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Czechoslovak Mathematical Journal, Vol. 59 (2009), No. 2, 453–479

Persistent URL: http://dml.cz/dmlcz/140491

Czechoslovak Mathematical Journal, 59 (134) (2009), 453–479

THE STRUCTURE OF IDEMPOTENT RESIDUATED CHAINS

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(Received September 2, 2007)

Abstract. In this paper we study some special residuated lattices, namely, idempotent residuated chains. After giving some properties of Green's relation $\mathcal{D}$ on the monoid reduct of an idempotent residuated chain, we establish a structure theorem for idempotent residuated chains. As an application, we give necessary and sufficient conditions for a band with an identity to be the monoid reduct of some idempotent residuated chain. Finally, based on the structure theorem for idempotent residuated chains, we obtain some characterizations of subdirectly irreducible, simple and strictly simple idempotent residuated chains.

Keywords: idempotent residuated lattice, chain, band

MSC 2010: 06F05, 20M10

1. Introduction

A *residuated lattice*, or a *residuated lattice-ordered monoid*, is an algebra $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /, e)$ such that $(L, \wedge, \vee)$ is a lattice, $(L, \cdot, e)$ is a monoid and multiplication is residuated with respect to the order by the division operations $\backslash, /$; i.e., for all $a, b, c \in L$,

$$a \cdot b \leqslant c \Leftrightarrow a \leqslant c/b \Leftrightarrow b \leqslant a\backslash c.$$

The last condition is equivalent to the fact that $(L, \wedge, \vee, \cdot, e)$ is a lattice-ordered monoid and for all $a, b \in L$, the sets $\{c \in L\colon\ ac \leqslant b\}$ and $\{c \in L\colon\ ca \leqslant b\}$ both contain largest elements.

In the late 1930's, M. Ward and R. F. Dilworth introduced the concept of residuated lattices in a more restrictive form which is a natural generalization of the notion of the ideal lattices of rings with identity. Since that time, there has been substantial research regarding some specific classes of residuated structures (see [1]–[5]).

This work is supported by a grant of NSF, China # 10471112 and a grant of Shaanxi Provincial Natural Science Foundation # 2005A15.

It is easy to see that the class of all residuated lattices denoted by $\mathcal{RL}$, is a variety. By an *idempotent residuated lattice* we mean a residuated lattice satisfying the additional identity $x^2 \approx x$. Idempotent residuated lattices are regarded as a common generalization of Brouwerian algebras and generalized Boolean algebras. The class of all *idempotent residuated lattices*, denoted by $\mathcal{IdRL}$, is a subvariety of the variety of all residuated lattices. N. Galatos studied minimal subvarieties of $\mathcal{IdRL}$ in [2]. He constructed uncountably many atoms in the subvariety lattice of $\mathcal{IdRL}$ by making use of a class of strictly simple idempotent residuated chains. By an *idempotent residuated chain* we mean an idempotent residuated lattice for which its lattice reduct is a chain. D. Stanovský in [1] studied *commutative idempotent residuated chains*, that is, idempotent residuated chains satisfying the additional identity $xy \approx yx$. He gave some characterizations of commutative idempotent residuated chains.

In the present paper we will investigate idempotent residuated chains. In order to obtain a structure theorem for idempotent residuated chains, we recall some properties of Green's relations on a band and the decomposition theorem for a band in Section 2. Also, some concepts and known results in the theory of residuated lattices are recalled. In Section 3, we give some basic properties of Green's relation $\mathcal{D}$ on the monoid reducts of an idempotent residuated chain. By making use of the above results, we describe a structure theorem for idempotent residuated chains in terms of a chain and a family of sets which satisfy some simple compatibility conditions in Section 4. As an application, we give necessary and sufficient conditions for a band with an identity to be the monoid reduct of some idempotent residuated chain in Section 5. Finally, based on the structure theorem for idempotent residuated chains, we obtain some characterizations of subdirectly irreducible, simple and strictly simple idempotent residuated chains in Section 6. In particular, we prove that there are precisely six distinct classes of strictly simple idempotent residuated chains.

2. Preliminaries

In what follows, we shall use the notion and notation from [3], [7]. Other undefined terms can be found in [2], [6]. We first recall some of the basic facts concerning Green's relations on the semigroup S. We refer to S^1 as the monoid obtained from S by adjoining an identity if necessary. Green's relations are the equivalence relations defined by

$$\mathcal{L} = \{(a,b) \in S \times S\colon\ S^1a = S^1b\}, \quad \mathcal{R} = \{(a,b) \in S \times S\colon\ aS^1 = bS^1\},$$
$$\mathcal{J} = \{(a,b) \in S \times S\colon\ S^1aS^1 = S^1bS^1\}, \quad \mathcal{H} = \mathcal{L} \cap \mathcal{R}, \mathcal{D} = \langle \mathcal{L} \cup \mathcal{R} \rangle,$$

where the final expression denotes the equivalence relation generated by $\mathcal{L}$ and $\mathcal{R}$. Evidently, $\mathcal{L}$ is a right congruence while $\mathcal{R}$ is a left congruence. Another property of Green's relations is as follows:

Lemma 2.1 [7]. *The relations $\mathcal{L}$ and $\mathcal{R}$ commute and $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$.*

A *band* is a semigroup in which each element is an idempotent; a *semilattice* is a commutative band. A *rectangular band* is a band S in which $xyz = xz$ for all $x, y, z \in S$. The following results are well known and can be proved by Proposition 2.1.4 and Theorem 4.1.3 of [7]. On any band S, Green's relations $\mathcal{D}$ and $\mathcal{J}$ coincide and constitute a congruence such that $S/\mathcal{D}$ is a semilattice. Let $\mathcal{D}^\natural \colon S \to S/\mathcal{D}$ be the natural homomorphism. We denote the semilattice $S/\mathcal{D}$ by Y, and for each α in Y we denote $\alpha(\mathcal{D}^\natural)^{-1}$ by D_α. Each D_α is both a $\mathcal{D}$-class of S and a rectangular band. Hence S is the disjoint union of the rectangular bands D_α $(\alpha \in Y)$, and the congruence property of $\mathcal{D}$ gives that $D_\alpha D_\beta = \{ab \colon a \in D_\alpha, b \in D_\beta\} \subseteq D_{\alpha\beta}$. We say that S is a semilattice of rectangular bands. Thus we have

Theorem 2.2 [7]. *Every band is a semilattice of rectangular bands.*

We recall some of the basic facts about $\mathcal{RL}$. Note that in a residuated lattice, the division operations are determined by multiplication and the order; in particular, $a\backslash b = \max\{c \in L \colon ac \leqslant b\}$ and $b/a = \max\{c \in L \colon ca \leqslant b\}$.

In a residuated lattice term, we assume that multiplication has priority over the division operations, which, in turn, have priority over the lattice operations. So, for example, we write $x/yz \wedge u\backslash v$ for $[x/(yz)] \wedge (u\backslash v)$.

Lemma 2.3 [5]. *Residuated lattices satisfy the following identities:*
(1) $x(y \vee z) \approx xy \vee xz$ *and* $(y \vee z)x \approx yx \vee zx$,
(2) $x\backslash(y \wedge z) \approx (x\backslash y) \wedge (x\backslash z)$ *and* $(y \wedge z)/x \approx (y/x) \wedge (z/x)$,
(3) $x/(y \vee z) \approx (x/y) \wedge (x/z)$ *and* $(y \vee z)\backslash x \approx (y\backslash x) \wedge (z\backslash x)$,
(4) $(x/y)y \leqslant x$ *and* $y(y\backslash x) \leqslant x$,
(5) $x(y/z) \leqslant xy/z$ *and* $(z\backslash y)x \leqslant z\backslash yx$,
(6) $(x/y)/z \approx x/zy$ *and* $z\backslash(y\backslash x) \approx yz\backslash x$,
(7) $x\backslash(y/z) \approx (x\backslash y)/z$,
(8) $x/e \approx x \approx e\backslash x$,
(9) $e \leqslant x/x$ *and* $e \leqslant x\backslash x$.
Moreover, if a residuated lattice has a least element $\perp$, then it has a greatest element $\top$ as well, and $\top = \perp/\perp = \perp\backslash\perp$.

For each element a in a residuated lattice $\mathbf{L}$ we define two unary polynomials $\varrho_a(x) = ax/a \wedge e$ and $\lambda_a(x) = a\backslash xa \wedge e$, the *right* and *left conjugate* of x by a. A subset S of a residuated lattice $\mathbf{L}$ is called *normal* if $\varrho_u(x), \lambda_u(x) \in S$ for all $u \in L$ and all $x \in S$. The closed interval $\{u \in L\colon\ x \leqslant u \leqslant y\}$ is denoted by $[x, y]$. As for posets, we call S *convex* if $[x, y] \subseteq S$ for all $x, y \in S$. We define $\kappa_u(x, y) = (u \wedge x) \vee y$. Note that for a sublattice S the property of being convex is equivalent to $\kappa_u(x, y) \in S$ for all $u \in L$ and $x, y \in S$. Thus a convex normal subalgebra is precisely a subalgebra of $\mathbf{L}$ that is closed under λ, ϱ and κ.

Theorem 2.4 [3], [5]**.** *The convex normal subalgebras of a residuated lattice* $\mathbf{L}$ *form a lattice,* $\mathbf{CN}(\mathbf{L})$*, which is isomorphic to the congruence lattice,* $\mathbf{CON}(\mathbf{L})$*, of* $\mathbf{L}$ *via* $S \mapsto \theta_S = \{(a, b) \in L^2 | (a/b \wedge e)(b/a \wedge e) \in S\}$ *and* $\theta \mapsto [e]_\theta$*, the* θ*-class of* e*.*

For a subset S of a residuated lattice $\mathbf{L}$ let $cn(S)$ denote the intersection of all convex normal subalgebras containing S. When $S = \{s\}$, we write $cn(s)$ rather than $cn(\{s\})$. Let

$$\begin{aligned}
\Delta(S) &= \{s \wedge e/s \wedge e\colon\ s \in S\},\\
\Gamma(S) &= \{\lambda_{u_1} \circ \varrho_{u_2} \circ \lambda_{u_3} \circ \cdots \circ \varrho_{u_{2n}}(s)\colon\ n \in \omega, u_i \in L, s \in S\}\\
\Pi(S) &= \{s_1 \cdot s_2 \cdots s_n\colon\ n \in \omega, s_i \in S\} \cup \{e\}.
\end{aligned}$$

Theorem 2.5 [3], [5]**.** *The convex normal subalgebra generated by a subset* S *in a residuated lattice* $\mathbf{L}$ *is*
$cn(S) = \{a \in L\colon\ x \leqslant a \leqslant x\backslash e$ *for some* $x \in \Pi\Gamma\Delta(S)\}$.

Lemma 2.6 [1]**.** *Let* $\mathbf{L}$ *be a lattice-ordered idempotent monoid and* $a, b \in L$*. Then the following statements are true:*
(1) $a \wedge b \leqslant ab \leqslant a \vee b$;
(2) *if* $a, b \geqslant e$*, then* $ab = a \vee b$;
(3) *if* $a, b \leqslant e$*, then* $ab = a \wedge b$;
(4) *if* $a \leqslant e \leqslant ab$*, then* $ab = b$;
(5) *if* $ab \leqslant e \leqslant a$*, then* $ab = b$.

3. Green's relation $\mathcal{D}$ on the monoid reduct of an idempotent residuated chain

It is well-known that Green's relations play a crucial role in the study of any semigroups. It is natural to find the important properties of Green's relation $\mathcal{D}$ on the monoid reduct of an idempotent residuated chain. This is the main aim of this section. For brevity, throughout this section we always assume that $\mathbf{L} =$

$(L, \wedge, \vee, \cdot, \backslash, /, e)$ is an idempotent residuated chain. We denote the $\mathcal{D}$-class containing the element a by D_a. To begin with, we verify the following fact which is the crucial observation enabling the study to get under way.

Proposition 3.1.

(1) *For all $a \in L$, D_a contains at most two elements. Moreover, if $b \in D_a$ and $b \neq a$, then either $a > e$, $b < e$ or $a < e$, $b > e$.*

(2) *For all $a \in L$, $(D_a, \cdot)$ is either a left zero semigroup or a right zero semigroup.*

P r o o f. (1) Let $a, b \in L$ and $a \mathcal{D} b$. Since $(L, \cdot)$ is a semilattice of rectangular bands, by Theorem 2.2, $(D_a, \cdot)$ is a rectangular band. If $a, b \leqslant e$, then by Lemma 2.6 (3), $a = aba = \min\{a, b\} = bab = b$. Similarly, if $a, b \geqslant e$, then $a = b$. Hence for $a \neq b$, either $a > e$, $b < e$ or $a < e$, $b > e$. Thus, for any $a \in L$, D_a contains at most two elements.

