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BOUNDED LINEAR FUNCTIONALS ON THE SPACE OF HENSTOCK-KURZWEIL INTEGRABLE FUNCTIONS

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Abstract. Applying a simple integration by parts formula for the Henstock-Kurzweil integral, we obtain a simple proof of the Riesz representation theorem for the space of Henstock-Kurzweil integrable functions. Consequently, we give sufficient conditions for the existence and equality of two iterated Henstock-Kurzweil integrals.

Keywords: Henstock-Kurzweil integral, bounded linear functional, bounded variation

MSC 2010: 26A39, 46E99

1. Introduction

It is well known that if f is Henstock-Kurzweil integrable on a compact interval $[a,b]$ of $\mathbb{R}$ and g is of bounded variation on $[a,b]$, then fg is Henstock-Kurzweil integrable on $[a,b]$ and the integration by parts formula holds; see, for example, [2, Chapter 11]. Denoting the space of Henstock-Kurzweil integrable functions by $\mathrm{HK}[a,b]$, it is not difficult to see that every function g of bounded variation on $[a,b]$ induces a bounded linear functional on the space $\mathrm{HK}[a,b]$. On the other hand, it is also known that if T is a bounded linear functional on $\mathrm{HK}[a,b]$, then there exist functions $g\colon [a,b] \longrightarrow \mathbb{R}$ and $g_0 \in BV[a,b]$ such that $g = g_0$ almost everywhere on $[a,b]$ and

$$T(f) = (\mathrm{HK}) \int_a^b f(t)g(t)\,\mathrm{d}t$$

for every $f \in \mathrm{HK}[a,b]$; see, for example, [6] for details.

In 1973, Kurzweil [5] proved an integration by parts formula for higher-dimensional Henstock-Kurzweil integral. More precisely, he proved that if f is Henstock-Kurzweil integrable on a compact interval E of a multidimensional Euclidean space and g is of bounded variation (in the sense of Hardy-Krause) on E, then fg is Henstock-Kurzweil integrable on E and the integration by parts formula holds. Furthermore, the function

$$T_g\colon \operatorname{HK}(E) \longrightarrow \mathbb{R}\colon f \mapsto (\mathrm{HK})\int_E f(t)g(t)\,\mathrm{d}t$$

is a bounded linear functional on $\operatorname{HK}(E)$. More recently, various authors [8], [12], [14], [17] have shown that the converse is also true; that is, if T is a bounded linear functional on $\operatorname{HK}(E)$, then there exist a function $g\colon E \longrightarrow \mathbb{R}$ and a function g_0 of bounded variation (in the sense of Hardy-Krause) on E with the following properties: $g = g_0$ almost everywhere on E and

$$T(f) = (\mathrm{HK})\int_E f(t)g(t)\,\mathrm{d}t \tag{1}$$

for every $f \in \operatorname{HK}(E)$. Nevertheless, the above proofs of (1) are non-elementary: either Kurzweil's multidimensional integration by parts formula or the measure theory is involved. One of the aims of this paper is to give a simpler proof of this representation theorem.

The paper is organised as follows. In Section 2 we state a number of useful results concerning functions of bounded variation (in the sense of Vitali), with proofs where necessary. In Section 3 we give a simple proof of the Riesz representation theorem for the space of Henstock-Kurzweil integrable functions; see Theorem 3.7 for details. In Section 4 we prove the corresponding Riesz representation theorem for the space of Cauchy-Lebesgue integrable functions. In Section 5 we employ our results to obtain a "Tonelli's theorem" for Henstock-Kurzweil integrals; see Theorem 5.10 for details.

2. Functions of bounded variation

Let $m \geqslant 1$ be an integer and let $\mathbb{R}^m$ denote the m-dimensional Euclidean space equipped with the maximum norm $\|\cdot\|$. For $\boldsymbol{x} \in \mathbb{R}^m$ and $r > 0$, set $B(\boldsymbol{x}, r) := \{\boldsymbol{y} \in \mathbb{R}^m\colon \|\boldsymbol{y} - \boldsymbol{x}\| < r\}$. An *interval* in $\mathbb{R}^m$ is a set of the form $[\boldsymbol{u}, \boldsymbol{v}] := \prod_{i=1}^{m} [u_i, v_i]$, where $\boldsymbol{u} = (u_1, \ldots, u_m)$, $\boldsymbol{v} = (v_1, \ldots, v_m)$ with $u_i, v_i \in \mathbb{R}$ and $u_i < v_i$ for $i = 1, \ldots, m$. Throughout this paper $[\boldsymbol{a}, \boldsymbol{b}] := \prod_{i=1}^{m} [a_i, b_i]$ denotes a fixed interval and $\mathcal{I}_m([\boldsymbol{a}, \boldsymbol{b}])$ the family of all subintervals of $[\boldsymbol{a}, \boldsymbol{b}]$.

