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BALANCED PATH DECOMPOSITION OF $\lambda K_{n,n}$ AND $\lambda K^*_{n,n}$

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Abstract. Let P_k denote a path with k edges and $\lambda K_{n,n}$ denote the λ-fold complete bipartite graph with both parts of size n. In this paper, we obtain the necessary and sufficient conditions for $\lambda K_{n,n}$ to have a balanced P_k-decomposition. We also obtain the directed version of this result.

Keywords: path decomposition, balanced decomposition, complete bipartite graph

MSC 2010: 05C38, 05C70

1. Introduction and preliminaries

Let $\mathscr{D}$ be a family of edge-disjoint subgraphs of a multigraph H. If every edge of H appears in some member of $\mathscr{D}$, then $\mathscr{D}$ is a *decomposition* of H. A decomposition $\mathscr{D}$ of a multigraph H is *balanced* if each vertex of H belongs to the same number of members in $\mathscr{D}$. For a multigraph G, a decomposition $\mathscr{D}$ of a multigraph H is a *G-decomposition* of H, if every member of $\mathscr{D}$ is isomorphic to G. For multidigraphs G and H, the following terms are similarly defined: a *decomposition* of H, a *balanced decomposition* of H and a *G-decomposition* of H.

Let G be a multigraph. We use the symbol G^* to denote the multidigraph obtained from G by replacing each edge e by two arcs with opposite directions. For a positive integer λ, λG denotes the multigraph obtained from G by replacing each edge e by λ edges each of which has the same endvertices as e. For a multidigraph G, λG is similarly defined. For a positive integer k, let P_k denote a path with k edges, and $\overrightarrow{P_k}$ a directed path with k arcs. Let K_n denote the complete graph on n vertices, and K_{n_1,n_2} the complete bipartite graph with parts of sizes n_1, n_2, respectively.

The balanced P_k-decomposition problem of λK_n was solved by Huang [3] and Hung and Mendelsohn [2], [4], independently. The balanced $\overrightarrow{P_k}$-decomposition problem of K^*_n for even k was solved by Bermond [1], [2]. Furthermore, Yu [6] obtained a necessary and sufficient condition for P_k-factorization of $\lambda K_{n,n}$ (the P_k-factorization is a special type of the balanced P_k-decomposition). Recently, Shyu [5] settled the P_k-decomposition problem of $\lambda K_{n,n}$ with the sole exception of $\lambda = 3$, $n = 15$ and $k = 27$. In this paper the balanced P_k-decomposition of $\lambda K_{n,n}$ and the balanced $\overrightarrow{P_k}$-decomposition of $\lambda K^*_{n,n}$ are investigated. We obtain the following results:

Theorem 2.6. *$\lambda K_{n,n}$ has a balanced P_k-decomposition if and only if $k \leqslant 2n - 1$ and $(k + 1)\lambda n \equiv 0 \pmod{2k}$.*

Theorem 2.7. *$\lambda K^*_{n,n}$ has a balanced $\overrightarrow{P_k}$-decomposition if and only if $k \leqslant 2n - 1$ and $\lambda n \equiv 0 \pmod{k}$.*

2. Balanced P_k-decompositions of $\lambda K_{n,n}$

In this section we investigate the balanced P_k-decomposition of $\lambda K_{n,n}$. A multigraph G is *r-regular* if each vertex of G is incident with r edges. Obviously $\lambda K_{n,n}$ is λn-regular. We begin with a necessary condition for the existence of a balanced decomposition.

Proposition 2.1 [1; pp. 45–46]**.** *Suppose that G is a multigraph of order n_1, size e_1, and H is a multigraph of order n_2, size e_2. If H has a balanced G-decomposition then $n_1 e_2 \equiv 0 \pmod{n_2 e_1}$.*

The above proposition implies a necessary condition for a regular multigraph to have a balanced decomposition.

Corollary 2.2. *Suppose that G is a multigraph of order n_1, size e_1. If an r-regular multigraph has a balanced G-decomposition, then $n_1 r \equiv 0 \pmod{2e_1}$.*

Now a necessary condition for a regular multigraph to have a balanced path decomposition follows.