(2) Let $a \in L$ and $b \in D_a$. Then by Lemma 2.1 there exists $c \in L$ such that $a \mathcal{L} c \mathcal{R} a$ and so $c \in D_a$. By (1), $|D_a| \leqslant 2$, it follows that either $c = a$ or $c = b$ and so $b \mathcal{L} a$ or $b \mathcal{R} a$. Thus $(D_a, \cdot)$ is either a left zero semigroup or a right zero semigroup. $\square$

On any band S the *natural ordering* $\leqslant_n$ is given by for all $a, b \in S$, $a \leqslant_n b$ if and only if $ab = ba = a$. By Theorem 2.2 $(L, \cdot)$ is a semilattice of rectangular bands. Hence $(L/\mathcal{D}, \cdot)$ is a semilattice. So we have the natural ordering on $(L/\mathcal{D}, \cdot)$ by defining for all $a, b \in L$, $D_a \leqslant^* D_b$ if and only if $D_a \cdot D_b = D_a$.

In the following we write $x \prec y (x \prec^* y)$ for the fact that x is covered by y; i.e., $x < y (x <^* y)$ and for every z, if $x \leqslant z \leqslant y (x \leqslant^* z \leqslant^* y)$, then $z = x$ or $z = y$. We now describe the relationship between the imposed ordering $\leqslant$, the natural ordering $\leqslant_n$ on L and the natural ordering $\leqslant^*$ on $L/\mathcal{D}$.

Proposition 3.2.

(1) *$(L/\mathcal{D}, \leqslant^*)$ is a chain with the greatest element D_e.*

(2) *If $a, b \in L$ such that $a \leqslant e$ and $b \leqslant e$, then $a \leqslant b$ if and only if $a \leqslant_n b$ if and only if $D_a \leqslant^* D_b$.*

(3) *If $a, b \in L$ such that $a \geqslant e$ and $b \geqslant e$, then $a \leqslant b$ if and only if $b \leqslant_n a$ if and only if $D_b \leqslant^* D_a$.*

(4) *If $a, b \in L$ such that $a \leqslant e$ and $b \geqslant e$, then $a <_n b$ if and only if $D_a <^* D_b$.*

(5) *If $a, b \in L$ such that $a \leqslant e$ and $b \geqslant e$, then $b <_n a$ if and only if $D_b <^* D_a$.*

(6) *If $a, b \in L$ such that $a < e$ and $b > e$, then a and b are non-comparable for $\leqslant_n$ if and only if $D_a = D_b$.*

Proof. (1) It is easy to see that D_e is the greatest element in $(L/\mathcal{D}, \leqslant^*)$. By Lemma 2.6, for all $a, b \in L$ we have $ab = a$ or $ab = b$. This implies that $D_a \cdot D_b = D_a$ or $D_a \cdot D_b = D_b$. Hence $D_a \leqslant^* D_b$ or $D_b \leqslant^* D_a$. This proves (1).

(2) Let $a, b \in L$ be such that $a \leqslant e$ and $b \leqslant e$. If $a \leqslant b$, then $a = ab = ba = a \wedge b$ by Lemma 2.6(3), and so $a \leqslant_n b$. If $a \leqslant_n b$, then $a = ab = ba$ and so $D_a = D_{ab} = D_a \cdot D_b$. Hence $D_a \leqslant^* D_b$. If $D_a \leqslant^* D_b$, then $D_a = D_{ab} = D_a \cdot D_b$, and so $ab \in D_a$. By Lemma 2.6(3) $ab = a \wedge b \leqslant e$, and so by Proposition 3.1(1) we have $ab = a$. Hence $a \leqslant b$.

(3) is established similarly.

(4) Let $a, b \in L$ be such that $a \leqslant e$ and $b \geqslant e$. If $a <_n b$, then $a = ab = ba$ and so $D_a = D_{ab} = D_a \cdot D_b$. Hence $D_a \leqslant^* D_b$. Suppose that $D_a = D_b$. Then a and b are non-comparable for $\leqslant_n$. It is a contradiction. Thus $D_a <^* D_b$. Conversely, if $D_a <^* D_b$, then $D_a = D_{ab} = D_a \cdot D_b$, and so $ab \in D_a$. Suppose that $ab \geqslant e$. Then by Lemma 2.6(4), $ab = b$. It is a contradiction. Hence $ab \leqslant e$ and so by Lemma 2.6(5), $ab = a$. Similarly, $ba = a$. Thus $a <_n b$.

(5): is similar to (4).

(6) Let $a, b \in L$ be such that $a < e$ and $b > e$. Suppose that a and b are non-comparable for $\leqslant_n$. If $D_a \neq D_b$, then (1) implies $D_a <^* D_b$ or $D_b <^* D_a$. So by (4) and (5) we have $a <_n b$ or $b <_n a$. It is a contradiction. Hence $D_a = D_b$. Conversely, let $D_a = D_b$. Then either $a = ab, b = ba$ or $a = ba, b = ab$. Hence a and b are non-comparable for $\leqslant_n$. □

Example 3.3. Let $L = \{b_1, b_2, b_5\} \cup \{a_3, a_4, a_5\} \cup \{e\}$. We define an order relation on L by $b_1 < b_2 < b_5 < e < a_5 < a_4 < a_3$; see Figure 1(1). Obviously, this is a total order on L. We also define a multiplication by

$$a_i \circ a_j = a_{\min\{i,j\}}, \quad b_i \circ b_j = b_{\min\{i,j\}},$$
$$a_i \circ b_j = \begin{cases} a_i & \text{if } i < j \text{ or } i = j = 5, \\ b_j & \text{if } j < i, \end{cases}$$

and

$$b_j \circ a_i = \begin{cases} a_i & \text{if } i < j, \\ b_j & \text{if } j < i \text{ or } i = j = 5. \end{cases}$$

Finally, we define two division operations on L, by $x/y = \max\{z|zy \leqslant x\}$ and $y\backslash x = \max\{z|yz \leqslant x\}$; note that maximum elements exist. Hence **L** is an idempotent residuated chain. Let $D_1 = \{b_1\}$, $D_2 = \{b_2\}$, $D_3 = \{a_3\}$, $D_4 = \{a_4\}$, $D_5 = \{b_5, a_5\}$, $D_e = \{e\}$. Then $L/\mathcal{D} = \{D_1, D_2, D_3, D_4, D_5, D_e\}$ and the natural ordering $\leqslant^*$ on $L/\mathcal{D}$ is as follows: $D_1 <^* D_2 <^* D_3 <^* D_4 <^* D_5 <^* D_e$; see Figure 1(2). The natural ordering $\leqslant_n$ on L is as follows: $b_1 <_n b_2 <_n a_3 <_n a_4 <_n b_5, a_5 <_n e$; see

Figure 1(3). By Proposition 3.2, we can obtain the imposed ordering $\leqslant$ on L from the natural ordering $\leqslant_n$ on L; see Figure 1(4).

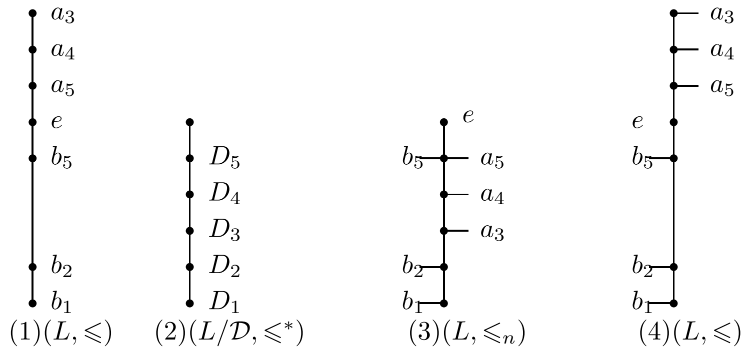


Figure 1

The next proposition describes some properties of the set of equivalence classes of $\mathcal{D}$ on L which play a crucial role in this paper.

Proposition 3.4.

(1) *If $a \in L$ and $a > e$, then there exists a unique element $b < e$ in L with $D_b <^* D_a$, satisfying the following condition:*

(RC) *if $D \in (L/\mathcal{D}, \leqslant^*)$ such that $D_b <^* D <^* D_a$ and $c \in D$, then $c > e$.*

(2) *If $b \in L$ and $b < e$, then there exists a unique element $a \geqslant e$ in L with $D_b <^* D_a$ satisfying the following condition:*

(RC′) *if $D \in (L/\mathcal{D}, \leqslant^*)$ such that $D_b <^* D <^* D_a$ and $c \in D$, then $c < e$.*

Proof. (1) Let $a \in L$ and $a > e$.

- Suppose that $(D_a, \cdot)$ is a left zero semigroup. Let $b = a\backslash e$. Then by Lemma 2.3(4), $ab \leqslant e$. By Lemma 2.6, $ab = b \leqslant e$. Hence $D_b \leqslant^* D_a$. Since $a \neq b$ and $(D_a, \cdot)$ is a left zero semigroup, we have $D_b <^* D_a$. If $D_b \prec^* D_a$, then b satisfies condition (RC). Assume that D_b is not covered by D_a. Suppose that there exists $D \in (L/\mathcal{D}, \leqslant^*)$ with $D_b <^* D <^* D_a$ and $c \in D$ such that $c < e$. By Lemma 2.6, $b = bc < c$. But $ac = c < e$. It is a contradiction. Hence b satisfies condition (RC).
- Assume that $(D_a, \cdot)$ is a right zero semigroup. Let $b = e/a$. Then by Lemma 2.3(4), $ba \leqslant e$. By Lemma 2.6, $ba = b$. Hence $D_b <^* D_a$. By a similar argument to the above, b satisfies condition (RC).

Up to now, we have proved the existence of b. The uniqueness of b is trivial.

(2): is similar to (1). $\square$

4. Construction

Let $(Y, \leqslant^*)$ be a chain with greatest element 1 and X be a subset of Y. Let $\mathscr{D} = \{D_\alpha\colon \alpha \in Y\}$ be a family of pairwise disjoint sets indexed by Y. Let partial mappings

$$\varphi\colon Y \to Y; \quad \alpha \mapsto \varphi(\alpha)$$

and

$$\phi\colon Y \to Y; \quad \alpha \mapsto \phi(\alpha)$$

be given. We call $(Y, X; \mathscr{D}; \varphi, \phi)$ an YD-system if the following conditions hold:

(YD1) for each $\alpha \in Y \setminus \{1\}$, D_α is either the set $\{a_\alpha, b_\alpha\}$ or one of $\{a_\alpha\}$ and $\{b_\alpha\}$, where $a_\alpha \neq b_\alpha$;

(YD1′) for $1 \in Y$, we have $D_1 = \{e\}$, where $e = a_1 = b_1$;

(YD2) $\operatorname{Dom}\varphi = \{\alpha \in Y \setminus \{1\}\colon a_\alpha \in D_\alpha\}$;

(YD2′) $\operatorname{Dom}\phi = \{\alpha \in Y \setminus \{1\}\colon b_\alpha \in D_\alpha\}$;

(YD3) if $\alpha \in \operatorname{Dom}\varphi$, then $\varphi(\alpha) <^* \alpha$ and $D_{\varphi(\alpha)}$ contains $b_{\varphi(\alpha)}$;

(YD3′) if $\alpha \in \operatorname{Dom}\phi$, then $\alpha <^* \phi(\alpha)$ and $D_{\phi(\alpha)}$ contains $a_{\phi(\alpha)}$;

(YD4) if $\beta \in Y$ such that $\varphi(\alpha) <^* \beta <^* \alpha$, then D_β contains exactly a_β;

(YD4′) if $\beta \in Y$ such that $\alpha <^* \beta <^* \phi(\alpha)$, then D_β contains exactly b_β.

Given an YD-system $(Y, X; \mathscr{D}; \varphi, \phi)$, put $L = \bigcup\limits_{\alpha \in Y} D_\alpha$. Define a binary relation $\leqslant$ on the set L as follows.