A *division* of $[\boldsymbol{a}, \boldsymbol{b}]$ is a finite collection $\{I_1, \ldots, I_p\}$ of non-overlapping intervals such that $\bigcup_{i=1}^{p} I_i = [\boldsymbol{a}, \boldsymbol{b}]$. For any given real-valued function g defined on $[\boldsymbol{a}, \boldsymbol{b}]$, the total variation of g over $[\boldsymbol{a},\boldsymbol{b}]$ is defined by

$$\operatorname{Var}(g,[\boldsymbol{a},\boldsymbol{b}]) := \sup\Big\{\sum_{[\boldsymbol{u},\boldsymbol{v}]\in P} |\,\Delta_g([\boldsymbol{u},\boldsymbol{v}])| : \ P \text{ is a division of } \ [\boldsymbol{a},\boldsymbol{b}]\Big\},$$

where

$$\Delta_g([\boldsymbol{u},\boldsymbol{v}]) := \sum_{\substack{\boldsymbol{t}\in[\boldsymbol{u},\boldsymbol{v}]\\ t_i\in\{u_i,v_i\}\ \forall\, i\in\{1,\dots,m\}}} g(\boldsymbol{t}) \prod_{i=1}^{m} \operatorname{sgn}\Big(t_i - \frac{u_i+v_i}{2}\Big)$$

for each $[\boldsymbol{u},\boldsymbol{v}] \in \mathcal{I}_m([\boldsymbol{a},\boldsymbol{b}])$.

Definition 2.1. A function $g\colon [\boldsymbol{a},\boldsymbol{b}] \longrightarrow \mathbb{R}$ is said to be of bounded variation (in the sense of Vitali) on $[\boldsymbol{a},\boldsymbol{b}]$ if $\operatorname{Var}(g,[\boldsymbol{a},\boldsymbol{b}])$ is finite.

The space of functions of bounded variation (in the sense of Vitali) on $[\boldsymbol{a},\boldsymbol{b}]$ is denoted by $BV[\boldsymbol{a},\boldsymbol{b}]$. Set

$$BV_0[\boldsymbol{a},\boldsymbol{b}] := \{g \in BV[\boldsymbol{a},\boldsymbol{b}] \colon \ g(\boldsymbol{x}) = 0 \text{ whenever } \boldsymbol{x} \in [\boldsymbol{a},\boldsymbol{b}] \setminus (\boldsymbol{a},\boldsymbol{b}]\},$$

where $(\boldsymbol{a},\boldsymbol{b}] := \prod\limits_{i=1}^{m} (a_i,b_i]$.

Let μ_m denote Lebesgue measure in $\mathbb{R}^m$. The following theorem, which asserts that every bounded linear functional on $C[\boldsymbol{a},\boldsymbol{b}]$ can be represented by Riemann-Stieltjes integration, is an m-dimensional analogue of [3, Theorem 2].

Theorem 2.2 (Riesz Representation Theorem). *Let $T\colon C[\boldsymbol{a},\boldsymbol{b}] \longrightarrow \mathbb{R}$ be a bounded linear functional. Then there exists $g \in BV_0[\boldsymbol{a},\boldsymbol{b}]$ such that*

$$T(F) = (RS)\int_{[\boldsymbol{a},\boldsymbol{b}]} F(\boldsymbol{x})\,\mathrm{d}g(\boldsymbol{x})$$

for every $F \in C[\boldsymbol{a},\boldsymbol{b}]$. Moreover, $\|T\| = \operatorname{Var}(g,[\boldsymbol{a},\boldsymbol{b}])$.

Proof. Let $B[\boldsymbol{a},\boldsymbol{b}]$ denote the space of bounded functions on $[\boldsymbol{a},\boldsymbol{b}]$ and assume that $B[\boldsymbol{a},\boldsymbol{b}]$ is equipped with the L^∞-norm $\|\cdot\|_{L^\infty[\boldsymbol{a},\boldsymbol{b}]}$, where

$$\|f\|_{L^\infty[\boldsymbol{a},\boldsymbol{b}]} = \inf\{M \in \mathbb{R}\colon \ |f(\boldsymbol{x})| \leqslant M \text{ for } \mu_m\text{-almost all } \boldsymbol{x} \in [\boldsymbol{a},\boldsymbol{b}]\}.$$

Let $B[\boldsymbol{a},\boldsymbol{b}]^*$ denote the dual space of $B[\boldsymbol{a},\boldsymbol{b}]$. By the Hahn-Banach Theorem, T has an extension $\widetilde{T} \in B[\boldsymbol{a},\boldsymbol{b}]^*$ with $\|T\| = \|\widetilde{T}\|$.