Corollary 2.3. *If an r-regular multigraph has a balanced P_k-decomposition, then $(k + 1)r \equiv 0 \pmod{2k}$.*

For our discussions in this section, we introduce the following terms and notations. For a positive integer n and an integer k, the notation $k \pmod{n}$ denotes the integer l with $0 \leqslant l \leqslant n - 1$ and $l \equiv k \pmod{n}$. For example, $22 \pmod 5$, $23 \pmod 5$, $24 \pmod 5$, $25 \pmod 5$, $26 \pmod 5$ denote 2, 3, 4, 0, 1, respectively. Let (A, B) be the bipartition of the bipartite graph $\lambda K_{n,n}$ where $A = \{a_0, a_1, \ldots, a_{n-1}\}$ and

$B = \{b_0, b_1, \ldots, b_{n-1}\}$. The subscripts of a_i and b_j will always be taken modulo n. For any edge a_ib_j $(0 \leqslant i, j \leqslant n-1)$ in $\lambda K_{n,n}$, the *label* of a_ib_j is $(j-i) \pmod n$. Thus the label of a_ib_j is $j-i$ if $0 \leqslant i \leqslant j \leqslant n-1$, and is $j-i+n$ if $0 \leqslant j < i \leqslant n-1$. Note that all the λ edges joining a_i and b_j have the same label.

Let G be a multigraph. For $x, y \in V(G)$ with $x \neq y$, we use $m_G(x,y)$ to denote the number of edges joining x and y in G. If xy is an edge of G, $m_G(x,y)$ is called the *multiplicity* of the edge xy in G.

Let G be a subgraph of $\lambda K_{n,n}$ with vertex set $V(G)$ and edge set $E(G)$, and let r be a nonnegative integer. Then $G+r$ denotes the subgraph of $\lambda K_{n,n}$ with vertex set $\{a_{i+r}\colon a_i \in V(G)\} \cup \{b_{j+r}\colon b_j \in V(G)\}$ and edge set $\{a_{i+r}b_{j+r}$ with multiplicity $\mu_{ij}\colon a_ib_j \in E(G)$ with multiplicity $\mu_{ij}\}$. Further G_{+r} denotes the subgraph of $\lambda K_{n,n}$ with vertex set $\{a_i\colon a_i \in V(G)\} \cup \{b_{j+r}\colon b_j \in V(G)\}$ and edge set $\{a_ib_{j+r}$ with multiplicity $\mu_{ij}\colon a_ib_j \in E(G)$ with multiplicity $\mu_{ij}\}$.

Suppose that $G_1, G_2, \ldots, G_t$ are subgraphs of a multigraph. We use $G_1+G_2+\ldots+G_t$ to denote the multigraph S with vertex set $V(S) = \bigcup\limits_{i=1}^{t} V(G_i)$, and for $x, y \in V(S)$ with $x \neq y$, $m_S(x,y) = \sum\limits_{i=1}^{t} m_{G_i}(x,y)$ (in case x or y is not in $V(G_i)$, we let $m_{G_i}(x,y) = 0$). The graph $G_1+G_2+\ldots+G_t$ is called the *edge sum* of $G_1, G_2, \ldots, G_t$, and is also denoted by $\sum\limits_{i=1}^{t} G_i$.

Lemma 2.4. *Suppose that Q is a subgraph of $\lambda K_{n,n}$ containing k edges which have the respective labels* $a \pmod n, (a+1) \pmod n, (a+2) \pmod n, \ldots, (a+k-1) \pmod n$. *Let t be a positive integer with $tk \leqslant \lambda n$. Then $\sum\limits_{i=0}^{t-1} Q_{+ik}$ is a subgraph of $\lambda K_{n,n}$ containing tk edges which have the respective labels* $a \pmod n, (a+1) \pmod n, (a+2) \pmod n, \ldots, (a+tk-1) \pmod n$.