Let $a \in D_\alpha$, $b \in D_\beta$. $a \leqslant b$ in L if one of the following conditions is satisfied:

(P1) $\alpha \leqslant^* \beta$ in Y and $a = b_\alpha$;

(P2) $\beta \leqslant^* \alpha$ in Y and $b = a_\beta$.

Lemma 4.1. *$(L, \leqslant)$ is a chain.*

P r o o f. We first prove that the binary relation $\leqslant$ is a partial order on L. Obviously, $\leqslant$ is reflexive. Let $a \in D_\alpha$, $b \in D_\beta$ be such that $a \leqslant b$ and $b \leqslant a$. We consider four cases:

(1) If $\alpha \leqslant^* \beta$, $a = b_\alpha$ and $\beta \leqslant^* \alpha$, $b = b_\beta$, then $\alpha = \beta$ and so $a = b$.

(2) If $\alpha \leqslant^* \beta$, $a = b_\alpha$ and $\alpha \leqslant^* \beta$, $a = a_\alpha$, then $\alpha = \beta = 1$ and $a = b = e$.

(3) If $\beta \leqslant^* \alpha$, $b = a_\beta$ and $\beta \leqslant^* \alpha$, $b = b_\beta$, then $\alpha = \beta = 1$ and $a = b = e$.

(4) If $\beta \leqslant^* \alpha$, $b = a_\beta$ and $\alpha \leqslant^* \beta$, $a = a_\alpha$, then $\alpha = \beta$ and so $a = b$.

Hence $\leqslant$ is antisymmetric. Let $a \in D_\alpha$, $b \in D_\beta$ and $c \in D_\gamma$ be such that $a \leqslant b$ and $b \leqslant c$. We consider four cases:

(1) If $\alpha \leqslant^* \beta$, $a = b_\alpha$ and $\beta \leqslant^* \gamma$, $b = b_\beta$, then $\alpha \leqslant^* \gamma$ and so $a \leqslant c$.

(2) If $\alpha \leqslant^* \beta$, $a = b_\alpha$ and $\gamma \leqslant^* \beta$, $c = a_\gamma$, then $a \leqslant c$, since $(Y, \leqslant^*)$ is a chain.

(3) If $\beta \leqslant^* \alpha$, $b = a_\beta$ and $\beta \leqslant^* \gamma$, $b = b_\beta$, then $\alpha = \beta = \gamma = 1$ and so $a = b = c = e$.

(4) If $\beta \leqslant^* \alpha$, $b = a_\beta$ and $\gamma \leqslant^* \beta$, $c = a_\gamma$, then $\gamma \leqslant^* \alpha$ and so $a \leqslant c$.

We have proved that $\leqslant$ is transitive. Therefore $\leqslant$ is a partial order on L. It is clear that $(L, \leqslant)$ is a chain. $\square$

We define a multiplication $\circ$ on the ordered set $(L, \leqslant)$ as follows:

$$a_\alpha \circ a_\beta = a_{\min\{\alpha,\beta\}}, \quad b_\alpha \circ b_\beta = b_{\min\{\alpha,\beta\}},$$

$$a_\alpha \circ b_\beta = \begin{cases} a_\alpha & \text{if } \alpha <^* \beta \text{ or } \alpha = \beta \in X, \\ b_\beta & \text{if } \beta <^* \alpha \text{ or } \alpha = \beta \notin X, \end{cases}$$

and

$$b_\beta \circ a_\alpha = \begin{cases} a_\alpha & \text{if } \alpha <^* \beta \text{ or } \alpha = \beta \notin X, \\ b_\beta & \text{if } \beta <^* \alpha \text{ or } \alpha = \beta \in X. \end{cases}$$

Lemma 4.2. *$(L, \leqslant, \circ)$ is a partially ordered band with an identity e.*

Proof. It is easy to see that $(L, \circ)$ is a band with an identity e. Let $a \in D_\alpha$, $b \in D_\beta$, $c \in D_\gamma$ and $a \leqslant b$ in L. Suppose that $\alpha \leqslant^* \beta$ and $a = b_\alpha$. We need to consider four cases:

(a) If $\alpha <^* \beta <^* \gamma$ or $\alpha <^* \gamma \leqslant^* \beta$, then by the definition of $\leqslant$, $a \leqslant c$. By the definition of multiplication, $a \circ c = c \circ a = a$, $b \circ c \in \{b, c\}$ and $c \circ b \in \{b, c\}$. Hence $a \circ c \leqslant b \circ c$ and $c \circ a \leqslant c \circ b$.

(b) $\alpha = \gamma \leqslant^* \beta$.

- If $\alpha \in X$, then $a \circ c = a = b_\alpha$, $c \circ a = c$ and $c \circ b = c$. It follows that $a \circ c \leqslant b \circ c$ and $c \circ a \leqslant c \circ b$.
- If $\alpha \notin X$, then $a \circ c = c$, $b \circ c = c$ and $c \circ a = a = b_\alpha$. Hence $a \circ c \leqslant b \circ c$ and $c \circ a \leqslant c \circ b$.

(c) If $\alpha = \beta <^* \gamma$, then $a \circ c = a \leqslant b = b \circ c$ and $c \circ a = a \leqslant b = c \circ b$.

(d) If $\gamma <^* \alpha \leqslant^* \beta$, then $a \circ c = b \circ c = c$ and $c \circ a = c \circ b = c$.

Similarly, if $\beta \leqslant^* \alpha$ and $b = a_\beta$, then $a \circ c \leqslant b \circ c$ and $c \circ a \leqslant c \circ b$. Thus $(L, \leqslant, \circ)$ is a partially ordered band with an identity e. $\square$

We define two division operations $\backslash$ and $/$ on L as follows: for $\alpha \in Y \setminus \{1\}$ and $\beta \in Y$, let

$$b_\alpha \backslash b_\beta = \begin{cases} a_{\phi(\alpha)} & \text{if } \alpha \leqslant^* \beta \text{ and } \alpha \notin X, \\ a_\alpha & \text{if } \alpha \leqslant^* \beta\ ,\ \alpha \in X \text{ and } D_\alpha \text{ contains } a_\alpha, \\ a_{\phi(\alpha)} & \text{if } \alpha \leqslant^* \beta\ ,\ \alpha \in X \text{ and } D_\alpha \text{ does not contain } a_\alpha, \\ b_\beta & \text{if } \beta <^* \alpha, \end{cases}$$

$$a_\alpha \backslash b_\beta = \begin{cases} b_{\varphi(\alpha)} & \text{if } \alpha \leqslant^* \beta \text{ and } \alpha \in X, \\ b_\alpha & \text{if } \alpha \leqslant^* \beta\ ,\ \alpha \notin X \text{ and } D_\alpha \text{ contains } b_\alpha, \\ b_{\varphi(\alpha)} & \text{if } \alpha <^* \beta\ ,\ \alpha \notin X \text{ and } D_\alpha \text{ does not contain } b_\alpha, \\ b_\beta & \text{if } \beta <^* \alpha, \end{cases}$$

$$b_\alpha \backslash a_\beta = \begin{cases} a_{\phi(\alpha)} & \text{if } \alpha <^* \beta \text{ and } \alpha \notin X, \\ a_\alpha & \text{if } \alpha <^* \beta\ ,\ \alpha \in X \text{ and } D_\alpha \text{ contains } a_\alpha, \\ a_{\phi(\alpha)} & \text{if } \alpha <^* \beta\ ,\ \alpha \in X \text{ and } D_\alpha \text{ does not contain } a_\alpha, \\ a_\beta & \text{if } \beta \leqslant^* \alpha, \end{cases}$$

$$a_\alpha \backslash a_\beta = \begin{cases} b_{\varphi(\alpha)} & \text{if } \alpha <^* \beta \text{ and } \alpha \in X, \\ b_\alpha & \text{if } \alpha <^* \beta\ ,\ \alpha \notin X \text{ and } D_\alpha \text{ contains } b_\alpha, \\ b_{\varphi(\alpha)} & \text{if } \alpha <^* \beta\ ,\ \alpha \notin X \text{ and } D_\alpha \text{ does not contain } b_\alpha, \\ a_\beta & \text{if } \beta \leqslant^* \alpha, \end{cases}$$

$$b_\beta / b_\alpha = \begin{cases} a_{\phi(\alpha)} & \text{if } \alpha \leqslant^* \beta \text{ and } \alpha \in X, \\ a_\alpha & \text{if } \alpha \leqslant^* \beta\ ,\ \alpha \notin X \text{ and } D_\alpha \text{ contains } a_\alpha, \\ a_{\phi(\alpha)} & \text{if } \alpha \leqslant^* \beta\ ,\ \alpha \notin X \text{ and } D_\alpha \text{ does not contain } a_\alpha, \\ b_\beta & \text{if } \beta <^* \alpha, \end{cases}$$

$$b_\beta / a_\alpha = \begin{cases} b_{\varphi(\alpha)} & \text{if } \alpha \leqslant^* \beta \text{ and } \alpha \notin X, \\ b_\alpha & \text{if } \alpha \leqslant^* \beta\ ,\ \alpha \in X \text{ and } D_\alpha \text{ contains } b_\alpha, \\ b_{\varphi(\alpha)} & \text{if } \alpha <^* \beta\ ,\ \alpha \in X \text{ and } D_\alpha \text{ does not contain } b_\alpha, \\ b_\beta & \text{if } \beta <^* \alpha, \end{cases}$$

$$a_\beta / b_\alpha = \begin{cases} a_{\phi(\alpha)} & \text{if } \alpha <^* \beta \text{ and } \alpha \in X, \\ a_\alpha & \text{if } \alpha <^* \beta\ ,\ \alpha \notin X \text{ and } D_\alpha \text{ contains } a_\alpha, \\ a_{\phi(\alpha)} & \text{if } \alpha <^* \beta\ ,\ \alpha \notin X \text{ and } D_\alpha \text{ does not contain } a_\alpha, \\ a_\beta & \text{if } \beta \leqslant^* \alpha, \end{cases}$$

$$a_\beta / a_\alpha = \begin{cases} b_{\varphi(\alpha)} & \text{if } \alpha <^* \beta \text{ and } \alpha \notin X, \\ b_\alpha & \text{if } \alpha <^* \beta\ ,\ \alpha \in X \text{ and } D_\alpha \text{ contains } b_\alpha, \\ b_{\varphi(\alpha)} & \text{if } \alpha <^* \beta\ ,\ \alpha \in X \text{ and } D_\alpha \text{ does not contain } b_\alpha, \\ a_\beta & \text{if } \beta \leqslant^* \alpha, \end{cases}$$

for $c \in L$, let

$$e \backslash c = c \qquad \text{and} \qquad c/e = c.$$

We denote by $IRLC(Y, X; \mathscr{D}; \varphi, \phi)$ the above $(L, \wedge, \vee, \circ, \backslash, /, e)$.

Theorem 4.3. $\mathbf{L} = IRLC(Y, X; \mathscr{D}; \varphi, \phi)$ *is an idempotent residuated chain.*

Proof. We need only to prove that for all $a, b \in L$, $a/b = \max\{c\colon\ c \circ b \leqslant a\}$ and $b\backslash a = \max\{c\colon\ b \circ c \leqslant a\}$. Let $a = b_\beta \in L, b = b_\alpha \in L \setminus \{e\}$. To show that $b\backslash a = \max\{c\colon\ b \circ c \leqslant a\}$, we need to consider the following cases:

(1) If $\alpha \leqslant^* \beta$ and $\alpha \notin X$ or $\alpha \leqslant^* \beta, \alpha \in X$ and D_α does not contain a_α, then $b_\alpha \backslash b_\beta = a_{\phi(\alpha)}$. Let $c \in D_\gamma$ be such that $b_\alpha \circ c \leqslant b_\beta$. If $c = b_\gamma$, then $c = b_\gamma < a_{\phi(\alpha)}$. If $c = a_\gamma$, then $\alpha <^* \gamma$. Hence by (YD4$'$), $\phi(\alpha) \leqslant^* \gamma$ and so $c = a_\gamma \leqslant a_{\phi(\alpha)}$.

(2) If $\alpha \leqslant^* \beta, \alpha \in X$ and D_α contains a_α, then $b_\alpha \backslash b_\beta = a_\alpha$. Let $c \in D_\gamma$ be such that $b_\alpha \circ c \leqslant b_\beta$. If $c = b_\gamma$, then $c \leqslant a_\alpha$. If $c = a_\gamma$, then $\alpha \leqslant^* \gamma$ and so $c = a_\gamma \leqslant a_\alpha$.

