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UNIFORM DECAY FOR A HYPERBOLIC SYSTEM WITH DIFFERENTIAL INCLUSION AND NONLINEAR MEMORY SOURCE TERM ON THE BOUNDARY

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Abstract. We prove the existence and uniform decay rates of global solutions for a hyperbolic system with a discontinuous and nonlinear multi-valued term and a nonlinear memory source term on the boundary.

Keywords: existence of solution, differential inclusion, memory source term, uniform decay

MSC 2010: 35L70, 35L85, 49J53

1. Introduction

In this paper we are concerned with the existence and uniform decay rates of solutions of a hyperbolic system with a differential inclusion and a memory source term on the boundary of the form

$$u'' - \operatorname{div}(a\nabla u) + |u|^{\gamma}u = 0 \text{ in } \Omega\times(0,\infty), \tag{1.1}$$

$$u = 0 \text{ on } \Gamma_1\times(0,\infty), \tag{1.2}$$

$$(a\nabla u)\cdot\nu + u' + \Xi = \int_0^t h(t-\tau)f(u(\tau))\,\mathrm{d}\tau \text{ on } \Gamma_0\times(0,\infty), \tag{1.3}$$

$$u(x,0) = u_0,\ \ u'(x,0) = u_1 \text{ in } \Omega, \tag{1.4}$$

$$\Xi \in \varphi(u(x,t)) \text{ a.e. } (x,t)\in\Gamma_0\times(0,\infty), \tag{1.5}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geqslant 2)$ with sufficiently smooth boundary $\Gamma = \partial\Omega$ such that $\Gamma = \Gamma_0\cup\Gamma_1, \overline{\Gamma}_0\cap\overline{\Gamma}_1 = \emptyset$ and Γ_0, Γ_1 have positive measures, $u' = \partial u/\partial t$, $u'' = \partial^2u/\partial t^2$, $a\in C^1(\overline{\Omega})$, f is a nonlinear function, ν is the unit outward normal to Γ and φ is a discontinuous and nonlinear set valued mapping arising by filling in the jumps of a function $b \in L^{\infty}_{\rm loc}(\mathbb{R})$. In the rest of the paper let us assume that

$$\frac{2}{n-1} < \gamma \leqslant \frac{2}{n-2} \quad \text{if } n \geqslant 3$$

and

$$\gamma > 2 \quad \text{if } n = 2.$$

The precise hypotheses on the above system will be given in the next section. Recently, a class of viscoelastic problems has been studied by many authors [2], [3], [10], [13]. M. Aassila [1] investigated the global existence of a solution to a system (1.1) and (1.4) with damping terms and the Dirichlet boundary conditions when $a(x) \equiv 1$. M. M. Cavalcanti et al. [3] studied the existence and uniform decay of solutions of the damped semilinear viscoelastic wave equation with the Dirichlet boundary conditions of the form

$$\begin{cases} u'' - \Delta u + \alpha u + \beta |u'|^{\varrho} u' + \delta |u|^{\varrho} u + \int_0^t h(t-\tau)\Delta u(\tau)\,\mathrm{d}\tau = 0 \text{ in } \Omega \times (0,\infty), \\ u(x,0) = u_0, \ \ u'(x,0) = u_1 \text{ in } \Omega, \end{cases}$$

where Ω is any bounded or finite measure domain in $\mathbb{R}^n$ and the constants α, β, ϱ and δ are positive and satisfy some conditions. Motivated by their works, we consider more general problems (1.1)–(1.5) with a discontinuous and nonlinear multi-valued term φ and a nonlinear memory source term on the boundary. The background of these variational problems is in physics, especially in solid mechanics, where non-monotone and multi-valued constitutive laws lead to differential inclusions. We refer to [5], [11], [12] to see the applications of such differential inclusions. In this paper we prove the existence of solutions of the variational inequality problems (1.1)–(1.5). Moreover, the uniform decay of the energy

$$E(t) = \frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\int_{\Omega} a(x)|\nabla u(x,t)|^2\,\mathrm{d}x + \frac{1}{\gamma+2}\|u(t)\|_{\gamma+2}^{\gamma+2}$$

is proved by assuming that μ (see assumption $(\mathrm{A}_2)^*$ below) is sufficiently small and the kernel h in the memory term decays exponentially. At this point it is important to mention that such differential inclusions were studied by some authors [4], [8], [9], [14], [15], but, as far as we are concerned, a differential inclusion acting on the boundary has never been considered and no decay rates in the present paper were obtained as in literature. Our paper is organized as follows: In Section 2, we give assumptions and state the main results. In Section 3, we prove the existence of solution of the problems (1.1)–(1.5) by using the Faedo-Galerkin method. Finally, in Section 4, we prove the uniform decay of energy by using the Lyapunov functional developed by Kormornik and Zuazua [6].