P r o o f. Let G be the multigraph $\sum\limits_{i=0}^{t-1} Q_{+ik}$. Since each Q_{+ik} $(i = 0, 1, \ldots, t-1)$ is a subgraph of $\lambda K_{n,n}$, G is a subgraph of $t\lambda K_{n,n}$. Further, since each Q_{+ik} $(i = 0, 1, \ldots, t-1)$ contains k edges, G contains tk edges. The fact that the edges of Q have labels $a \pmod n, (a+1) \pmod n, \ldots, (a+k-1) \pmod n$ implies that the edges of Q_{+k} have labels $(a+k) \pmod n, (a+k+1) \pmod n, \ldots, (a+2k-1) \pmod n$, the edges of Q_{+2k} have labels $(a+2k) \pmod n, (a+2k+1) \pmod n, \ldots, (a+3k-1) \pmod n, \ldots$, and the edges of $Q_{+(t-1)k}$ have labels $(a+(t-1)k) \pmod n, (a+(t-1)k+1) \pmod n, \ldots, (a+tk-1) (\bmod n)$. Thus the edges of G have labels $a \pmod n, (a+1) \pmod n, \ldots, (a+tk-1) \pmod n$. Now we show that G is in fact a subgraph of $\lambda K_{n,n}$. In G, there are either $\lfloor tk/n \rfloor$ or $\lceil tk/n \rceil$ edges (multiplicities being considered) with labels i for each $i = 0, 1, 2, \ldots, n-1$. Thus each edge in G has multiplicity $\leqslant \lceil tk/n \rceil \leqslant \lambda$, which implies that G is a subgraph of $\lambda K_{n,n}$. $\square$

Lemma 2.5. *Suppose that G is a subgraph of $\lambda K_{n,n}$ containing exactly λ_i edges (multiplicities are counted) with label i, $i = 0, 1, \ldots, n-1$. Then $\sum_{r=0}^{n-1}(G+r)$ is a subgraph of $\lambda K_{n,n}$ with the property that each edge with label i has multiplicity λ_i.*

Proof. Let $S = \sum_{r=0}^{n-1}(G+r)$. Suppose that e is an edge with label i. Let $e = a_k b_{k+i}$, for some $0 \leqslant k \leqslant n-1$. Then

$$m_S(a_k, b_{k+i}) = \sum_{r=0}^{n-1} m_{G+r}(a_k, b_{k+i}) = \sum_{r=0}^{n-1} m_G(a_{k-r}, b_{k+i-r}) = \lambda_i.$$

Thus each edge in S with label i has multiplicity λ_i. Since $\lambda_i \leqslant \lambda$, S is a subgraph of $\lambda K_{n,n}$. □

Letting $\lambda_0 = \lambda_1 = \lambda_2 = \ldots = \lambda_{n-1} = \lambda$ in Lemma 2.5, we have

Lemma 2.6. *Suppose that G is a subgraph of $\lambda K_{n,n}$ containing exactly λ edges (multiplicities being counted) with labels i, $i = 0, 1, \ldots, n-1$. Then $\sum_{r=0}^{n-1}(G+r) = \lambda K_{n,n}$.*

The following lemma is trivial.

Lemma 2.7. *Let G be a subgraph of $\lambda K_{n,n}$, and let G have v vertices in $A = \{a_0, a_1, \ldots, a_{n-1}\}$. Suppose that $G, G+1, G+2, \ldots, G+(n-1)$ are distinct subgraphs of $\lambda K_{n,n}$. Let $F = \{G+r \colon r = 0, 1, 2, \ldots, n-1\}$. Then for each $a \in A$, a belongs to v members in F.* □

Now we prove the main result of this section.

Theorem 2.8. *$\lambda K_{n,n}$ has a balanced P_k-decomposition if and only if $k \leqslant 2n-1$ and $(k+1)\lambda n \equiv 0 \pmod{2k}$.*

Proof. (Necessity) The required inequality is trivial. The required congruence relation follows from Corollary 2.3 since $\lambda K_{n,n}$ is a λn-regular multigraph.
(Sufficiency) The assumption $(k+1)\lambda n \equiv 0 \pmod{2k}$ implies $\lambda n \equiv 0 \pmod{k}$. Let $\lambda n = pk$ where p is a positive integer. We distinguish two cases: Case 1. k is odd, Case 2. k is even.

Case 1. k is odd.

Let Q be the walk $b_{\frac{k-1}{2}} a_{\frac{k-1}{2}} b_{\frac{k+1}{2}} a_{\frac{k-3}{2}} b_{\frac{k+3}{2}} a_{\frac{k-5}{2}} \ldots b_{k-2} a_1 b_{k-1} a_0$. Since $\frac{k+1}{2} \leqslant n$, we see that the vertices $b_{\frac{k-1}{2}}, b_{\frac{k+1}{2}}, b_{\frac{k+3}{2}}, \ldots, b_{k-2}, b_{k-1}$ are distinct, and so are the vertices $a_{\frac{k-1}{2}}, a_{\frac{k-3}{2}}, a_{\frac{k-5}{2}}, \ldots, a_1, a_0$. Thus Q is a path of length k. We see that Q