(3) If $\beta <^* \alpha$, then $b_\alpha \backslash b_\beta = b_\beta$. Let $c \in D_\gamma$ be such that $b_\alpha \circ c \leqslant b_\beta$. If $c = a_\gamma$, then $b_\alpha \circ a_\gamma = \{b_\alpha, a_\gamma\}$. But $b_\beta < b_\alpha$ and $b_\beta < a_\gamma$. It is a contradiction. Hence $c = b_\gamma$. So $b_\alpha \circ b_\gamma = b_{\alpha \wedge \gamma} \leqslant b_\beta$. It follows that $\alpha \wedge \gamma \leqslant^* \beta$. Since $\beta <^* \alpha$, $\gamma = \alpha \wedge \gamma \leqslant^* \beta$. Thus $c = b_\gamma \leqslant b_\beta$.

Similarly we can show that for all $a, b \in L$, $a/b = \max\{c\colon\ c \circ b \leqslant a\}$ and $b\backslash a = \max\{c\colon\ b \circ c \leqslant a\}$. Hence $IRLC(Y, X; \mathscr{D}; \varphi, \phi)$ is an idempotent residuated chain.

□

In the remainder of this section, we prove that any idempotent residuated chain is isomorphic to some $IRLC(Y, X; \mathscr{D}; \varphi, \phi)$. For convenience, in what follows, let $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /, e)$ be an idempotent residuated chain. By Proposition 3.2(1), $(L/\mathcal{D}, \leqslant^*)$ is a chain with greatest element D_e. Let $X = \{D_a \in L/\mathcal{D}\colon\ (D_a, \cdot)$ is a left zero semigroup$\}$. Put $\mathscr{D} = \{D_a\colon\ D_a \in L/\mathcal{D}\}$. By Proposition 3.4(1)and (2), we can define two partial mappings φ and ϕ as follows:

$$\varphi\colon\ L/\mathcal{D} \to L/\mathcal{D};\quad D_a \mapsto \max\{D_b \in L/\mathcal{D} | b < e \text{ and } D_b <^* D_a\}$$

and

$$\phi\colon\ L/\mathcal{D} \to L/\mathcal{D};\quad D_b \mapsto \min\{D_a \in L/\mathcal{D} | a \geqslant e \text{ and } D_b <^* D_a\},$$

where $\operatorname{Dom}\varphi = \{D_a \in L/\mathcal{D}\colon\ a > e\}$ and $\operatorname{Dom}\phi = \{D_b \in L/\mathcal{D}\colon\ b < e\}$.

Lemma 4.4. $(L/\mathcal{D}, X; \mathscr{D}; \varphi, \phi)$ *is an* YD-*system.*

Proof. For the sake of simplicity, we identify $L/\mathcal{D}$ with Y which contains a greatest element 1, and denote $\mathscr{D} = \{D_\alpha\colon\ \alpha \in Y\}$. By Proposition 3.1(1), for each $\alpha \in Y \setminus \{1\}$ we have $D_\alpha = \{a_\alpha, b_\alpha\}$ or one of $\{a_\alpha\}$ and $\{b_\alpha\}$, where $a_\alpha > e, b_\alpha < e$. Let $x \in D_1$, then $e = exe = x$. Hence $D_1 = \{e\}$, where $e = a_1 = b_1$. This shows (YD1) and (YD1$'$).

By the definitions of φ and ϕ, conditions (YD2–4) and (YD2$'$–4$'$) hold. □

Theorem 4.5. **L** *is equal to* $IRLC(L/\mathcal{D}, X; \mathscr{D}; \varphi, \phi)$.

Proof. For convenience, we denote by $\leqslant_1$ the imposed ordering on $IRLC(L/\mathcal{D}, X; \mathscr{D}; \varphi, \phi)$. We need only to prove that $a \cdot b = a \circ b$ and $\leqslant = \leqslant_1$ for all $a, b \in L$. Let $a, b \in L$. We need to consider four cases:

(1) If $a \leqslant e, b \leqslant e$, then $a \cdot b = a \wedge b$ by Lemma 2.6(3). If $a \leqslant b$, then by Proposition 3.2, $D_a \leqslant^* D_b$ and so by the definition of $\circ$, $a \circ b = a$. Hence $a \cdot b = a \circ b = a$. If $a \geqslant b$, then by Proposition 3.2, $D_b \leqslant^* D_a$. and so by the definition of $\circ$, $a \circ b = b$. Hence $a \cdot b = a \circ b = b$.

(2) If $a \geqslant e, b \geqslant e$, then by a similar argument to (1), $a \cdot b = a \circ b$.

(3) $a \geqslant e$ and $b \leqslant e$.

- If $D_a <^* D_b$ or $D_a = D_b$ and $(D_a \cdot)$ is a left zero semigroup, then $a \circ b = a$ and by Lemma 2.6, $a \cdot b \in \{a, b\}$. Hence $a \cdot b = a$ and so $a \cdot b = a \circ b$.
- If $D_b <^* D_a$ or $D_a = D_b$ and $(D_a \cdot)$ is a right zero semigroup, then $a \circ b = b$ and by Lemma 2.6, $a \cdot b \in \{a, b\}$. Hence $a \cdot b = b$ and so $a \cdot b = a \circ b$.

(4) If $a \leqslant e$ and $b \geqslant e$, then by a similar argument to (3), $a \circ b = a \cdot b$.

Let $a, b \in L$ be such that $a \leqslant b$; we need to consider three cases:

(1) If $a \leqslant e, b \leqslant e$, then by Proposition 3.2(2), $D_a \leqslant^* D_b$ and so by the definition of $\leqslant_1$, $a \leqslant_1 b$.

(2) If $a \geqslant e, b \geqslant e$, then by Proposition 3.2(3), $D_b \leqslant^* D_a$ and so by the definition of $\leqslant_1$, $a \leqslant_1 b$.

(3) If $a \leqslant e$ and $b \geqslant e$, then $a \leqslant_1 b$.

Let $a, b \in L$ be such that $a \leqslant_1 b$. We need to consider two cases:

(1) $D_a \leqslant^* D_b$ and $a \leqslant e$. If $b \leqslant e$, then by Proposition 3.2(2), $a \leqslant b$. If $b \geqslant e$, then $a \leqslant b$.

(2) $D_b \leqslant^* D_a$ and $b \geqslant e$. If $a \geqslant e$, then by Proposition 3.2(3), $a \leqslant b$. If $a \leqslant e$, then $a \leqslant b$.

Hence the orders $\leqslant$ and $\leqslant_1$ coincide. Thus **L** is equal to $IRLC(L/\mathcal{D}, X; \mathscr{D}; \varphi, \phi)$. □

As a conclusion, we have, by Theorems 4.3 and 4.5, the following theorem.

Theorem 4.6. *Let* $(Y, X; \mathscr{D}; \varphi, \phi)$ *be an* YD-*system. Then* $IRLC(Y, X; \mathscr{D}; \varphi, \phi)$ *is an idempotent residuated chain. Conversely, any such residuated lattice can be constructed in this manner.*

5. Application

In this section we use the above theorem to obtain a structure theorem for commutative idempotent residuated chains and establish necessary and sufficient conditions for a band with an identity to be the monoid reduct of some idempotent residuated chain. In addition, we give some characterizations of idempotent residuated chains.

Theorem 5.1. *Let $(Y, X; \mathscr{D}; \varphi, \phi)$ be an* YD*-system. If $\mathscr{D}$ is a family of pairwise disjoint singletons, then $IRLC(Y, X; \mathscr{D}; \varphi, \phi)$ is a commutative idempotent residuated chain. Conversely, any such residuated lattice can be constructed in this manner.*

Proof. We need only to prove that $IRLC(Y, X; \mathscr{D}; \varphi, \phi)$ is commutative. Now let $a, b \in IRLC(Y, X; \mathscr{D}; \varphi, \phi)$. Then there exist $\alpha, \beta \in Y$ such that $a \in D_\alpha$, $b \in D_\beta$. If $\alpha = \beta$, then because for all $\alpha \in Y$, D_α contains exactly one element, it follows that $a = b$. Hence $ab = ba$. If $\alpha <^* \beta$, then $ab = ba = a$. If $\beta <^* \alpha$, then $ab = ba = b$. We have proved that $IRLC(Y, X; \mathscr{D}; \varphi, \phi)$ is commutative. □

Theorem 5.2. *A band* $\mathbf{L}$ *with identity e is the monoid reduct of some idempotent residuated chain if and only if* $\mathbf{L}$ *satisfies the following conditions:*

(TR1) *$(L/\mathcal{D}, \leqslant^*)$ is a chain where the partial order $\leqslant^*$ is the natural ordering on $L/\mathcal{D}$;*

(TR2) *each $\mathcal{D}$-class of L contains at most two elements;*

(TR3) *$ab = ba = a$ for all $a, b \in L$ with $D_a <^* D_b$;*

(TR4) *if $D \in L/\mathcal{D} \setminus \{D_e\}$ and D contains exactly one element, then D satisfies one of the following conditions:*

(D1) *there exists $D^+ \in L/\mathcal{D}$ satisfying*

(i) *$D <^* D^+$;*

(ii) *if $D' \in L/\mathcal{D}$ such that $D <^* D' <^* D^+$, then D' contains exactly one element;*

(iii) *if $D^+ \neq D_e$, then there exists $D'' \in L/\mathcal{D}$ such that $D'' \prec^* D^+$;*

(D2) *there exists $D^* \in L/\mathcal{D}$ satisfying*

(i) *$D^* <^* D$;*

(ii) *if $D' \in L/\mathcal{D}$ such that $D^* <^* D' <^* D$, then D' contains exactly one element;*

(iii) *there exists $D'' \in L/\mathcal{D}$ such that $D^* \prec^* D''$;*

(TR5) *if $D \in L/\mathcal{D} \setminus \{D_e\}$ and D contains two elements, then D satisfies conditions* (D1) *and* $(D2)$.

Proof. In the forward direction this is obvious. Conversely, suppose that L is a band with an identity e and satisfies conditions (TR1–5). For the sake of simplicity, we identify $(L/\mathcal{D}, \leqslant^*)$ with $(Y, \leqslant^*)$ which has a greatest element 1. Now let $L = \bigcup_{\alpha \in Y} D_\alpha$ be the semilattice decomposition of L over rectangular bands D_α for $\alpha \in Y$. Put $\mathscr{D} = \{D_\alpha \colon \alpha \in Y\}$ and $Y_1 = \{\alpha \in Y \colon D_\alpha$ contains two elements$\}$. Let $D_1 = \{e\}$, where $e = a_1 = b_1$. For any $\alpha \in Y_1$, we define $D_\alpha = \{a_\alpha, b_\alpha\}$. Let $\alpha \in Y_1$. We distinguish two cases.

Case I. For all $\beta \in Y$ such that $\alpha <^* \beta$, D_β contains only one element. By condition (TR5), we can choose $\alpha^* \in Y$ and 1 is interpreted as α^+. Hence we obtain the closed interval $H_\alpha = [\alpha^*, 1]$ and there exists an element γ in Y such that $\alpha^* \prec^* \gamma$. For any $\beta \in Y$ such that $\alpha <^* \beta <^* 1$, let $D_\beta = \{b_\beta\}$. For any $\gamma' \in Y$ such that $\alpha^* <^* \gamma' <^* \alpha$, let $D_{\gamma'} = \{a_{\gamma'}\}$. If D_{α^*} contains exactly one element, then we define $D_{\alpha^*} = \{b_{\alpha^*}\}$.

Case II. There exists $\beta \in Y$ such that $\alpha <^* \beta$ and D_β contains two elements. By condition (TR5), we can choose α^* and α^+ in Y. Hence we obtain the closed interval $H_\alpha = [\alpha^*, \alpha^+]$ and there exist element β, γ in Y such that $\beta \prec^* \alpha^+$ and $\alpha^* \prec^* \gamma$. For any $\beta' \in Y$ such that $\alpha <^* \beta' <^* \alpha^+$, let $D_{\beta'} = \{b_{\beta'}\}$. If D_{α^+} contains exactly one element, then we define $D_{\alpha^+} = \{a_{\alpha^+}\}$. For any $\gamma' \in Y$ such that $\alpha^* <^* \gamma' <^* \alpha$, let $D_{\gamma'} = \{a_{\gamma'}\}$. If D_{α^*} contains exactly one element, then we define $D_{\alpha^*} = \{b_{\alpha^*}\}$.