Let $g(\boldsymbol{x}) := \widetilde{T}(\chi_{(\boldsymbol{a},\boldsymbol{x}]})$. Then we can follow the proof of Riesz's theorem (cf. [4]) to get

$$\operatorname{Var}(g,[\boldsymbol{a},\boldsymbol{b}]) \leqslant \|T\| < \infty$$

and

$$T(F) = (RS) \int_{[\boldsymbol{a},\boldsymbol{b}]} F(\boldsymbol{x})\,\mathrm{d}g(\boldsymbol{x})$$

for every $F \in C[\boldsymbol{a},\boldsymbol{b}]$. It is now easy to check that $\operatorname{Var}(g,[\boldsymbol{a},\boldsymbol{b}]) = \|T\|$. The proof is complete. $\square$

Remark 2.3. Theorem 2.2 can be proved without using the Hahn-Banach Theorem; consult [3, Theorem 2].

3. The Henstock-Kurzweil integral

A *partial partition* of the interval $[\boldsymbol{a},\boldsymbol{b}]$ is a collection $\{(I_1,\boldsymbol{t}_1),\dots,(I_p,\boldsymbol{t}_p)\}$ $[\boldsymbol{a},\boldsymbol{b}]$, where $I_1,\dots,I_p$ are nonoverlapping intervals and $\boldsymbol{t}_i \in I_i \subset [\boldsymbol{a},\boldsymbol{b}]$ for $i=1,\dots,p$. If δ is a gauge (i.e. a positive function) on a set $Z \subseteq [\boldsymbol{a},\boldsymbol{b}]$, we say that a partial partition $\{(I_1,\boldsymbol{t}_1),\dots,(I_p,\boldsymbol{t}_p)\}$ of $[\boldsymbol{a},\boldsymbol{b}]$ is δ-fine whenever $\boldsymbol{t}_i \in Z$ and $\operatorname{diam}(I_i) < \delta(\boldsymbol{t}_i)$ for $i=1,\dots,p$, where $\operatorname{diam}(A)$ denotes the diameter of a set $A \subset \mathbb{R}^m$.

Lemma 3.1 (cf. [7, Lemma 6.2.6])**.** *If δ is a gauge on $[\boldsymbol{a},\boldsymbol{b}]$, then there exists a δ-fine partial partition $\{(I_1,\boldsymbol{t}_1),\dots,(I_p,\boldsymbol{t}_p)\}$ of $[\boldsymbol{a},\boldsymbol{b}]$ such that $\bigcup_{i=1}^{p} I_i = [\boldsymbol{a},\boldsymbol{b}]$.*

Definition 3.2. A function $f\colon [\boldsymbol{a},\boldsymbol{b}] \longrightarrow \mathbb{R}$ is said to be *Henstock-Kurzweil integrable* on $[\boldsymbol{a},\boldsymbol{b}]$ if there exists $A \in \mathbb{R}$ with the following property: given $\varepsilon > 0$ there exists a gauge δ on $[\boldsymbol{a},\boldsymbol{b}]$ such that

$$\left|\sum_{i=1}^{p} f(\boldsymbol{t}_i)\mu_m(I_i) - A\right| < \varepsilon \tag{2}$$

for each δ-fine partial partition $\{(I_1,\boldsymbol{t}_1),\dots,(I_p,\boldsymbol{t}_p)\}$ of $[\boldsymbol{a},\boldsymbol{b}]$ with $\bigcup_{i=1}^{p} I_i = [\boldsymbol{a},\boldsymbol{b}]$. Here A is called the Henstock-Kurzweil integral of f over $[\boldsymbol{a},\boldsymbol{b}]$, and we write A as $(\mathrm{HK})\int_{[\boldsymbol{a},\boldsymbol{b}]} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$.

The collection of all functions that are Henstock-Kurzweil integrable on $[\boldsymbol{a},\boldsymbol{b}]$ will be denoted by $\mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$. The following properties are known for the Henstock-Kurzweil integral; see [7] for the proofs, where the term “Kurzweil-Henstock integral” is used to describe this integral.