consists of edges with labels $0, 1, 2, \ldots, (k-1) \pmod n$. Let G be the edge sum $Q + Q_{+k} + Q_{+2k} + \ldots + Q_{+(p-1)k}$. By Lemma 2.4, G is a subgraph of $\lambda K_{n,n}$ consisting of edges with labels $0, 1, 2, \ldots, (pk-1) \pmod n$, and hence with labels $0, 1, 2, \ldots, (\lambda n - 1) \pmod n$. Thus for each $i \in \{0, 1, \ldots, n-1\}$, G contains exactly λ edges (multiplicities being counted) with label i. Thus

$$\begin{aligned}\lambda K_{n,n} &= \sum_{r=0}^{n-1}(G+r) \quad \text{(by Lemma 2.6)}\\ &= \sum_{r=0}^{n-1}((Q + Q_{+k} + \ldots + Q_{+(p-1)k}) + r)\\ &= \sum_{r=0}^{n-1}((Q+r) + (Q_{+k}+r) + \ldots + (Q_{+(p-1)k}+r)).\end{aligned}$$

Thus $\lambda K_{n,n}$ can be decomposed into the following paths of length k: $Q_{+ik} + r$ $(i = 0, 1, \ldots, p-1; r = 0, 1, \ldots, n-1)$. Let $F = \{Q_{+ik} + r \colon i = 0, 1, \ldots, p-1; r = 0, 1, \ldots, n-1\}$. Then F is a P_k-decomposition of $\lambda K_{n,n}$. Now we check that the decomposition F is balanced. Since Q has $\frac{k+1}{2}$ vertices in A, so does each Q_{+ik} $(i = 1, 2, \ldots, p-1)$. By Lemma 2.7, for each $a \in A, a$ belongs to $p\frac{k+1}{2}$ members in F. Similarly, since Q has $\frac{k+1}{2}$ vertices in B, for each $b \in B$, b belongs to $p\frac{k+1}{2}$ members in F. Thus F is balanced.

Case 2. k is even.

Since $(k+1)\lambda n \equiv 0 \pmod{2k}$ and $\lambda n = pk$, we have $(k+1)p \equiv 0 \pmod 2$, which implies $p \equiv 0 \pmod 2$. Let Q be the walk $a_{\frac{k}{2}} b_{\frac{k}{2}} a_{\frac{k}{2}-1} b_{\frac{k}{2}+1} a_{\frac{k}{2}-2} b_{\frac{k}{2}+2} \ldots a_1 b_{k-1} a_0$. Since $k \leqslant 2n-1$ and k is even, we have $\frac{k}{2} + 1 \leqslant n$, which implies that the vertices $a_{\frac{k}{2}}, a_{\frac{k}{2}-1}, a_{\frac{k}{2}-2}, \ldots, a_1$, a_0 are distinct, and so are the vertices $b_{\frac{k}{2}}, b_{\frac{k}{2}+1}, b_{\frac{k}{2}+2}, \ldots,$ b_{k-1}. Hence Q is a path of length k. We see that Q consists of edges with labels $0, 1, 2, \ldots, (k-1) \pmod n$. Since $pk = \lambda n$, we have $\frac{p}{2}k \leqslant \lambda n$. By Lemma 2.4, $Q + Q_{+k} + Q_{+2k} + \ldots, Q_{+(\frac{p}{2}-1)k}$ is a subgraph of $\lambda K_{n,n}$ of which the edges have labels $0, 1, 2, \ldots, (\frac{p}{2}k - 1) \pmod n$.

Let R be the walk $b_{(\frac{p}{2}+\frac{1}{2})k-1} a_{\frac{k}{2}-1} b_{(\frac{p}{2}+\frac{1}{2})k} a_{\frac{k}{2}-2} \ldots b_{(\frac{p}{2}+1)k-2} a_0 b_{(\frac{p}{2}+1)k-1}$. Then R is a path of length k consisting of edges with labels $\frac{p}{2}k \pmod n, (\frac{p}{2}k+1) \pmod n, (\frac{p}{2}k+2) \pmod n, \ldots, ((\frac{p}{2}+1)k-1) \pmod n$. Again, by Lemma 2.4, $R + R_{+k} + R_{+2k} + \ldots, R_{+(\frac{p}{2}-1)k}$ is a subgraph of $\lambda K_{n,n}$ the edges of which have labels $\frac{p}{2}k \pmod n, (\frac{p}{2}k+1) \pmod n, (\frac{p}{2}k+2) \pmod n, \ldots, (pk-1) \pmod n$.