2. Assumptions and main results

Throughout the paper we denote

$$V = \{u \in H^1(\Omega)\colon\ u = 0 \text{ on } \Gamma_1\}, \quad (u,v) = \int_\Omega u(x)v(x)\,\mathrm{d}x,$$
$$(u,v)_{\Gamma_0} = \int_{\Gamma_0} u(x)v(x)\,\mathrm{d}\Gamma, \quad \|u\|^2_{2,\Gamma_0} = \int_{\Gamma_0} |u(x)|^2\,\mathrm{d}\Gamma.$$

Let us denote by V^* the dual space of V, by $\|\cdot\|_*$ the norm of V^* and by $\langle\cdot,\cdot\rangle$ the dual pairing between V and V^*. For simplicity, we denote $\|\cdot\|_{L^2(\Omega)}, \|\cdot\|_{L^p(\Omega)} (1 \leqslant p \leqslant \infty)$ and $\|\cdot\|_{2,\Gamma_0}$ by $\|\cdot\|, \|\cdot\|_p$ and $\|\cdot\|_{\Gamma_0}$, respectively. Let λ_0 and λ be the smallest positive constants such that

$$\|u\|^2 \leqslant \lambda_0\|\nabla u\|^2, \quad \|u\|^2_{\Gamma_0} \leqslant \lambda\|\nabla u\|^2, \quad \forall u \in V. \tag{2.1}$$

We formulate the following assumptions:

(A_1) *Assumptions on* a

Let $a \in C^1(\overline{\Omega})$ satisfy $a(x) \geqslant a_0 > 0$ in Ω for some a_0.

For short notation, define $a(u,v) = \sum\limits_{j=1}^{n} \int_\Omega a(x)\partial u/\partial x_j \partial v/\partial x_j\,\mathrm{d}x$. By the above assumption on a, we have

$$a_0\|\nabla u\|^2 \leqslant a(u,u) \leqslant a_1\|\nabla u\|^2, \ \ \forall u \in V \text{ for some } a_1 > 0.$$

(A_2) *Assumptions on* b

Let $b\colon\ \mathbb{R} \to \mathbb{R}$ be a locally bounded function satisfying

$$|b(s)| \leqslant \mu_0(1 + |s|) \ \ \forall s \in \mathbb{R} \text{ for some } \mu_0 > 0.$$

In order to get the uniform decay rates for the solutions of problem (1.1)–(1.5) we shall use the following stronger hypothesis:

(A_2)* $|b(s)| \leqslant \mu|s|$ and $b(s)s \geqslant \mu_1 s^2$, where $\mu_1 > 0$ and $0 < \mu < 1$.

The multi-valued function $\varphi\colon\ \mathbb{R} \to 2^{\mathbb{R}}$ is obtained by filling in the jumps of the function $b\colon\ \mathbb{R} \to \mathbb{R}$ by means of the functions $\underline{b}_\varepsilon, \overline{b}_\varepsilon, \underline{b}, \overline{b}\colon\ \mathbb{R} \to \mathbb{R}$ as follows:

$$\underline{b}_\varepsilon(t) = \operatorname*{ess\,inf}_{|s-t|\leqslant\varepsilon} b(s),\ \overline{b}_\varepsilon(t) = \operatorname*{ess\,sup}_{|s-t|\leqslant\varepsilon} b(s);$$
$$\underline{b}(t) = \lim_{\varepsilon\to 0^+} \underline{b}_\varepsilon(t),\ \overline{b}(t) = \lim_{\varepsilon\to 0^+} \overline{b}_\varepsilon(t);$$
$$\varphi(t) = [\underline{b}(t), \overline{b}(t)].$$

We shall need a regularization of b defined by

$$b^m(t) = m \int_{-\infty}^{\infty} b(t-\tau)\varrho(m\tau)\,\mathrm{d}\tau,$$

where $\varrho \in C_0^\infty((-1,1))$, $\varrho \geqslant 0$ and $\int_{-1}^{1} \varrho(\tau)\,\mathrm{d}\tau = 1$.