Let $\alpha, \beta \in Y_1$ be such that $\alpha <^* \beta$. If $|H_\alpha \cap H_\beta| > 2$, then we choose α^+ to be γ which covers β^*. Thus we obtain a family of closed intervals $\{H_\alpha \colon \alpha \in Y_1\}$ such that for $\alpha, \beta \in Y_1$ with $\alpha \neq \beta$, $|H_\alpha \cap H_\beta| \leqslant 2$.

We arbitrarily choose $\alpha \in Y \setminus \bigcup_{\delta \in Y_1} H_\delta$ such that $\alpha \neq 1$. We need to consider three cases:

(1) We can choose α^* in $Y \setminus \bigcup_{\delta \in Y_1} H_\delta$. So we obtain the set $H_\alpha = \{\beta \in Y \colon [\alpha^*, \beta] \subseteq Y \setminus \bigcup_{\delta \in Y_1} H_\delta\}$. For each $\beta \in H_\alpha$, let

$$D_\beta = \begin{cases} \{a_\beta\} & \text{if } \beta \neq \alpha^*, \\ \{b_\beta\} & \text{if } \beta = \alpha^*. \end{cases}$$

(2) We can only choose α^+ in $Y \setminus \bigcup_{\delta \in Y_1} H_\delta$. So we obtain the set $H_\alpha = \{\beta \in Y \colon [\beta, \alpha^+] \subseteq Y \setminus \bigcup_{\delta \in Y_1} H_\delta\}$. For each $\beta \in H_\alpha$, let

$$D_\beta = \begin{cases} \{b_\beta\} & \text{if } \beta \neq \alpha^+, \\ \{a_\beta\} & \text{if } \beta = \alpha^+. \end{cases}$$

(3) We cannot choose α^+, α^* in $Y \setminus \bigcup_{\delta \in Y_1} H_\delta$.

- If there exists $\gamma \in Y_1$ such that $\gamma <^* \alpha$ and $[\gamma, \alpha] \cap (\bigcup_{\delta \in Y_1} H_\delta) \subseteq H_\gamma$, then there exists $\zeta \in H_\gamma$ such that $\zeta \prec^* \gamma^+$. Hence, when ζ is interpreted as α^*, we have the set $H_\alpha = \{\beta \in Y \colon [\alpha^*, \beta] \subseteq (Y \setminus \bigcup_{\delta \in Y_1} H_\delta) \cup \{\alpha^*, \gamma^+\}\}$. For each $\beta \in H_\alpha$ such that $\gamma^+ <^* \beta$, let $D_\beta = \{a_\beta\}$.
- If there does not exist $\gamma \in Y_1$ such that $\gamma <^* \alpha$ and $[\gamma, \alpha] \cap (\bigcup_{\delta \in Y_1} H_\delta) \subseteq H_\gamma$, then by (TR4) there exists $\beta \in Y_1$ such that $\alpha <^* \beta$ and $[\alpha, \beta] \cap (\bigcup_{\delta \in Y_1} H_\delta) \subseteq H_\beta$, thereby there exists $\eta \in H_\beta$ such that $\beta^* \prec^* \eta$. Hence, when η is interpreted as α^+, we have the set $H_\alpha = \{\gamma \in Y \colon [\gamma, \alpha^+] \subseteq (Y \setminus \bigcup_{\delta \in Y_1} H_\delta) \cup \{\alpha^+, \beta^*\}\}$. For each $\gamma \in H_\alpha$ such that $\gamma <^* \beta^*$, let $D_\gamma = \{b_\gamma\}$.

We repeat the above proceeding, replacing $\bigcup_{\delta \in Y_1} H_\delta$ by corresponding subsets of Y. Then we obtain a family of sets $\{H_\alpha \colon \alpha \in Y_2\}$ such that for all $\alpha, \beta \in Y_1 \cup Y_2$ with $\alpha \neq \beta$, $|H_\alpha \cap H_\beta| \leqslant 2$. By Zorn's Lemma, $Y \setminus \{1\} \subseteq \bigcup_{\alpha \in Y_1 \cup Y_2} H_\alpha$. Now we define two partial mappings φ and ϕ from Y to Y. Put $\operatorname{Dom} \varphi = \{\alpha \in Y \setminus \{1\} \colon a_\alpha \in D_\alpha\}$ and $\operatorname{Dom} \phi = \{\alpha \in Y \setminus \{1\} \colon b_\alpha \in D_\alpha\}$. For any $\alpha \in \operatorname{Dom} \varphi$ there exists $\beta \in Y_1 \cup Y_2$ such that $\alpha \in H_\beta$ and $\alpha \neq \beta^*$. Let

$$\varphi(\alpha) = \begin{cases} \beta^* & \text{if } \alpha \neq \beta^+, \\ \gamma \ (\gamma \prec^* \beta^+) & \text{if } \alpha = \beta^+. \end{cases}$$

For any $\alpha \in \operatorname{Dom} \phi$ there exists $\beta \in Y_1 \cup Y_2$ such that $\alpha \in H_\beta$ and $\alpha \neq \beta^+$. Let

$$\phi(\alpha) = \begin{cases} \beta^+ & \text{if } \alpha \neq \beta^*, \\ \gamma \ (\beta^* \prec^* \gamma) & \text{if } \alpha = \beta^*. \end{cases}$$

It is easy to see that φ and ϕ are well defined. Let

$$X = \{\alpha \in Y \colon (D_\alpha, \cdot) \text{ is a left zero semigroup}\}.$$

It is easy to see that $(Y, X; \mathscr{D}; \varphi, \phi)$ is an YD-system. By Theorem 4.3, $\mathbf{L} = IRLC(Y, X; \mathscr{D}; \varphi, \phi)$ is an idempotent residuated chain. By condition (TR3) and the definition of multiplication of $IRLC(Y, X; \mathscr{D}; \varphi, \phi)$, $(\mathbf{L}, \cdot)$ is the monoid reduct of $IRLC(Y, X; \mathscr{D}; \varphi, \phi)$. $\square$

The following result is an immediate consequence of Theorem 5.2.

Corollary 5.3. *A finite band* $\mathbf{L}$ *with an identity* e *is the monoid reduct of some idempotent residuated chain if and only if* $\mathbf{L}$ *satisfies conditions* (TR1–3) *and the following condition:*

(FTR) *there exists* $a \in L$ *such that* $ab = ba = a$ *for all* $b \in L$.

Corollary 5.4. *A semilattice* $\mathbf{L}$ *with an identity* e *is the monoid reduct of some commutative idempotent residuated chain if and only if for all* $a, b \in L$, $ab = a$ *or* $ab = b$.

Proof. By Theorem 5.2, in the forward direction this is clear. Conversely, let $\mathbf{L}$ be a semilattice with an identity e. So we have the natural ordering on $(L, \cdot)$: for all $a, b \in L$, $a \leqslant b$ if and only if $a \cdot b = a$. For all $a, b \in L$, we have $ab = a$ or $ab = b$. This implies that $a \leqslant b$ or $b \leqslant a$. Thus $(L, \leqslant)$ is a chain and by Exercise 4.7.18 of [7], $(L, \leqslant, \cdot)$ is a partially ordered band with an identity e. We define a division operation $\backslash$ on L as follows:

$$a \backslash b = \begin{cases} e & \text{if } a \leqslant b, \\ b & \text{if } \ b < a. \end{cases}$$

It is easy to see that $(L, \wedge, \vee, \cdot, \backslash, /, e)$ is a commutative idempotent residuated chain. □

The remainder of this section is devoted to describing idempotent residuated chains.

Proposition 5.5. *Let* $\mathbf{L} = (L, \wedge, \vee, \cdot, e)$ *be a structure such that* $(L, \wedge, \vee)$ *is a chain and* $(L, \cdot, e)$ *is a band with an identity* e. *Then* $\mathbf{L}$ *is a lattice-ordered monoid if and only if* $\mathbf{L}$ *satisfies the following conditions:*

(LO1) *if* $a, b \geqslant e$, *then* $ab = a \vee b$;

(LO2) *if* $a, b \leqslant e$, *then* $ab = a \wedge b$;

(LO3) *if* $a \leqslant e \leqslant b$, *then* (1) *either* $ab = a$ *or* $ab = b$, (2) *either* $ba = a$ *or* $ba = b$.

Proof. By Lemma 2.6, it is easy to see that if $\mathbf{L}$ is a lattice-ordered monoid, then $\mathbf{L}$ satisfies (LO1–3).

Conversely, suppose that $\mathbf{L}$ satisfies (LO1–3). Let $a, b \in L$ and $a \leqslant b$. We need to prove that $ac \leqslant bc$ and $ca \leqslant cb$ for every $c \in L$. We consider three cases:

(1) $a, b \geqslant e$. If $c \geqslant e$, then $ac = ca = a \vee c$ and $bc = cb = b \vee c$. Since $a \leqslant b \leqslant b \vee c$ and $c \leqslant b \vee c$, we have $a \vee c \leqslant b \vee c$. This implies $ac \leqslant bc$ and $ca \leqslant cb$. If $c < e$, then $ac \in \{a, c\}$ and $bc \in \{b, c\}$. Hence the only bad situation is $ac = a$ and $bc = c$. We prove that it is actually impossible. Suppose that $ac = a$ and $bc = c$. Then

$D_a \leqslant^* D_c \leqslant^* D_b$ and $a \neq b$. Since $aba = a \vee b = b$, we have $D_b <^* D_a$. It is a contradiction. This implies $ac \leqslant bc$. Similarly, we can prove that $ca \leqslant cb$.

(2) $a, b \leqslant e$. By a similar argument to (1), $ac \leqslant bc$ and $ca \leqslant cb$.

(3) $a \leqslant e \leqslant b$. If $c \leqslant e$, then $ac = ca = a \wedge c$ and $bc, cb \in \{b, c\}$. Hence $ac \leqslant bc$ and $ca \leqslant cb$. If $c > e$, then $ac, ca \in \{a, c\}$ and $bc = cb = b \vee c$. Hence $ac \leqslant bc$ and $ca \leqslant cb$. $\square$

Corollary 5.6 [1]. *Let* $\mathbf{L} = (L, \wedge, \vee, \cdot, e)$ *be a structure such that* $(L, \wedge, \vee)$ *is a chain and* $(L, \cdot, e)$ *is a semilattice with an identity* e. *Then* $\mathbf{L}$ *is a lattice-ordered monoid if and only if* $\mathbf{L}$ *satisfies* (LO1), (LO2) *and* (LO3).

By Proposition 3.4 and the proof of Theorem 4.6, the next fact is straightforward:

Proposition 5.7. *Let* $\mathbf{L} = (L, \wedge, \vee, \cdot, e)$ *be a lattice-ordered monoid such that* $(L, \wedge, \vee)$ *is a chain and* $(L, \cdot, e)$ *is a band with an identity* e. *Then* $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /, e)$ *is a residuated lattice for some* $\backslash$ *and* $/$ *if and only if* $\mathbf{L}$ *satisfies the following conditions:*

(1) *if* $a \in L$ *and* $a > e$, *then there exists an element* $b < e$ *in* L *with* $D_b <^* D_a$ *satisfying condition* (RC);

(2) *if* $b \in L$ *and* $b < e$, *then there exists an element* $a \geqslant e$ *in* L *with* $D_b <^* D_a$ *satisfying condition* (RC$'$).

Corollary 5.8. *Let* $\mathbf{L} = (L, \wedge, \vee, \cdot, e)$ *be a finite lattice-ordered monoid such that* $(L, \wedge, \vee)$ *is a chain and* $(L, \cdot, e)$ *is a band with an identity* e. *Then* $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /, e)$ *is a residuated lattice for some* $\backslash$ *and* $/$ *if and only if* $\mathbf{L}$ *satisfies the following condition:*

(FBR) *if* $\bot$ *is the least element of* L, *then* $\bot a = a\bot = \bot$ *for all* $a \in L$.

Corollary 5.9. *Let* $\mathbf{L} = (L, \wedge, \vee, \cdot, e)$ *be a lattice-ordered monoid such that* $(L, \wedge, \vee)$ *is a chain and* $(L, \cdot, e)$ *is a semilattice with an identity* e. *Then* $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, e)$ *is a residuated lattice for some* $\backslash$ *if and only if* $\mathbf{L}$ *satisfies the following conditions:*

(1) *if* $a \in L$ *and* $a > e$, *then there exists an element* $b < e$ *in* L *with* $D_b <^* D_a$ *satisfying condition* (RC);

(2) *if* $b \in L$ *and* $b < e$, *then there exists an element* $a \geqslant e$ *in* L *with* $D_b <^* D_a$ *satisfying condition* (RC$'$).