Let $G = Q + Q_{+k} + Q_{+2k} + \ldots + Q_{+(\frac{p}{2}-1)k} + R + R_{+k} + R_{+2k} + \ldots + R_{+(\frac{p}{2}-1)k}$. Then the edges in G are with labels $0, 1, \ldots, (pk-1) \pmod n$. Since $pk = \lambda n$, G

contains exactly λ edges with label i for each $i = 0, 1, 2, \dots, n-1$. Thus

$$\begin{aligned}\lambda K_{n,n} &= \sum_{r=0}^{n-1}(G+r) \quad \text{(by Lemma 2.6)}\\ &= \sum_{r=0}^{n-1}((Q+Q_{+k}+\ldots+Q_{+(\frac{p}{2}-1)k}+R+R_{+k}+\ldots+R_{+(\frac{p}{2}-1)k})+r)\\ &= \sum_{r=0}^{n-1}((Q+r)+(Q_{+k}+r)+\ldots+(Q_{+(\frac{p}{2}-1)k}+r)\\ &\quad +(R+r)+(R_{+k}+r)+\ldots+(R_{+(\frac{p}{2}-1)k}+r)).\end{aligned}$$

Hence $\lambda K_{n,n}$ is decomposed into the following paths of length k: $Q_{+ik}+r$ ($i = 0, 1, \dots, \frac{p}{2}-1; r = 0, 1, \dots, n-1$), and $R_{+ik}+r$ ($i = 0, 1, \dots, \frac{p}{2}-1; r = 0, 1, \dots, n-1$).

Let $F_1 = \{Q_{+ik}+r\colon\ i = 0, 1, \dots, \frac{p}{2}-1; r = 0, 1, \dots, n-1\}$, $F_2 = \{R_{+ik}+r\colon\ i = 0, 1, \dots, \frac{p}{2}-1; r = 0, 1, \dots, n-1\}$, and let $F = F_1 \cup F_2$. Then F is a P_k-decomposition of $\lambda K_{n,n}$. Now we check that the decomposition F is balanced. Since Q has $\frac{k}{2}+1$ vertices in A and $\frac{k}{2}$ vertices in B, by an argument similar to Case 1, for each $a \in A$, a belongs to $\frac{p}{2}(\frac{k}{2}+1)$ members in F_1, and for each $b \in B$, b belongs to $\frac{p}{2}\frac{k}{2}$ members in F_1. Similarly, since R has $\frac{k}{2}$ vertices in A and $\frac{k}{2}+1$ vertices in B, for each $a \in A$, a belongs to $\frac{p}{2}\frac{k}{2}$ members in F_2, and for each $b \in B$, b belongs to $\frac{p}{2}(\frac{k}{2}+1)$ members in F_2. Consequently, for each $x \in A \cup B$, x belongs to $\frac{p}{2}(k+1)$ members in F. Hence F is balanced. □

3. Balanced $\overrightarrow{P_k}$-decompositions of $\lambda K^*_{n,n}$

In this section we investigate the balanced $\overrightarrow{P_k}$-decompositions of $\lambda K^*_{n,n}$. We introduce some terms and notations which are similar to those in Section 2. Let (A, B) be the bipartition of $\lambda K^*_{n,n}$ where $A = \{a_0, a_1, \dots, a_{n-1}\}$ and $B = \{b_0, b_1, \dots, b_{n-1}\}$, and the subscripts of a_i and b_j will always be taken modulo n. Now label the arcs in $\lambda K^*_{n,n}$. First, assign labels $0, 1, 2, \dots, n-1$ to arcs of the form $\overrightarrow{a_ib_j}$. For $0 \leqslant i, j \leqslant n-1$, the *label* of $\overrightarrow{a_ib_j}$ is $(j-i) \pmod n$. Next we assign labels $\overline{0}, \overline{1}, \overline{2}, \dots, \overline{n-1}$ to arcs of the form $\overrightarrow{b_ja_i}$ by the following rule: when the label of $\overrightarrow{a_ib_j}$ is t, assign $\overrightarrow{b_ja_i}$ the label $\overline{t}$. For example in $3K^*_{6,6}$, the labels of $\overrightarrow{a_2b_4}$ and $\overrightarrow{a_3b_1}$ are 2 and 4, respectively, and the labels of $\overrightarrow{b_4a_2}$ and $\overrightarrow{b_1a_3}$ are $\overline{2}$ and $\overline{4}$, respectively.

