = \{(i,j) \in T \colon x_{ij} \in J_1\}$ and $\mu(D_1) = 1$. Again dividing J_1 into two equal parts and proceeding as above we can find a set D_2 and an interval J_2 such that $D_2 = \{(i,j) \in T \colon x_{ij} \in J_2\}$ and $\mu(D_2) = 1$. Proceeding in this way we obtain a sequence of closed and bounded intervals $\{J_n\}$ such that $J_1 \supseteq J_2 \supseteq \ldots \supseteq J_n \ldots, J_n = [a_n, b_n]$ and $\lim\limits_{n\to\infty}(b_n - a_n) = 0$ and the sets $D_n = \{(i,j) \in T \colon x_{ij} \in J_n\}$ and $\mu(D_n) = 1$ for all $n \in \mathbb{N}$.

Then by the nested intervals theorem there exists $\xi \in \bigcap\limits_{n\in\mathbb{N}} J_n$. Let $\varepsilon > 0$ be given. Let $M'(\varepsilon) = \{(i,j) \in \mathbb{N} \times \mathbb{N} \colon |x_{ij} - \xi| < \varepsilon\}$. Now we have $J_m \subseteq (\xi - \varepsilon, \xi + \varepsilon)$ for sufficiently large $m \in \mathbb{N}$. Therefore $\mu(M'(\varepsilon)) = 1$. Since $\{(i,j) \in \mathbb{N} \times \mathbb{N} \colon |x_{ij} - \xi| \geqslant \varepsilon\} \subseteq (T \setminus M'(\varepsilon)) \cup T^c$, so $\mu(\{(i,j) \in \mathbb{N} \times \mathbb{N} \colon |x_{ij} - \xi| \geqslant \varepsilon\}) = 0$ and so $x \in C_\mu$. This proves that $S_2 = C_\mu \cup D_\mu$.

Conversely, suppose that $S_2 = C_\mu \cup D_\mu$. Let μ be not defined on the whole of $\wp(\mathbb{N} \times \mathbb{N})$. Then there exists $A \subseteq \mathbb{N} \times \mathbb{N}$ such that $\mu(A)$ and $\mu(\mathbb{N} \times \mathbb{N} \setminus A)$ are not defined. Let us construct a sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ by $x_{ij} = \chi_A(i,j)$ for all $(i,j) \in \mathbb{N} \times \mathbb{N}$. Clearly x cannot belong to D_μ. Also, for every $\xi \in \mathbb{R}$ and $0 < \varepsilon < 1$ the set $\{(i,j) \in \mathbb{N} \times \mathbb{N} \colon |x_{ij} - \xi| \geqslant \varepsilon\}$ is equal to A or $\mathbb{N} \times \mathbb{N} \setminus A$ or $\mathbb{N} \times \mathbb{N}$ and as none of them have measure equal to zero, so x is not μ-statistically convergent and so $x \notin C_\mu$. Thus $x \notin D\mu \cup C_\mu$ which contradicts that $S_2 = C_\mu \cup D_\mu$. Hence μ must be defined on the whole of $\wp(\mathbb{N} \times \mathbb{N})$. This completes the proof of the theorem. □

Remark 5. Just as we extended the idea of divergence to $+\infty$ to μ-statistical divergence to $+\infty$ and divergence to $+\infty$ in μ-density, the same can also be done for divergence to $-\infty$. All the definitions and results proved so far can be obtained in a similar fashion with necessary modifications which are very obvious.

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Authors' addresses: Pratulananda Das, Department of Mathematics, Jadavpur University, Kolkata-700032, West Bengal, India, e-mail: `pratulananda@yahoo.co.in`; Santanu Bhunia, Department of Mathematics, F.C. College, Diamond Harbour, South 24 Prgs., Pin-743331, West Bengal, India, e-mail: `santo_812004@yahoo.co.in`.