Theorem 3.3.
(a) $\mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$ is a linear space.
(b) If $f \in \mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$, then $f \in \mathrm{HK}(J)$ for each $J \in \mathcal{I}_m([\boldsymbol{a},\boldsymbol{b}])$.
(c) If $f \in \mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$, then the interval function $J \mapsto (\mathrm{HK})\int_J f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$ is additive on $\mathcal{I}_m([\boldsymbol{a},\boldsymbol{b}])$. This interval function is known as the *indefinite Henstock-Kurzweil integral*, or in short the indefinite HK-integral, of f.
(d) If $f \in \mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$, then for each $\varepsilon > 0$ there exists $\eta > 0$ such that $\left|(\mathrm{HK})\int_J f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right| < \varepsilon$ whenever $J \in \mathcal{I}_m([\boldsymbol{a},\boldsymbol{b}])$ and $\mu_m(J) < \eta$.
(e) If $f \in L^1[\boldsymbol{a},\boldsymbol{b}]$ and f is real-valued, then $f \in \mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$ and $\int_{[\boldsymbol{a},\boldsymbol{b}]} f(\boldsymbol{x})\,\mathrm{d}\mu_m(\boldsymbol{x}) = (\mathrm{HK})\int_{[\boldsymbol{a},\boldsymbol{b}]} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$.
(f) If $\{f, |f|\} \subset \mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$, then $f \in L^1[\boldsymbol{a},\boldsymbol{b}]$.

For the rest of this paper, the space $\mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$ will be equipped with the semi-norm $\|\cdot\|_{\mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]}$, where

$$\|f\|_{\mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]} := \sup\left\{\left|(\mathrm{HK})\int_I f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right| : I \in \mathcal{I}_m([\boldsymbol{a},\boldsymbol{b}])\right\}.$$

The following theorem, which is an improvement of Theorem 3.3(e), is also important.

Theorem 3.4 ([9, Theorem 6]). *$L^1[\boldsymbol{a},\boldsymbol{b}]$ is dense in* $\mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$.

For further properties of the space $\mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$, consult, for example, [11], [14], [18], [19].

As a consequence of Theorem 3.4 and the absolute continuity of the indefinite Lebesgue integrals we obtain the following result of Kurzweil [5].

Corollary 3.5. *If $f \in \mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$, then the map*

$$\boldsymbol{x} \mapsto (\mathrm{HK})\int_{[\boldsymbol{x},\boldsymbol{b}]} f(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t}$$

is continuous on $[\boldsymbol{a},\boldsymbol{b}]$.

The following theorem is a simple version of Kurzweil's multiple integration by parts formula (cf. [5, Theorem 2.10]).

Theorem 3.6 ([16, Theorem 4.8]). *If $f \in \mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$ and $g \in BV_0[\boldsymbol{a},\boldsymbol{b}]$, then $fg \in \mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$ and*

$$(3)\qquad (\mathrm{HK})\int_{[\boldsymbol{a},\boldsymbol{b}]} f(\boldsymbol{x})g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = (RS)\int_{[\boldsymbol{a},\boldsymbol{b}]}\left\{(\mathrm{HK})\int_{[\boldsymbol{x},\boldsymbol{b}]} f(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t}\right\}\mathrm{d}g(\boldsymbol{x}).$$

We observe that when $m = 1$, the following result of Alexiewicz [1] is known.

Theorem. *Let* $m = 1$ *and let* T *be a bounded linear functional on* $\mathrm{HK}[a,b]$. *Then there exists* $g \in BV[a,b]$ *such that*

$$T(f) = (\mathrm{HK}) \int_a^b f(t)g(t)\,\mathrm{d}t$$

for every $f \in \mathrm{HK}[a,b]$.

As a simple application of Theorem 3.6 we obtain the following refinement of [8, Theorem 3.2] and the above-mentioned result of Alexiewicz.

Theorem 3.7. *If* T *is a bounded linear functional on* $\mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$, *then there exists* $g \in BV_0[\boldsymbol{a},\boldsymbol{b}]$ *such that* $\|T\| = \mathrm{Var}(g,[\boldsymbol{a},\boldsymbol{b}])$ *and*

$$T(f) = (\mathrm{HK}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f(\boldsymbol{t})g(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t}$$

for every $f \in \mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$.

Proof. Since the function $\boldsymbol{x} \mapsto (\mathrm{HK}) \int_{[\boldsymbol{x},\boldsymbol{b}]} f(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t}$ is continuous on $[\boldsymbol{a},\boldsymbol{b}]$, the theorem follows from the Hahn-Banach Theorem, Theorems 2.2 and 3.6. The proof is complete. $\square$

Theorem 3.8. *Let* $g\colon [\boldsymbol{a},\boldsymbol{b}] \longrightarrow \mathbb{R}$. *If* $fg \in \mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$ *for every* $f \in \mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$, *then the linear functional*

$$f \mapsto (\mathrm{HK}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f(\boldsymbol{t})g(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t}$$

is $\|\cdot\|_{\mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]}$*-bounded.*

Proof. Since the proof is similar to that of [10, Theorem 4.4], we give only a sketch of the proof.

Suppose that the linear functional

$$f \mapsto (\mathrm{HK}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f(\boldsymbol{t})g(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t}$$

is not $\|\cdot\|_{\mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]}$-bounded. Following the argument of [10, Theorem 4.4], we can construct a function $f \in \mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$ such that $fg \notin \mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$. This contradiction completes the proof. $\square$

The following theorem is an m-dimensional analogue of a result of Sargent [20].