Suppose that G is a multidigraph. For $x, y \in V(G)$ with $x \neq y$, we use $m_G(x, y)$ to denote the number of arcs from x to y in G. If $\overrightarrow{xy}$ is an arc of G, $m_G(x, y)$ is called the *multiplicity* of $\overrightarrow{xy}$ in G.

Let G be a subdigraph of $\lambda K^*_{n,n}$ with vertex set $V(G)$ and arc set $E(G)$, and let r be a nonnegative integer. Then $G+r$ denotes the subdigraph of $\lambda K^*_{n,n}$ with vertex set $\{a_{i+r}\colon a_i \in V(G)\} \cup \{b_{j+r}\colon b_j \in V(G)\}$ and arc set $\{\overrightarrow{a_{i+r}b_{j+r}}$ with multiplicity $\mu_{ij}\colon \overrightarrow{a_ib_j} \in E(G)$ with multiplicity $\mu_{ij}\} \cup \{\overrightarrow{b_{j+r}a_{i+r}}$ with multiplicity $\varrho_{ji}\colon \overrightarrow{b_ja_i} \in E(G)$ with multiplicity $\varrho_{ji}\}$. Further, G_{+r} denotes the subdigraph of $\lambda K^*_{n,n}$ with vertex set $\{a_i\colon a_i \in V(G)\} \cup \{b_{j+r}\colon b_j \in V(G)\}$ and arc set $\{\overrightarrow{a_ib_{j+r}}$ with multiplicity $\mu_{ij}\colon \overrightarrow{a_ib_j} \in E(G)$ with multiplicity $\mu_{ij}\} \cup \{\overrightarrow{b_{j+r}a_i}$ with multiplicity $\varrho_{ji}\colon \overrightarrow{b_ja_i} \in E(G)$ with multiplicity $\varrho_{ji}\}$.

Suppose that $G_1, G_2, \ldots, G_t$ are subdigraphs of a multidigraph. We use $G_1 + G_2 + \ldots + G_t$ to denote the multidigraph S with vertex set $V(S) = \bigcup_{i=1}^{t} V(G_i)$, and for $x, y \in V(S)$ with $x \neq y$, $m_S(x,y) = \sum_{i=1}^{t} m_{G_i}(x,y)$ (in case x or y is not in $V(G_i)$, we let $m_{G_i}(x,y) = 0$). The graph $G_1 + G_2 + \ldots + G_t$ is called the *arc sum* of $G_1, G_2, \ldots, G_t$, and is also denoted by $\sum_{i=1}^{t} G_i$.

Now consider the balanced $\overrightarrow{P_k}$-decomposition of $\lambda K^*_{n,n}$. The following three lemmas being similar to Lemmas 2.5–2.7, we omit the proofs.

Lemma 3.1. *Let G be a subdigraph of $\lambda K^*_{n,n}$. Suppose that for $\alpha = 0, 1, \ldots, n-1, \overline{0}, \overline{1}, \ldots, \overline{n-1}$, G contains exactly λ_α arcs (multiplicities being counted) with label α where $\lambda_\alpha \leqslant \lambda$. Then $\sum_{r=0}^{n-1}(G+r)$ is a subdigraph of $\lambda K^*_{n,n}$ with the property that each arc with label α has multiplicity λ_α.*

Letting $\lambda_\alpha = \lambda$ for $\alpha = 0, 1, \ldots, n-1, \overline{0}, \overline{1}, \ldots, \overline{n-1}$ in Lemma 3.1, we have

Lemma 3.2. *Suppose that G is a subdigraph of $\lambda K^*_{n,n}$. For $\alpha = 0, 1, \ldots, n-1, \overline{0}, \overline{1}, \ldots, \overline{n-1}$, G contains exactly λ arcs (multiplicities being counted) with label α. Then $\sum_{r=0}^{n-1}(G+r) = \lambda K^*_{n,n}$.*

Lemma 3.3. *Let G be a subdigraph of $\lambda K^*_{n,n}$, and let G have v vertices in $A = \{a_0, a_1, \ldots, a_{n-1}\}$. Suppose that $G, G+1, G+2, \ldots, G+(n-1)$ are distinct subdigraphs of $\lambda K^*_{n,n}$. Let $F = \{G+r\colon r = 0, 1, 2, \ldots, n-1\}$. Then for each $a \in A$, a belongs to v members in F.*

Now we prove the main result of this section.