Corollary 5.10 [1]. *Let* $\mathbf{L} = (L, \wedge, \vee, \cdot, e)$ *be a finite lattice-ordered monoid such that* $(L, \wedge, \vee)$ *is a chain and* $(L, \cdot, e)$ *is a semilattice with an identity* e*. Then* $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, e)$ *is a residuated lattice for some* $\backslash$ *if and only if* $\mathbf{L}$ *satisfies* (FBR).

6. Subdirectly irreducible idempotent residuated chains

The main aim of this section is to characterize subdirectly irreducible, simple and strictly simple idempotent residuated chains.

We have the following residuated lattice terms: $l(x) = x\backslash e, r(x) = e/x$. Moreover, we consider binary relations defined by

$$x \xrightarrow{r} y \Leftrightarrow r(x) = y,$$
$$x \xrightarrow[l]{} y \Leftrightarrow l(x) = y,$$
$$x \longrightarrow y \Leftrightarrow r(x) = y \text{ or } l(x) = y.$$

By the structure of idempotent residuated chains we have the following crucial lemma:

Lemma 6.1. *Let* $\mathbf{L} = IRLC(Y, X; \mathscr{D}; \varphi, \phi)$ *be an idempotent residuated chain. Let* $\alpha \in Y$ *be such that* D_α *contains two elements.*

(1) *If* $(D_\alpha, \cdot)$ *is a left zero semigroup, then*

$$b_{\varphi(\alpha)} \xleftarrow[l]{} a_\alpha \rightleftarrows b_\alpha \xrightarrow{r} a_{\phi(\alpha)}.$$

(2) *If* $(D_\alpha, \cdot)$ *is a right zero semigroup, then*

$$b_{\varphi(\alpha)} \xleftarrow{r} a_\alpha \rightleftarrows b_\alpha \xrightarrow[l]{} a_{\phi(\alpha)}.$$

Proof. This is easy. □

Let $\mathbf{C}_2$ be the two-element idempotent residuated chain, $C_2 = \{\bot, \top\}$, $\top = e$. Let $\mathbf{C}_3$ be the three-element idempotent residuated chain, $C_3 = \{\bot, e, \top\}$, $\bot < e < \top$. Note that C_2 is the only two-element idempotent residuated chain. The next proposition gives some characterizations of convex normal subalgebras of idempotent residuated chains.

Proposition 6.2. *Let* $\mathbf{L} = IRLC(Y, X; \mathscr{D}; \varphi, \phi)$ *be an idempotent residuated chain.*

(1) *Let* $\alpha \in Y \setminus \{1\}$ *and* $a_\alpha \in D_\alpha$. *Then*

  (i) $\varphi(\alpha) \prec^* \phi(\varphi(\alpha))$;

  (ii) *if* $|D_{\varphi(\alpha)}| = |D_{\phi(\varphi(\alpha))}| = 1$, *then* $\overline{\mathbf{A}}_\alpha = \{b_{\varphi(\alpha)}, e, a_{\phi(\varphi(\alpha))}\} \cong \mathbf{C}_3$;

  (iii) $\mathbf{A}_\alpha = [b_{\varphi(\alpha)}, a_{\phi(\varphi(\alpha))}]$ *is a convex normal subalgebra of* $\mathbf{L}$ *if and only if* $|D_{\varphi(\alpha)}| = 1$.

(2) *Let* $b_\alpha \in D_\alpha$. *Then*

  (i) *if* $\phi(\alpha) \neq 1$, *then* $\varphi(\phi(\alpha)) \prec^* \phi(\alpha)$;

  (ii) *if* $\phi(\alpha) \neq 1$ *and* $|D_{\phi(\alpha)}| = |D_{\varphi(\phi(\alpha))}| = 1$, *then* $\overline{\mathbf{B}}_\alpha = \{b_{\varphi(\phi(\alpha))}, e, a_{\phi(\alpha)}\} \cong \mathbf{C}_3$;

  (iii) *if* $\phi(\alpha) = 1$ *and* $|D_\alpha| = 1$, *then* $\widetilde{\mathbf{B}_\alpha} = \{b_\alpha, e\} \cong \mathbf{C}_2$;

  (iv) $\mathbf{B}_\alpha = [b_\alpha, a_{\phi(\alpha)}]$ *is a convex normal subalgebra of* $\mathbf{L}$ *if and only if* $|D_\alpha| = 1$.

Proof. (1) Suppose that there exists $\beta \in Y$ such that $\varphi(\alpha) <^* \beta <^* \phi(\varphi(\alpha))$. By (YD4′), D_β contains exactly b_β. Since $\varphi(\alpha) <^* \beta <^* \alpha$, D_β contains exactly a_β. It is a contradiction. That is, (i) holds. Let $|D_{\varphi(\alpha)}| = |D_{\phi(\varphi(\alpha))}| = 1$. Then by Lemma 6.1 and (i), $b_{\varphi(\alpha)}\backslash e = e/b_{\varphi(\alpha)} = b_{\varphi(\alpha)}\backslash a_{\phi(\varphi(\alpha))} = a_{\phi(\varphi(\alpha))}/b_{\varphi(\alpha)} = a_{\phi(\varphi(\alpha))}/a_{\phi(\varphi(\alpha))} = a_{\phi(\varphi(\alpha))}\backslash a_{\phi(\varphi(\alpha))} = b_{\varphi(\alpha)}/b_{\varphi(\alpha)} = b_{\varphi(\alpha)}\backslash b_{\varphi(\alpha)} = a_{\phi(\varphi(\alpha))}$ and $a_{\phi(\varphi(\alpha))}\backslash e = e/a_{\phi(\varphi(\alpha))} = b_{\varphi(\alpha)}$. It is easy to see that $b_{\varphi(\alpha)}a_{\phi(\varphi(\alpha))} = a_{\phi(\varphi(\alpha))}b_{\varphi(\alpha)} = b_{\varphi(\alpha)}$. Hence $\overline{\mathbf{A}}_\alpha = \{b_{\varphi(\alpha)}, e, a_{\phi(\varphi(\alpha))}\} \cong \mathbf{C}_3$. This proves (ii). Assume that $|D_{\varphi(\alpha)}| = 1$. By the proof of Theorem 4.6 and Lemma 6.1, for all $x, y \in [b_{\varphi(\alpha)}, a_{\phi(\varphi(\alpha))}]$ and $u \in L$, we have $\{xy, yx, x \wedge y, x \vee y, x\backslash y, y\backslash x, x/y, y/x, \varrho_u(x), \lambda_u(x), \kappa_u(x,y)\} \subseteq [b_{\varphi(\alpha)}, a_{\phi(\varphi(\alpha))}]$. Hence $\mathbf{A}_\alpha$ is a convex normal subalgebra of $\mathbf{L}$. Conversely, let $\mathbf{A}_\alpha = [b_{\varphi(\alpha)}, a_{\phi(\varphi(\alpha))}]$ be a convex normal subalgebra of $\mathbf{L}$. Suppose that $|D_{\varphi(\alpha)}| = 2$. By Lemma 6.1, $b_{\varphi(\varphi(\alpha))} \in cn(b_\varphi(\alpha)) \subseteq \mathbf{A}_\alpha$. But by hypothesis, $b_{\varphi(\varphi(\alpha))} \notin \mathbf{A}_\alpha$, it's a contradiction. This shows (iii).

(2) is established similarly. □

We are now ready to give some characterizations of subdirectly irreducible idempotent residuated chains.

Theorem 6.3. *Let* $\mathbf{L} = IRLC(Y, X; \mathscr{D}; \varphi, \phi)$ *be a nontrivial idempotent residuated chain. Then the following statements are equivalent:*

(1) $\mathbf{L}$ *is subdirectly irreducible;*

(2) $\mathbf{L}$ *satisfies one of the following conditions:*

  (SI1) *there exists* $\alpha \in Y$ *such that* $\alpha \prec^* 1$;

  (SI2) *there exists* $\alpha \in Y \setminus \{1\}$ *such that for all* $\beta \in Y \setminus \{1\}$ *with* $\alpha <^* \beta$, *either* $D_\beta = \{a_\beta\}$ *or* $D_\beta = \{a_\beta, b_\beta\}$ *and* $\{\gamma \in [\alpha, \beta] \colon |D_\gamma| = 2\}$ *is a finite set;*

(3) *there exists $u \in L$ such that $u < e$ and for all $v \in L \setminus \{u\}$ with $uv = vu = u$, either $v \geqslant e$ or $v < e, |D_v| = 2$ and $[u, v]$ is a finite set.*

P r o o f. (1) $\Rightarrow$ (2) Suppose that $\mathbf{L}$ is subdirectly irreducible, and $\mathbf{L}$ does not satisfy conditions (SI1) and (SI2). By Theorem 2.4, there exists a convex normal subalgebra $\mathbf{S}$ of $\mathbf{L}$ such that $\mathbf{S}$ is the minimum convex normal subalgebra in $\mathbf{CN}(\mathbf{L}) - \{e\}$. We choose $b_\alpha \in S \setminus \{e\}$. Suppose that there exists $\beta \in Y \setminus \{1\}$ with $\alpha <^* \beta$ such that D_β contains exactly b_β. By Proposition 6.2(2) and Theorem 2.5, $cn(b_\beta) = \mathbf{B}_\beta = [b_\beta, a_{\phi(\beta)}] \subseteq S$. Since $b_\alpha \notin [b_\beta, a_{\phi(\beta)}]$ and $\mathbf{B}_\beta$ is a convex normal subalgebra of $\mathbf{L}$, it is a contradiction. Suppose that there exists $\beta \in Y \setminus \{1\}$ with $\alpha <^* \beta$ such that $|D_\beta| = 2$ and $\{\gamma \in [\alpha, \beta] \colon |D_\gamma| = 2\}$ is infinite set. Then since $cn(b_\beta) = S$, by Theorem 2.5 and Lemma 6.1 there is a sequence of transitions

$$b_\beta = z_1 \to z_2 \to \ldots \to z_n = b_\alpha,$$

where the relation $\to$ is defined above. Hence $|\{\gamma \in [\alpha, \beta] \colon |D_\gamma| = 2\}| \leqslant n$. It is a contradiction.

(2) $\Rightarrow$ (1) If $\mathbf{L}$ satisfies (SI1), then there exists $\alpha \in Y$ such that $\alpha \prec^* 1$. Let $s \in D_\alpha$. By Lemma 6.1 and Proposition 6.2 it is easy to see that $cn(s)$ is the minimum convex normal subalgebra in $\mathbf{CN}(\mathbf{L}) - \{e\}$. Hence $\mathbf{L}$ is subdirectly irreducible. Suppose that $\mathbf{L}$ satisfies (SI2). We arbitrarily choose β in $Y \setminus \{1\}$ such that $\alpha <^* \beta$. If $D_\beta = \{a_\beta, b_\beta\}$, then since $\{\gamma \in [\alpha, \beta] \colon |D_\gamma| = 2\}$ is a finite set, Lemma 6.1implies that for all $x \in [b_\beta, a_\beta]$ and $x \neq e$ there is a sequence of transitions

$$x = z_1 \to z_2 \to \ldots \to z_n = b_\beta,$$

where the relation $\to$ is defined above. So $b_\beta \in cn(x)$. Hence $cn(b_\beta)$ is the minimum convex normal subalgebra in $\mathbf{CN}(\mathbf{L}) - \{e\}$. This implies that $\mathbf{L}$ is subdirectly irreducible. If $D_\beta = \{a_\beta\}$, then, by similar arguments to the above $b_{\varphi(\beta)} \in cn(x)$ for all $x \in [b_{\varphi(\beta)}, a_\beta]$ and $x \neq e$. Since $a_\beta \in cn(b_{\varphi(\beta)})$, we have $a_\beta \in cn(x)$. Hence $cn(a_\beta)$ is the minimum convex normal subalgebra in $\mathbf{CN}(\mathbf{L}) - \{e\}$. This shows that $\mathbf{L}$ is subdirectly irreducible.