Theorem 3.9 (cf. [8, Theorem 5.1]). *Let $g\colon [\boldsymbol{a},\boldsymbol{b}] \longrightarrow \mathbb{R}$. If $fg \in \mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$ for every $f \in \mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$, then there exists $g_0 \in BV_0[\boldsymbol{a},\boldsymbol{b}]$ such that $g = g_0$ μ_m-almost everywhere on $[\boldsymbol{a},\boldsymbol{b}]$.*

P r o o f. This is a consequence of Theorems 3.8 and 3.7. □

4. The Cauchy-Lebesgue integral

The aim of this section is to study the Cauchy-Lebesgue integral, which is the Cauchy extension of the Lebesgue integral.

Definition 4.1 (cf. [10]). An interval function $F\colon \mathcal{I}_m[\boldsymbol{a},\boldsymbol{b}] \longrightarrow \mathbb{R}$ is said to be continuous if

$$\lim_{\substack{\mu_m(I)\to 0\\ I\in\mathcal{I}_m([\boldsymbol{a},\boldsymbol{b}])}} F(I) = 0.$$

Definition 4.2 (cf. [10]). A function $f\colon [\boldsymbol{a},\boldsymbol{b}] \longrightarrow \mathbb{R}$ is said to be Cauchy-Lebesgue integrable on $[\boldsymbol{a},\boldsymbol{b}]$ if there exist an additive continuous interval function F and a finite set $Q \subset [\boldsymbol{a},\boldsymbol{b}]$ such that $f \in L^1(I)$ and $F(I) = \int_I f$ for every interval $I \in \mathcal{I}_m([\boldsymbol{a},\boldsymbol{b}])$ satisfying $I \cap Q = \emptyset$. In this case, we write $F([\boldsymbol{a},\boldsymbol{b}])$ as $(\mathrm{CL})\int_{[\boldsymbol{a},\boldsymbol{b}]} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$.

It is easy to prove the following theorem.

Theorem 4.3. *If $f \in \mathrm{CL}[\boldsymbol{a},\boldsymbol{b}]$, then $f \in \mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$ and*

$$(\mathrm{CL})\int_{[\boldsymbol{a},\boldsymbol{b}]} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = (\mathrm{HK})\int_{[\boldsymbol{a},\boldsymbol{b}]} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

In view of Theorem 4.3 we can equip the space $\mathrm{CL}[\boldsymbol{a},\boldsymbol{b}]$ with the norm $\|\cdot\|_{\mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]}$. In order to prove an analogous version of Theorem 3.7 for the space $\mathrm{CL}[\boldsymbol{a},\boldsymbol{b}]$, we need the following results.

Lemma 4.4 ([15, Lemma 2.3]). *If $f \in \mathrm{CL}[\boldsymbol{a},\boldsymbol{b}]$, $g \in L^\infty[\boldsymbol{a},\boldsymbol{b}]$ and $fg \in \mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$, then $fg \in \mathrm{CL}[\boldsymbol{a},\boldsymbol{b}]$ and*

$$(\mathrm{CL})\int_{[\boldsymbol{a},\boldsymbol{b}]} f(\boldsymbol{x})g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = (\mathrm{HK})\int_{[\boldsymbol{a},\boldsymbol{b}]} f(\boldsymbol{x})g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

The following theorem is a consequence of Theorem 3.6 and Lemma 4.4.

Theorem 4.5 ([16, Remark 4.11(ii)]). *If $f \in \mathrm{CL}[\boldsymbol{a},\boldsymbol{b}]$ and $g \in BV_0[\boldsymbol{a},\boldsymbol{b}]$, then $fg \in \mathrm{CL}[\boldsymbol{a},\boldsymbol{b}]$ and*

$$(4)\qquad (\mathrm{CL})\int_{[\boldsymbol{a},\boldsymbol{b}]} f(\boldsymbol{x})g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = (RS)\int_{[\boldsymbol{a},\boldsymbol{b}]}\left\{(\mathrm{CL})\int_{[\boldsymbol{x},\boldsymbol{b}]} f(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t}\right\}\mathrm{d}g(\boldsymbol{x}).$$

Following the proof of Theorem 3.7 we get a refinement of [10, Corollary 4.6].

Theorem 4.6. *If T is a bounded linear functional on $\mathrm{CL}[\boldsymbol{a},\boldsymbol{b}]$, then there exists $g \in BV_0[\boldsymbol{a},\boldsymbol{b}]$ such that $\|T\| = \mathrm{Var}(g,[\boldsymbol{a},\boldsymbol{b}])$ and*

$$T(f) = (\mathrm{CL})\int_{[\boldsymbol{a},\boldsymbol{b}]} f(\boldsymbol{t})g(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t}$$

for all $f \in \mathrm{CL}[\boldsymbol{a},\boldsymbol{b}]$.