Theorem 3.4. *$\lambda K^*_{n,n}$ has a balanced $\overrightarrow{P_k}$-decomposition if and only if $k \leqslant 2n-1$ and $\lambda n \equiv 0 \pmod k$.*

P r o o f. (Necessity) The required inequality is trivial. Now we prove the required congruence relation. Removing the directions from the arcs of directed paths in the balanced $\overrightarrow{P_k}$-decomposition of $\lambda K^*_{n,n}$, we obtain a balanced P_k-decomposition of $2\lambda K_{n,n}$. By the necessity condition of Theorem 2.8, $(k+1)2\lambda n \equiv 0 \pmod{2k}$, and hence $\lambda n \equiv 0 \pmod k$.

(Sufficiency) Let $\lambda n = pk$ where p is a positive integer. We distinguish two cases: Case 1. $2k \mid (k+1)\lambda n$, Case 2. $2k \nmid (k+1)\lambda n$.

C a s e 1. $2k \mid (k+1)\lambda n$.

Since $k \leqslant 2n-1$, by Theorem 2.8 there exists a balanced P_k-decomposition of $\lambda K_{n,n}$. Replacing each edge in $\lambda K_{n,n}$ by two arcs with opposite directions, we obtain $\lambda K^*_{n,n}$, and any P_k in $\lambda K_{n,n}$ becomes two $\overrightarrow{P_k}$'s with opposite directions in $\lambda K^*_{n,n}$. Thus we obtain a balanced $\overrightarrow{P_k}$-decomposition of $\lambda K^*_{n,n}$.

C a s e 2. $2k \nmid (k+1)\lambda n$.

Since $\lambda n = pk$ and $2k \nmid (k+1)\lambda n$, we have $2 \nmid (k+1)p$, which implies that p is odd and k is even.

Let Q be the directed walk $a_{\frac{k}{2}} b_{\frac{k}{2}} a_{\frac{k}{2}-1} b_{\frac{k}{2}+1} a_{\frac{k}{2}-2} b_{\frac{k}{2}+2} \ldots a_1 b_{k-1} a_0$. Since $\frac{k}{2}+1 \leqslant n$, we see that the vertices $a_{\frac{k}{2}}, a_{\frac{k}{2}-1}, a_{\frac{k}{2}-2}, \ldots, a_1, a_0$ are distinct, and so are the vertices $b_{\frac{k}{2}}, b_{\frac{k}{2}+1}, b_{\frac{k}{2}+2}, \ldots, b_{k-1}$. Hence Q is a directed path of length k. We see that the arcs of Q have labels $0, \overline{1}, 2, \overline{3}, \ldots, (k-2) \pmod n, \overline{(k-1) \pmod n}$, the arcs of Q_{+k} have labels $k \pmod n$, $\overline{(k+1) \pmod n}$, $\ldots$, $(2k-2) \pmod n$, $\overline{(2k-1) \pmod n}$, the arcs of Q_{+2k} have labels $2k \pmod n$, $\overline{(2k+1) \pmod n}$, $\ldots$, $(3k-2) \pmod n$, $\overline{(3k-1) \pmod n}$, $\ldots$, and the arcs of $Q_{+(p-1)k}$ have labels $(p-1)k \pmod n$, $\overline{((p-1)k+1) \pmod n}$, $\ldots$, $(pk-2) \pmod n$, $\overline{(pk-1) \pmod n}$. Thus the arcs of $Q + Q_{+k} + Q_{+2k} + \ldots + Q_{+(p-1)k}$ have labels $0, \overline{1}, 2, \overline{3}, \ldots, (pk-2) \pmod n$, $\overline{(pk-1) \pmod n}$.