Remark 2.1. It is easy to show that b^m is continuous for all $m \in \mathbb{N}$ and $\underline{b}_\varepsilon, \overline{b}_\varepsilon, \underline{b}, \overline{b}, b^m$ satisfy the same condition (A_2) or $(\mathrm{A}_2)^*$ possibly with different constants if b satisfies (A_2) or $(\mathrm{A}_2)^*$. So, in the sequel, we denote the different constants by the same symbols as the original ones.

(A_3) *Assumptions on* f

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying

$$|f(s)| \leqslant \alpha(1+|s|), \quad \forall s \in \mathbb{R}$$

for a positive constant α.

(A_4) *Assumptions on the kernel* h

Let $h\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a continuously differentiable function verifying

$$-\xi_1 h(t) \leqslant h'(t) \leqslant -\xi_2 h(t), \quad \forall t \geqslant t_0$$

for some $\xi_1 > 0$, $\xi_2 > 0$, $t_0 > 0$, where $h(0) = 0$ and $1 - \lambda {a_0}^{-1} \int_0^\infty h(s)\,\mathrm{d}s = l > 0$.

Definition. A function $u(x,t)$ is a solution to problem (1.1)–(1.5) if for every $T > 0$, u satisfies

$$u \in L^\infty(0,T;V),\; u' \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;L^2(\Gamma_0)),\; u'' \in L^2(0,T;V^*),$$

and there exists $\Xi \in L^2(0,T;L^2(\Gamma_0))$ such that the following relations hold:

$$\begin{aligned}
&\int_0^T \{\langle u'', v\rangle + a(u(t),v) + (|u(t)|^\gamma u(t), v) + (u'(t),v)_{\Gamma_0} + (\Xi, v)_{\Gamma_0}\}\,\mathrm{d}t \\
&\qquad = \int_0^T \int_0^t h(t-\tau)(f(u(\tau)),v)_{\Gamma_0}\,\mathrm{d}\tau\,\mathrm{d}t, \quad \forall v \in V, \\
&\Xi(x,t) \in \varphi(u(x,t)) \text{ a.e. } (x,t) \in \Gamma_0 \times (0,T), \\
&u(x,0) = u_0, \quad u'(x,0) = u_1 \text{ on } \Omega.
\end{aligned}$$

Now we are in a position to state our results.

Theorem 2.1. *Assume the conditions* (A_1)–(A_4) *hold. Then for every* $(u_0, u_1) \in V \times L^2(\Omega)$ *there exists a solution of problem* (1.1)–(1.5).

Theorem 2.2. *Assume the conditions* $(\mathrm{A}_1), (\mathrm{A}_2)^*, (\mathrm{A}_3)$ *and* (A_4) *hold and* $(u_0, u_1) \in V \times L^2(\Omega)$. *Then, if we assume* $\|\nabla a\|_\infty / a_0 \leqslant \mu$ *and consider* $\|h\|_{L^1(0,\infty)}$ *and* μ *(given in* $(\mathrm{A}_2)^*$*) sufficiently small, the energy determined by the solutions of problem* (1.1)–(1.5) *decays exponentially, that is,*

$$E(t) \leqslant C_3 \exp\Big(-\frac{2}{3} C_2 t\Big) \quad \text{a.e. } t \geqslant t_0,$$

for some positive constants C_2 *and* C_3.