Theorem 4.7. *Let $g\colon [\boldsymbol{a},\boldsymbol{b}] \longrightarrow \mathbb{R}$. If $fg \in \mathrm{CL}[\boldsymbol{a},\boldsymbol{b}]$ for every $f \in \mathrm{CL}[\boldsymbol{a},\boldsymbol{b}]$, then there exists $g_0 \in BV_0[\boldsymbol{a},\boldsymbol{b}]$ such that $g = g_0$ μ_m-almost everywhere on $[\boldsymbol{a},\boldsymbol{b}]$.*

Proof. The proof is similar to that of Theorem 3.9. We omit it. □

Theorem 4.8. *Let $g\colon [\boldsymbol{a},\boldsymbol{b}] \longrightarrow \mathbb{R}$. The following statements are equivalent.*

(i) *If $f \in \mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$, then $fg \in \mathrm{HK}[\boldsymbol{a},\boldsymbol{b}]$.*
(ii) *If $f \in \mathrm{CL}[\boldsymbol{a},\boldsymbol{b}]$, then $fg \in \mathrm{CL}[\boldsymbol{a},\boldsymbol{b}]$.*

Proof. The implication "(i) $\Longrightarrow$ (ii)" is a consequence of Theorem 3.9 and Lemma 4.4. The converse follows from Theorems 4.7, 3.3(e) and 3.6. □

5. An application to iterated Henstock-Kurzweil integrals

For the rest of this paper we let r and s be positive integers. For $q \in \{r,s\}$ we let E_q be a compact interval in $\mathbb{R}^q$. If f and g are functions defined on E_r and E_s respectively, we let

$$(f \otimes g)(\boldsymbol{x},\boldsymbol{y}) = f(\boldsymbol{x})g(\boldsymbol{y}).$$

The main result (Theorem 5.10) is motivated by the following problem in [15]:

Problem 5.1. Let f and g be Henstock-Kurzweil integrable on intervals $E_r \subset \mathbb{R}^r$ and $E_s \subset \mathbb{R}^s$ respectively. Is $f \otimes g$ Henstock-Kurzweil integrable on the interval $E_r \times E_s$?

For the case when $r = 1$ or $s = 1$, it is known that $f \otimes g \in \mathrm{HK}(E_r \times E_s)$ whenever $f \in \mathrm{HK}(E_r)$ and $g \in \mathrm{HK}(E_s)$; see [13, Theorem 4.5]. If, in addition, h belongs to $BV_0(E_r \times E_s)$, then it follows from Theorem 3.6 that $(f \otimes g)h \in \mathrm{HK}(E_r \times E_s)$; Fubini's theorem for the Henstock-Kurzweil integral yields

$$\begin{aligned}(5)\qquad (\mathrm{HK}) \int_{E_r \times E_s} & f(\boldsymbol{x})g(\boldsymbol{y})h(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}(\boldsymbol{x},\boldsymbol{y}) \\ &= (\mathrm{HK}) \int_{E_r} f(\boldsymbol{x}) \Big\{ (\mathrm{HK}) \int_{E_s} g(\boldsymbol{y})\ h(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \Big\}\,\mathrm{d}\boldsymbol{x} \\ &= (\mathrm{HK}) \int_{E_s} g(\boldsymbol{y}) \Big\{ (\mathrm{HK}) \int_{E_r} f(\boldsymbol{x})\ h(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{x} \Big\}\,\mathrm{d}\boldsymbol{y}.\end{aligned}$$

While it is unclear whether (5) holds when $r, s > 1$ (cf. Problem 5.1), a weaker result is known.

Theorem 5.2 ([13, Theorem 4.6])**.** *If $f \in \mathrm{CL}(E_r)$ and $g \in \mathrm{HK}(E_s)$, then $f \otimes g \in \mathrm{HK}(E_r \times E_s)$ and*

$$\begin{aligned}(\mathrm{HK}) \int_{E_r \times E_s} & (f \otimes g)(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}(\boldsymbol{x},\boldsymbol{y}) \\ &= \Big\{ (\mathrm{CL}) \int_{E_r} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \Big\} \Big\{ (\mathrm{HK}) \int_{E_s} g(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \Big\}.\end{aligned}$$

In this section, we shall prove that another result holds for the function $(\boldsymbol{x},\boldsymbol{y}) \mapsto f(\boldsymbol{x})g(\boldsymbol{y})h(\boldsymbol{x},\boldsymbol{y})$; see Theorem 5.10 for details. We need some lemmas.