(2) $\Rightarrow$ (3) Suppose that $\mathbf{L}$ satisfies (SI1). Then there exists $\alpha \in Y$ such that $\alpha \prec^* 1$. Suppose that D_α contains b_α. Let $u = b_\alpha$. If $v \in D_\beta$ such that $u \neq v$ and $uv = vu = u$, then $u <_n v$. So $D_u <^* D_v$ by Proposition 3.2. Hence $\alpha <^* \beta$. Thus $\beta = 1$ and $v = e$. Suppose that $D_\alpha = \{a_\alpha\}$. Let $u = b_{\varphi(\alpha)}$. If $v \in D_\beta$ such that $u \neq v$ and $uv = vu = u$, then $u <_n v$. So by Proposition 3.2, $D_u <^* D_v$. Hence $\varphi(\alpha) <^* \beta \leqslant 1$. Thus $v \geqslant e$. Assume $\mathbf{L}$ satisfies (SI2). Then there exists $\alpha \in Y \setminus \{1\}$ such that for all $\beta \in Y \setminus \{1\}$ with $\alpha <^* \beta$, either $D_\beta = \{a_\beta\}$ or $D_\beta = \{a_\beta, b_\beta\}$ and $\{\gamma \in [\alpha, \beta] \colon |D_\gamma| = 2\}$ is a finite set. Suppose that D_α contains b_α. Let $u = b_\alpha$. If

$v \in D_{\beta'}$ such that $u \neq v$ and $uv = vu = u$, then $u <_n v$. So by Proposition 3.2, $D_u <^* D_v$ and $\alpha <^* \beta'$. Thus either $v \geqslant e$ or $v < e$, $|D_v| = |D_{\beta'}| = 2$ and $[u, v]$ is a finite set. Suppose that $D_\alpha = \{a_\alpha\}$. Let $u = b_{\varphi(\alpha)}$. If $v \in D_{\beta'}$ such that $u \neq v$ and $uv = vu = u$, then $u <_n v$. So by Proposition 3.2, $D_u <^* D_v$ and $\varphi(\alpha) <^* \beta'$. Thus either $v \geqslant e$ or $v < e$, $|D_v| = |D_{\beta'}| = 2$ and $[u, v]$ is a finite set.

(3) $\Rightarrow$ (2) Suppose that $u \in L$ such that $u < e$ and for all $v \in L \setminus \{u\}$ with $uv = vu = u$, either $v \geqslant e$ or $v < e, |D_v| = 2$ and $[u, v]$ is a finite set. Assume **L** does not satisfy (SI1). Then there exists $v \in L \backslash \{e\}$ such that $D_u <^* D_v$. Let $D_v = D_\alpha$. Thus for all $\beta \in Y \setminus \{1\}$ with $\alpha <^* \beta$, either $D_\beta = \{a_\beta\}$ or $D_\beta = \{a_\beta, b_\beta\}$ and $\{\gamma \in [\alpha, \beta] \colon |D_\gamma| = 2\}$ is a finite set. □

As a consequence of Theorem 6.3, we have:

Corollary 6.4. *If* **L** *is a finite idempotent residuated chain, then* **L** *is subdirectly irreducible.*

By Theorem 6.3 and Corollary 2.3 of [1] we have:

Corollary 6.5. *Let* $\mathbf{L} = IRLC(Y, X; \mathscr{D}; \varphi, \phi)$ *be a nontrivial commutative idempotent residuated chain. Then the following statements are equivalent:*

(1) **L** *is subdirectly irreducible;*

(2) **L** *satisfies* (SI1) *or the following condition:*

  (SIC) *there exists* $\alpha \in Y \setminus \{1\}$ *such that for all* $\beta \in Y \setminus \{1\}$ *with* $\alpha <^* \beta$, $D_\beta = \{a_\beta\}$;

(3) *There exists* $a \in L \setminus \{e\}$ *such that for all* $b \in L \setminus \{a\}$ *such that* $ab = a$, $b \geqslant e$;

(4) e *is completely join-irreducible.*

By the proof of Theorem 6.3 we can similarly obtain some characterizations of simple idempotent residuated chain.

Theorem 6.6. *Let* $\mathbf{L} = IRLC(Y, X; \mathscr{D}; \varphi, \phi)$ *be a nontrivial idempotent residuated chain. Then the following statements are equivalent:*

(1) **L** *is simple;*

(2) *for any* $\alpha \in Y \setminus \{1\}$ *such that* α *is not the least element of* Y, *either* $D_\alpha = \{a_\alpha\}$ *or* $D_\alpha = \{a_\alpha, b_\alpha\}$ *and for* $\beta \in Y \setminus \{1\}$ *with* $\alpha \leqslant^* \beta$, $\{\gamma \in [\alpha, \beta] \colon |D_\gamma| = 2\}$ *is a finite set;*

(3) *for each* $u \in L$ *such that* u *is not the least element of* L, *either* $u \geqslant e$ *or* $u < e$, $|D_u| = 2$ *and for* $v \in L$ *with* $u \leqslant v < e$, $[u, v]$ *is a finite set.*

This implies the next fact

Corollary 6.7. *Let $\mathbf{L} = IRLC(Y, X; \mathscr{D}; \varphi, \phi)$ be a nontrivial commutative idempotent residuated chain. Then the following statements are equivalent:*

(1) $\mathbf{L}$ *is simple;*

(2) Y *has a least element* β *and for any* $\alpha \in Y \setminus \{1, \beta\}$, $D_\alpha = \{a_\alpha\}$*;*

(3) *there exists a unique element* $b \in L$ *such that* $b < e$.

Example 6.8. Let $Y = \{0, 1, 2, 3, 4, \} \cup \{\top\}$ where $0 <^* 1 <^* 2 <^* 3 <^* 4 <^* \top$; see Figure 2(1). Let $X = \{0, 1, 3, 4\} \subseteq Y$. Let $A = \{b_0, b_1, b_2, a_2, a_3, a_4, e\}$; see Figure 2(2). We define an order relation on A by $a_2 > a_3 > a_4 > e > b_2 > b_1 > b_0$; see Figure 2(3). We can define a multiplication operation and two division operations on A described in Section 4. Then by Theorem 4.3, $\mathbf{A}$ is an idempotent residuated chain and by Theorem 6.3, $\mathbf{A}$ is subdirectly irreducible. Note by Theorem 6.6, $\mathbf{A}$ is not simple.

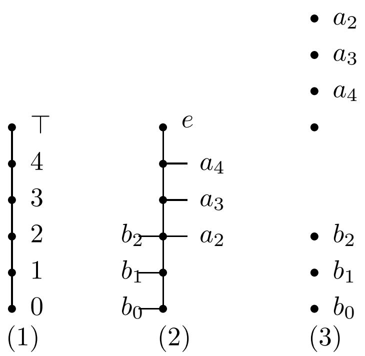


Figure 2

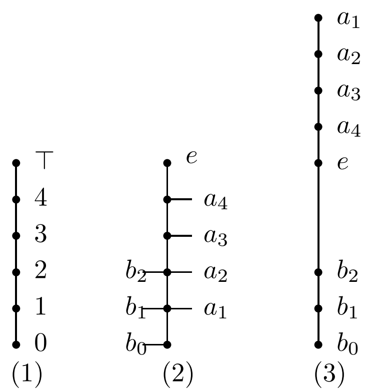


Figure 3

A nontrivial algebra $\mathbf{L}$ is called *strictly simple*, if it lacks nontrivial proper subalgebras and congruences. By Theorem 2.4 the absence of nontrivial proper subalgebras in a residuated lattice is enough to establish strict simplicity.

Example 6.9. Let $Y = \{0, 1, 2, 3, 4\} \cup \{\top\}$ where $0 <^* 1 <^* 2 <^* 3 <^* 4 <^* \top$; see Figure 3(1). Let $X = \{0, 1, 3, 4\} \subseteq Y$. Let $B = \{b_0, b_1, a_1, b_2, a_2, a_3, a_4, e\}$; see Figure 3(2). We define an order relation on B by $a_1 > a_2 > a_3 > a_4 > e > b_2 > b_1 > b_0$; see Figure 3(3). We can define a multiplication operation and two division operations on B described in Section 4. Then by Theorem 4.3, $\mathbf{B}$ is an idempotent residuated chain and by Theorem 6.6, $\mathbf{B}$ is simple. It is easy to see that $\mathbf{C} = \{b_0, b_1, a_1, b_2, a_2, a_3, e\}$ is a subalgebra of $\mathbf{B}$. Hence $\mathbf{B}$ is not strictly simple.

We now describe some characterizations of strictly simple idempotent residuated chains.

Theorem 6.10. *Let $\mathbf{L} = IRLC(Y, X; \mathscr{D}; \varphi, \phi)$ be a nontrivial idempotent residuated chain. Then the following statements are equivalent:*

(1) $\mathbf{L}$ *is strictly simple;*

(2) $\mathbf{L}$ *satisfies one of the following conditions:*

(ST1) *if Y has not a least element and $Y \setminus \{1\}$ has not a greatest element, then for all $\alpha \in Y \setminus \{1\}$, $|D_\alpha| = 2$ and for $\alpha' \in Y \setminus \{1\}$ with $\alpha \leqslant^* \alpha'$, $[\alpha, \alpha']$ is a finite set;*

(ST2) *if Y has not a least element and $Y \setminus \{1\}$ has a greatest element γ, then D_γ contains a_γ and for all $\alpha \in Y \setminus \{1, \gamma\}$, $|D_\alpha| = 2$ and for $\alpha' \in Y \setminus \{1\}$ with $\alpha \leqslant^* \alpha'$, $[\alpha, \alpha']$ is a finite set;*

(ST3) *if Y has a least element β and $Y \setminus \{1\}$ has not a greatest element, then for all $\alpha \in Y \setminus \{1, \beta\}$, $|D_\alpha| = 2$ and for $\alpha' \in Y \setminus \{1\}$ with $\alpha \leqslant^* \alpha'$, $[\alpha, \alpha']$ is a finite set;*

(ST4) *if Y has a least element β and $Y \setminus \{1\}$ has a greatest element γ, then for $\gamma \neq \beta$, D_γ contains a_γ and for any $\alpha \in Y \setminus \{1, \gamma, \beta\}$, $|D_\alpha| = 2$ and for $\alpha' \in Y \setminus \{1\}$ with $\alpha \leqslant^* \alpha'$, $[\alpha, \alpha']$ is a finite set;*

(3) *for $u, v \in L$ with $u \leqslant v < e$, $[u, v]$ is a finite set and for each $a \in L$ such that a is not the least element of L, either $|D_a| = 2$ or $a \geqslant e$ and for all $b \in L \backslash \{e\}$, $ab = ba = b$.*

P r o o f. (1) $\Rightarrow$ (2) Let $\mathbf{L}$ be strictly simple. We consider four cases:

(a) Y has not a least element and $Y \setminus \{1\}$ has not a greatest element. Suppose that there exists $\alpha \in Y \setminus \{1\}$, $|D_\alpha| = 1$.

- If D_α contains b_α, then by Theorem 6.6, $\mathbf{L}$ is not simple. It is a contradiction.
- Suppose that D_α contains a_α. We denote the subalgebra generated by a_α by $\mathbf{A}$. By the proof of Theorem 4.6 and Lemma 6.1, for all $x \in A$, $x \leqslant b_{\varphi(\alpha)}$ or $x \geqslant a_\alpha$. It is a contradiction.

Thus $\mathbf{L}$ satisfies (ST1).

(b) Y has not a least element and $Y \setminus \{1\}$ has a greatest element γ. We need to show that D_γ contains a_γ. Suppose that D_γ contains exactly b_γ. By Proposition 6.2(2), $\widetilde{\mathbf{B}_\gamma} = \{b_\gamma, e\} \cong \mathbf{C}_2$. It is a contradiction. Hence $\mathbf{L}$ satisfies (ST2).

(c) Y has a least element β and $Y \setminus \{1\}$ has not a greatest element. By a similar argument to (a), condition (ST3) holds in $\mathbf{L}$.

(d) Y has a least element β and $Y \setminus \{1\}$ has a greatest element γ. By a similar argument to (b), condition (ST4) holds in $\mathbf{L}$.

(2) $\Rightarrow$ (1) It follows from Theorem 6.6 and Lemma 6.1.