Let R be the directed walk $b_{\frac{k}{2}-1} a_{\frac{k}{2}-1} b_{\frac{k}{2}} a_{\frac{k}{2}-2} \ldots b_{k-2} a_0 b_{k-1}$. Then R is a directed path of length k. We see that the arcs of R have labels $\overline{0}, 1, \overline{2}, 3, \ldots, \overline{(k-2) \pmod n}$, $(k-1) \pmod n$, the arcs of R_{+k} have labels $\overline{k \pmod n}$, $(k+1) \pmod n$, $\ldots$, $\overline{(2k-2) \pmod n}$, $(2k-1) \pmod n$, the arcs of R_{+2k} have labels $\overline{2k \pmod n}$, $(2k+1) \pmod n, \ldots, \overline{(3k-2) \pmod n}, (3k-1) \pmod n, \ldots,$ and the arcs of $R_{+(p-1)k}$ have labels $\overline{(p-1)k \pmod n}$, $((p-1)k+1) \pmod n, \ldots, \overline{(pk-2) \pmod n}$, $(pk-1) \pmod n$. Thus the arcs of $R + R_{+k} + R_{+2k} + \ldots + R_{+(p-1)k}$ have labels $\overline{0}, 1, \overline{2}, 3, \ldots, \overline{(pk-2) \pmod n}, (pk-1) \pmod n$.

Let $G = Q + Q_{+k} + Q_{+2k} + \ldots + Q_{+(p-1)k} + R + R_{+k} + R_{+2k} + \ldots + R_{+(p-1)k}$. From above we see that the arcs in G have labels $0, 1, \ldots, (pk-1) \pmod n$ and

$\overline{0}, \overline{1}, \ldots, \overline{(pk-1) \pmod n}$. Since $pk = \lambda n$, G contains exactly λ edges with label α for each $\alpha = 0, 1, \ldots n-1, \overline{0}, \overline{1}, \ldots, \overline{n-1}$. Thus

$$\begin{aligned}\lambda K^*_{n,n} &= \sum_{r=0}^{n-1} (G + r) \quad \text{(by Lemma 3.2)} \\ &= \sum_{r=0}^{n-1} ((Q + Q_{+k} + \ldots + Q_{+(p-1)k} + R + R_{+k} + \ldots + R_{+(p-1)k}) + r) \\ &= \sum_{r=0}^{n-1} ((Q + r) + (Q_{+k} + r) + \ldots + (Q_{+(p-1)k} + r) \\ &\quad + (R + r) + (R_{+k} + r) + \ldots + (R_{+(p-1)k} + r).\end{aligned}$$

Hence $\lambda K^*_{n,n}$ is decomposed into the following directed paths of length k: $Q_{+ik} + r$ ($i = 0, 1, \ldots, p-1; r = 0, 1, \ldots, n-1$), $R_{+ik} + r$ ($i = 0, 1, \ldots, p-1; r = 0, 1, \ldots, n-1$).

Let $F_1 = \{Q_{+ik} + r \colon i = 0, 1, \ldots, p-1; r = 0, 1, \ldots, n-1\}$, $F_2 = \{R_{+ik} + r \colon i = 0, 1, \ldots, p-1; r = 0, 1, \ldots, n-1\}$, and $F = F_1 \cup F_2$. Then F is a $\overrightarrow{P_k}$-decomposition of $\lambda K^*_{n,n}$. Now we check that the decomposition F is balanced. Since Q has $\frac{k}{2} + 1$ vertices in A and $\frac{k}{2}$ vertices in B, by Lemma 3.3, for each $a \in A$, a belongs to $p(\frac{k}{2} + 1)$ members in F_1, and for each $b \in B$, b belongs to $p\frac{k}{2}$ members in F_1. Similarly, since R has $\frac{k}{2}$ vertices in A and $\frac{k}{2} + 1$ vertices in B, for each $a \in A$, a belongs to $p\frac{k}{2}$ members in F_2, and for each $b \in B$, b belongs to $p(\frac{k}{2} + 1)$ members in F_2. Thus for each $x \in A \cup B$, x belongs to $p(k+1)$ members in F. Hence F is balanced. $\square$

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References

[1] *J.-C. Bermond*: Cycles dans les graphes et G-configurations. Thesis, University of Paris XI (Orsay), Paris, 1975.

[2] *J. Bosák*: Decompositions of Graphs. Kluwer, Dordrecht, Netherlands, 1990.

[3] *C. Huang*: On Handcuffed designs. Dept. of C. and O. Research Report CORR75-10, University of Waterloo.

[4] *S. H. Y. Hung and N. S. Mendelsohn*: Handcuffed designs. Discrete Math. *18* (1977), 23–33.

[5] *T.-W. Shyu*: Path decompositions of $\lambda K_{n,n}$. Ars Comb. *85* (2007), 211–219.

[6] *M.-L. Yu*: On path factorizations of complete multipartite graphs. Discrete Math. *122* (1993), 325–333.

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