Lemma 5.3. *If $g \in \mathrm{HK}(E_s)$ and $h \in BV_0(E_r \times E_s)$, then $(\mathrm{HK}) \int_{E_s} g(\boldsymbol{y})h(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}$ exists for all $\boldsymbol{x} \in E_r$. Moreover, the function*

$$\boldsymbol{x} \mapsto (\mathrm{HK}) \int_{E_s} g(\boldsymbol{y})h(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}$$

belongs to $L^\infty(E_r)$.

P r o o f. We observe that if $\boldsymbol{x} \in E_r$ is fixed, then the function $\boldsymbol{y} \mapsto h(\boldsymbol{x},\boldsymbol{y})$ belongs to $BV_0(E_s)$. An appeal to Theorem 3.6 gives the first part of the theorem.

Next we infer from Theorems 5.2, 3.6 and Fubini's theorem that the function

$$\boldsymbol{x} \mapsto (\mathrm{HK}) \int_{E_s} g(\boldsymbol{y})h(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}$$

is Henstock-Kurzweil integrable on E_r. In particular, the function

$$\boldsymbol{x} \mapsto (\mathrm{HK}) \int_{E_s} g(\boldsymbol{y}) h(\boldsymbol{x}, \boldsymbol{y}) \, \mathrm{d}\boldsymbol{y}$$

is μ_r-measurable.

Finally, we let $f_0 \in L^1(E_r)$ be given. Clearly it suffices to prove that the function

$$\boldsymbol{x} \mapsto f_0(\boldsymbol{x}) \left\{ (\mathrm{HK}) \int_{E_s} g(\boldsymbol{y}) h(\boldsymbol{x}, \boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} \right\}$$

belongs to $L^1(E_r)$. Using Theorems 5.2, 3.6 and Fubini's theorem again, we see that $f_0 \in L^1(E_r)$ implies

$$(\mathrm{HK}) \int_{E_r} f_0(\boldsymbol{x}) \left\{ (\mathrm{HK}) \int_{E_s} g(\boldsymbol{y}) h(\boldsymbol{x}, \boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} \right\} \mathrm{d}\boldsymbol{x}$$

exists. Now, since the function

$$\boldsymbol{x} \mapsto (\mathrm{HK}) \int_{E_s} g(\boldsymbol{y}) h(\boldsymbol{x}, \boldsymbol{y}) \, \mathrm{d}\boldsymbol{y}$$

is μ_r-measurable and $|f_0| \in L^1(E_r)$, a similar argument shows that

$$(\mathrm{HK}) \int_{E_r} \left| f_0(\boldsymbol{x}) \left\{ (\mathrm{HK}) \int_{E_s} g(\boldsymbol{y}) h(\boldsymbol{x}, \boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} \right\} \right| \mathrm{d}\boldsymbol{x}$$

exists. It is now clear that the lemma holds. □

Lemma 5.4. *If $f \in \mathrm{CL}(E_r)$, $g \in \mathrm{HK}(E_s)$ and $h \in BV_0(E_r \times E_s)$, then*

$$(\mathrm{HK}) \int_{E_r \times E_s} f(\boldsymbol{x}) g(\boldsymbol{y}) h(\boldsymbol{x}, \boldsymbol{y}) \, \mathrm{d}(\boldsymbol{x}, \boldsymbol{y}) \tag{6}$$

and

$$(\mathrm{CL}) \int_{E_r} f(\boldsymbol{x}) \left\{ (\mathrm{HK}) \int_{E_s} g(\boldsymbol{y}) h(\boldsymbol{x}, \boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} \right\} \mathrm{d}\boldsymbol{x} \tag{7}$$

exist and coincide.

Proof. We infer from Theorems 5.2 and 3.6 that the Henstock-Kurzweil integral (6) exists. Hence, by Fubini's theorem, the iterated Henstock-Kurzweil integral

$$(\mathrm{HK}) \int_{E_r} f(\boldsymbol{x}) \left\{ (\mathrm{HK}) \int_{E_s} g(\boldsymbol{y}) h(\boldsymbol{x}, \boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} \right\} \mathrm{d}\boldsymbol{x} \tag{8}$$

exists and is equal to the Henstock-Kurzweil integral (6). As a consequence of Lemmas 5.3 and 4.4, the Cauchy-Lebesgue integral (7) exists and is equal to the Henstock-Kurzweil integral (8). The proof is complete. □

The following lemma is a consequence of Lemma 5.4 and Theorem 4.8.

Lemma 5.5. *If $f \in \mathrm{HK}(E_r)$, $g \in \mathrm{HK}(E_s)$ and $h \in BV_0(E_r \times E_s)$, then the iterated Henstock-Kurzweil integral*

$$(\mathrm{HK})\int_{E_r} f(\boldsymbol{x})\left\{(\mathrm{HK})\int_{E_s} g(\boldsymbol{y})h(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right\}\mathrm{d}\boldsymbol{x}$$

exists.