(2) $\Leftrightarrow$ (3) This is clear. $\square$

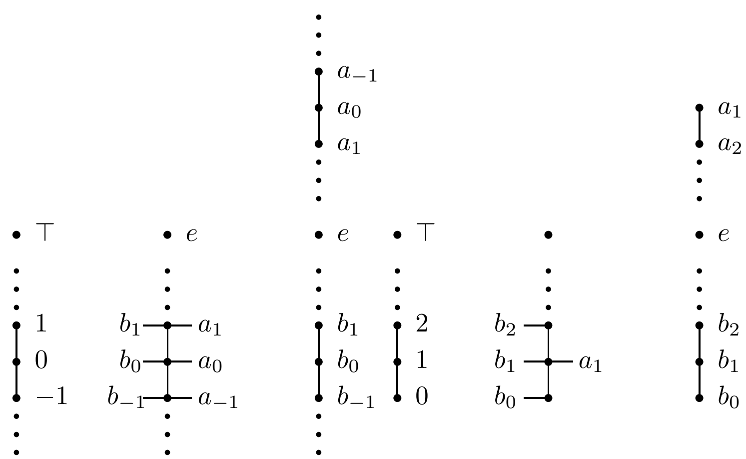
Corollary 6.11. *If* $\mathbf{L} = IRLC(Y, X; \mathscr{D}; \varphi, \phi)$ *is an idempotent residuated chain and strictly simple, then* $|L| \leqslant \omega$.

P r o o f. Suppose that $\mathbf{L}$ is an idempotent residuated chain and strictly simple. If Y has not a least element, then by Theorem 6.10 and Proposition 6.2, for each $\alpha \in Y \setminus \{1\}$, there exist $\beta, \gamma \in Y$ such that $\beta \prec^* \alpha \prec^* \gamma$. Hence $|Y| \leqslant \omega$. Thus $|L| \leqslant \omega$. If Y has a least element β, then by a similar argument to the above, $|Y| \leqslant \omega$ and so $|L| \leqslant \omega$. □

As a consequence of Theorem 6.10, we have:

Corollary 6.12. *Let* $\mathbf{L} = IRLC(Y, X; \mathscr{D}; \varphi, \phi)$ *be a nontrivial commutative idempotent residuated chain. Then* $\mathbf{L}$ *is strictly simple if and only if* $\mathbf{L} \cong \mathbf{C}_2$ *or* $\mathbf{L} \cong \mathbf{C}_3$.

Example 6.13. Let $Y = \mathbb{Z} \cup \{\top\}$, $S \subseteq \mathbb{Z}$. We define an order relation $\leqslant^*$ on Y by: for all $i, j \in \mathbb{Z}$, $i \leqslant^* j$ is the same as on $\mathbb{Z}$ for the usual order; for all $i \in \mathbb{Z}$, $i <^* \top$; see Figure 4(1). Let $N_S = \{a_i \colon i \in Y \setminus \{\top\}\} \cup \{b_i \colon i \in Y \setminus \{\top\}\} \cup \{e\}$; see Figure 4(2). We define an order relation $\leqslant$ on N_S by $b_i < b_j < e < a_k < a_l$, for all $i, j, k, l \in \mathbb{Z}$ such that $i <^* j$ and $k^* > l$; see Figure 4(3). We can define a multiplication operation and two division operations on N_S described in Section 4. Then by Theorem 4.3, $\mathbf{N}_S$ is an idempotent residuated chain. By Theorem 6.10, $\mathbf{N}_S$ is strictly simple.



(1) (2) (3) (1) (2) (3)

Figure 4 Figure 5

Example 6.14. Let $Y = \mathbb{N} \cup \{\top\}$, $T \subseteq \mathbb{N}$. We define an order relation $\leqslant^*$ on Y by: for all $i, j \in \mathbb{N}$, $i \leqslant^* j$ is the same as on $\mathbb{N}$ for the usual order; for all $i \in \mathbb{N}$, $i <^* \top$; see

Figure 5(1). Let $N_T = \{a_i\colon\ i \in Y \setminus \{\top, 0\}\} \cup \{b_i\colon\ i \in Y \setminus \{\top\}\} \cup \{e\}$; see Figure 5(2). We define an order relation $\leqslant$ on N_T by $b_0 < b_i < b_j < e < a_k < a_l < a_1$, for all $i, j, k, l \in \mathbb{N}$ such that $0 <^* i <^* j$ and $k^* > l^* > 1$; see Figure 5(3). We can define a multiplication operation and two division operations on N_T described in Section 4. Then by Theorem 4.3, $\mathbf{N}_T$ is an idempotent residuated chain. By Theorem 6.10, $\mathbf{N}_T$ is strictly simple.

Example 6.15. Let $Y = \mathbb{Z}^- \cup \{\top\}$, $X \subseteq \mathbb{Z}^-$, $N_X = \{a_i\colon\ i \in Y \setminus \{\top\}\} \cup \{b_i\colon\ i \in Y \setminus \{\top\}\} \cup \{e\}$. We define an order relation $\leqslant^*$ on Y by: for all $i, j \in \mathbb{Z}^-$, $i \leqslant^* j$ is the same as on $\mathbb{Z}^-$ for the usual order; for all $i \in \mathbb{Z}^-$, $i <^* \top$. We define an order relation $\leqslant$ on N_X by $b_i < b_j < b_{-1} \prec e \prec a_{-1} < a_k < a_l$, for all $i, j, k, l \in \mathbb{Z}^-$ such that $i <^* j <^* -1$ and $-1^* > k^* > l$; see Figure 6. We can define a multiplication operation and two division operations on N_X described in Section 4. Then by Theorem 4.3, $\mathbf{N}_X$ is an idempotent residuated chain. Let $\widetilde{N_X} = \{a_i\colon\ i \in Y \setminus \{\top\}\} \cup \{b_i\colon\ i \in Y \setminus \{-1, \top\}\} \cup \{e\}$. We define an order relation $\leqslant$ on $\widetilde{N_X}$ by $b_i < b_j < b_{-2} \prec e \prec a_{-1} < a_k < a_l$, for all $i, j, k, l \in \mathbb{Z}^-$ such that $i <^* j <^* -2$ and $-1^* > k^* > l$; see Figure 7. We can define a multiplication operation and two division operations on $\widetilde{N_X}$ described in Section 4. Then by Theorem 4.3, $\widetilde{\mathbf{N}_X}$ is an idempotent residuated chain. By Theorem 6.10, $\mathbf{N}_X$ and $\widetilde{\mathbf{N}_X}$ are strictly simple.

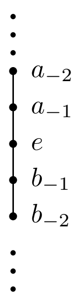


Figure 6

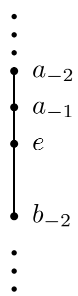


Figure 7

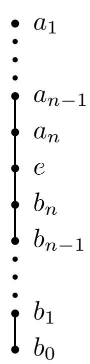


Figure 8

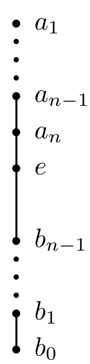


Figure 9

Example 6.16. Let $Y = \{0, 1, \ldots, n\} \cup \{\top\}$, $M \subseteq \{0, 1, \ldots, n\}$, $n_M = \{a_i\colon\ i \in Y \setminus \{\top, 0\}\} \cup \{b_i\colon\ i \in Y \setminus \{\top\}\} \cup \{e\}$. We define an order relation $\leqslant^*$ on Y by: for all $i, j \in \{0, 1, \ldots, n\}$, $i \leqslant^* j$ is the same as on $\mathbb{N}$ for the usual order; for all $i \in \{0, 1, \ldots, n\}$, $i <^* \top$. We define an order relation $\leqslant$ on n_M by $b_0 \leqslant b_i \leqslant b_j \leqslant b_n \prec e \prec a_n \leqslant a_k \leqslant a_l \leqslant a_1$, for all $i, j, k, l \in \{0, 1, \ldots, n\}$ such that $0 \leqslant^* i \leqslant^* j \leqslant^* n$ and $n^* \geqslant k^* \geqslant l^* \geqslant 1$; see Figure 8. We can define a multiplication operation and two division operations on n_M described in Section 4.

Then by Theorem 4.3, $\mathbf{n}_M$ is an idempotent residuated chain. Let $\widetilde{n_M} = \{a_i \colon i \in Y \setminus \{\top, 0\}\} \cup \{b_i \colon i \in Y \setminus \{\top, n\}\} \cup \{e\}$. We define an order relation $\leqslant$ on $\widetilde{n_M}$ by $b_0 \leqslant b_i \leqslant b_j \leqslant b_{n-1} \prec e \prec a_n \leqslant a_k \leqslant a_l \leqslant a_1$, for all $i, j, k, l \in \{0, 1, \ldots, n\}$ such that $0 \leqslant^* i \leqslant^* j \leqslant^* n-1$ and $n^* \geqslant k^* \geqslant l^* \geqslant 1$; see Figure 9. We can define a multiplication operation and two division operations on $\widetilde{n_M}$ described in Section 4. Then by Theorem 4.3, $\widetilde{\mathbf{n}_M}$ is an idempotent residuated chain. By Theorem 6.10, $\mathbf{n}_M$ and $\widetilde{\mathbf{n}_M}$ are strictly simple.

Let $\mathbf{K} = \{\mathbf{N}_S \colon S \subseteq \mathbb{Z}\} \cup \{\mathbf{N}_T \colon T \subseteq \mathbb{N}\} \cup \{\mathbf{N}_X \colon X \subseteq \mathbb{Z}^-\} \cup \{\widetilde{\mathbf{N}_X} \colon X \subseteq \mathbb{Z}^-\} \cup \{\mathbf{n}_M \colon n \in \mathbb{N}, M \subseteq \{0, 1, \ldots, n\}\} \cup \{\widetilde{\mathbf{n}_M} \colon n \in \mathbb{N}, M \subseteq \{0, 1, \ldots, n\}\}$. We may prove the following characterization of strictly simple idempotent residuated chains by using Theorem 6.10.

Theorem 6.17. *Let* $\mathbf{L} = IRLC(Y, X; \mathscr{D}; \varphi, \phi)$ *be a nontrivial idempotent residuated chain. Then* $\mathbf{L}$ *is strictly simple if and only if* $\mathbf{L} \in \mathbf{I}(\mathbf{K})$.

Proof. Let $\mathbf{L} = IRLC(Y, X; \mathscr{D}; \varphi, \phi)$ be strictly simple. Then by Theorem 6.10 and Lemma 6.1, for all $a, b \in L \setminus \{e\}$, (a, b) is in the transitive closure of the relation $\rightarrow$ defined above. Suppose that $a \in D_\alpha, b \in D_\beta$. Then the closed interval $[\alpha, \beta]$ of Y is a finite set. By Theorem 6.10, we need to consider four cases:

(a) If Y has not a least element and $Y \setminus \{1\}$ has not a greatest element, then by the proof of Corollary 6.11, $Y \cong \mathbb{Z} \cup \{\top\}$. Hence $\mathbf{L} \in \mathbf{I}(\{\mathbf{N}_S \colon S \subseteq \mathbb{Z}\})$.

(b) If Y has not a least element and $Y \setminus \{1\}$ has a greatest element, then by the proof of Corollary 6.11, $Y \cong \mathbb{Z}^- \cup \{\top\}$. Hence $\mathbf{L} \in \mathbf{I}(\{\mathbf{N}_X \colon X \subseteq \mathbb{Z}^-\} \cup \{\widetilde{\mathbf{N}_X} \colon X \subseteq \mathbb{Z}^-\})$.

(c) If Y has a least element and $Y \setminus \{1\}$ has not a greatest element, then by the proof of Corollary 6.11, $Y \cong \mathbb{N} \cup \{\top\}$. Hence $\mathbf{L} \in \mathbf{I}(\{\{\mathbf{N}_T \colon T \subseteq \mathbb{N}\})$.

(d) If Y has a least element and $Y \setminus \{1\}$ has a greatest element, then by the proof of Corollary 6.11, $Y \cong \{0, 1, \ldots, n\} \cup \{\top\}$ where $n \in \mathbb{N}$. Hence $\mathbf{L} \in \mathbf{I}(\{\mathbf{n}_M \colon n \in \mathbb{N}, M \subseteq \{0, 1, \ldots, n\}\} \cup \{\widetilde{\mathbf{n}_M} \colon n \in \mathbb{N}, M \subseteq \{0, 1, \ldots, n\}\})$.

Thus $\mathbf{L} \in \mathbf{I}(\mathbf{K})$.

Conversely, it's easy to see that if $\mathbf{L} \in \mathbf{I}(\mathbf{K})$, then $\mathbf{L}$ is strictly simple. $\square$

Acknowledgment. The authors would like to thank the referee heartily for his/her careful reading and valuable suggestions which lead to a substantial improvement of this paper.

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