Lemma 5.6. *If $g \in \mathrm{HK}(E_s)$ and $h \in BV_0(E_r \times E_s)$, then the functional*

$$S_g\colon \mathrm{HK}(E_r) \longrightarrow \mathbb{R}\colon f \mapsto (\mathrm{HK})\int_{E_r} f(\boldsymbol{x})\left\{(\mathrm{HK})\int_{E_s} g(\boldsymbol{y})h(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right\}\mathrm{d}\boldsymbol{x}$$

is linear and bounded.

P r o o f. This is a consequence of Lemma 5.5 and Theorem 3.8. □

The proof of the following lemma is similar to that of Lemma 5.5.

Lemma 5.7. *If $f \in \mathrm{HK}(E_r)$, $g \in \mathrm{HK}(E_s)$ and $h \in BV_0(E_r \times E_s)$, then the iterated Henstock-Kurzweil integral*

$$(\mathrm{HK})\int_{E_s} g(\boldsymbol{y})\left\{(\mathrm{HK})\int_{E_r} f(\boldsymbol{x})h(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\right\}\mathrm{d}\boldsymbol{y}$$

exists.

On the other hand, the proof of the following lemma is more involved than that of Lemma 5.6.

Lemma 5.8. *If $g \in \mathrm{HK}(E_s)$ and $h \in BV_0(E_r \times E_s)$, then the functional*

$$T_g\colon \mathrm{HK}(E_r) \longrightarrow \mathbb{R}\colon f \mapsto (\mathrm{HK})\int_{E_s} g(\boldsymbol{y})\left\{(\mathrm{HK})\int_{E_r} f(\boldsymbol{x})h(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\right\}\mathrm{d}\boldsymbol{y}$$

is linear and bounded.

P r o o f. According to Theorem 3.4 there exists a sequence $\{g_n\}_{n=1}^{\infty}$ in $L^1(E_s)$ such that

$$\lim_{n\to\infty} \|g_n - g\|_{\mathrm{HK}(E_s)} = 0.$$

For each $f \in \mathrm{HK}(E_r)$ we argue as in the proof of Lemma 5.6 to conclude that the function $\boldsymbol{y} \mapsto (\mathrm{HK})\int_{E_r} f(\boldsymbol{x})h(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}$ induces a bounded linear functional on

$\mathrm{HK}(E_s)$. Therefore T_g is bounded on $\mathrm{HK}(E_r)$:

$$\begin{aligned}|T_g(f)| &= \lim_{n\to\infty}\left|(\mathrm{HK})\int_{E_s} g_n(\boldsymbol{y})\left\{(\mathrm{HK})\int_{E_r} f(\boldsymbol{x})h(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\right\}\mathrm{d}\boldsymbol{y}\right| \\ &= \lim_{n\to\infty}\left|(\mathrm{HK})\int_{E_r\times E_s} (f\otimes g_n)(\boldsymbol{x},\boldsymbol{y})h(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}(\boldsymbol{x},\boldsymbol{y})\right| \text{(by Theorems 5.2 and 3.6)} \\ &\leqslant \|f\|_{\mathrm{HK}(E_r)}\|g\|_{\mathrm{HK}(E_s)}\|h\|_{BV_0(E_r\times E_s)},\end{aligned}$$

where the last inequality holds by Theorem 3.6 and our choice of $\{g_n\}_{n=1}^{\infty}$. The proof is complete. □

Lemma 5.9. *Let $g\in\mathrm{HK}(E_s)$ and let $h\in BV_0(E_r\times E_s)$. If S_g and T_g are given as in Lemmas* 5.6 *and* 5.8 *respectively, then*

$$S_g(f_0)=T_g(f_0)$$

for every $f_0\in\mathrm{CL}(E_r)$.

P r o o f. This follows from Lemma 5.4 and Fubini's theorem. The proof is complete. □

Theorem 5.10 (Main Theorem)**.** *If $f\in\mathrm{HK}(E_r)$, $g\in\mathrm{HK}(E_s)$ and $h\in BV_0(E_r\times E_s)$, then the iterated Henstock-Kurzweil integrals*

$$(\mathrm{HK})\int_{E_r} f(\boldsymbol{x})\left\{(\mathrm{HK})\int_{E_s} g(\boldsymbol{y})h(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right\}\mathrm{d}\boldsymbol{x},$$

$$(\mathrm{HK})\int_{E_s} g(\boldsymbol{y})\left\{(\mathrm{HK})\int_{E_r} f(\boldsymbol{x})h(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\right\}\mathrm{d}\boldsymbol{y}$$

exist and coincide.

P r o o f. This follows from Lemmas 5.5–5.9 and Theorem 3.4. □

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