3. Proof of theorem 2.1

In this section we are going to show the existence of solutions to problem (1.1)–(1.5) using the Faedo-Galerkin approximation. To this end we represent by $\{w_j\}_{j\geqslant 1}$ a basis in V which is orthonormal in $L^2(\Omega)$. Let V_m be the space generated by $w_1, \ldots, w_m$. We may choose (u_{0m}) and (u_{1m}) in V_m such that

$$u_{0m} \to u_0 \text{ in } V \quad \text{and} \quad u_{1m} \to u_1 \text{ in } L^2(\Omega).$$

Let

$$u_m(t) = \sum_{j=1}^{m} g_{jm}(t) w_j$$

be the solution to the Cauchy problem

$$\begin{aligned} (3.1)\qquad & (u_m''(t), w) + a(u_m(t), w) + (|u_m(t)|^\gamma u_m(t), w) \\ & \quad + (u_m'(t), w)_{\Gamma_0} + (b^m(u_m(t)), w)_{\Gamma_0} \\ & \quad = \int_0^t h(t-\tau)(f(u_m(\tau)), w)_{\Gamma_0} \, \mathrm{d}\tau, \ \forall w \in V_m, \\ (3.2)\qquad & u_m(0) = u_{0m}, \ u_m'(0) = u_{1m}. \end{aligned}$$

By standard methods of differential equations, we can prove the existence of a solution to (3.1)–(3.2) on an interval $[0, t_m)$. Then, this solution can be extended to the closed interval $[0, T]$ by using the a priori estimate below.

Step 1*: A priori estimate.*

Replacing w by $u_m'(t)$ in (3.1) and noting that $h(0)=0$, we get

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big\{\frac{1}{2}\|u_m'(t)\|^2+\frac{1}{\gamma+2}\|u_m(t)\|_{\gamma+2}^{\gamma+2}+\frac{1}{2}\int_\Omega a(x)|\nabla u_m'(x,t)|^2\,\mathrm{d}x\Big\} + \|u_m'(t)\|_{\Gamma_0}^2+(b^m(u_m(t)),u_m'(t))_{\Gamma_0} = \frac{\mathrm{d}}{\mathrm{dt}}\int_0^t h(t-\tau)(f(u_m(\tau)),u_m(t))_{\Gamma_0}\,\mathrm{d}\tau - \int_0^t h'(t-\tau)(f(u_m(\tau)),u_m(t))_{\Gamma_0}\,\mathrm{d}\tau. \tag{3.3}$$

Assumption (A_2) and Eq. (2.1) yield that

$$-(b^m(u_m(t)),u_m'(t))_{\Gamma_0}\leqslant\frac{1}{2}\|u_m'(t)\|_{\Gamma_0}^2+C(1+\|\nabla u_m(t)\|^2). \tag{3.4}$$

Here and in the sequel C denotes generic constants independent of m. By assumption (A_3) and Eq. (2.1), we get

$$(f(u_m(\tau)),u_m(t))_{\Gamma_0}\leqslant C(1+\|\nabla u_m(\tau)\|^2+\|\nabla u_m(t)\|^2).$$

Thus

$$-\int_0^t h'(t-\tau)(f(u_m(\tau)),u_m(t))_{\Gamma_0}\,\mathrm{d}\tau \leqslant C\Big(\|h'\|_{L^1(0,\infty)}+\int_0^t|h'(t-\tau)|\|\nabla u_m(\tau)\|^2\,\mathrm{d}\tau+\|h'\|_{L^1(0,\infty)}\|\nabla u_m(t)\|^2\Big). \tag{3.5}$$

Combining estimates (3.3)–(3.5), integrating over $(0,t)$ and noting that $a_0\leqslant a(x)\leqslant a_1$, we get

$$\frac{1}{2}\|u_m'(t)\|^2+\frac{1}{\gamma+2}\|u_m(t)\|_{\gamma+2}^{\gamma+2}+\frac{a_0}{2}\|\nabla u_m(t)\|^2+\frac{1}{2}\int_0^t\|u_m'(s)\|_{\Gamma_0}^2\,\mathrm{d}s \leqslant\frac{1}{2}\|u_{1m}\|^2+\frac{1}{\gamma+2}\|u_{0m}\|_{\gamma+2}^{\gamma+2}+\frac{a_1}{2}\|\nabla u_{0m}\|^2+\int_0^t h(t-\tau)(f(u_m(\tau)),u_m(t))_{\Gamma_0}\,\mathrm{d}\tau + C\int_0^t\Big(1+\|\nabla u_m(s)\|^2+\int_0^s|h'(s-\tau)|\|\nabla u_m(\tau)\|^2\,\mathrm{d}\tau\Big)\,\mathrm{d}s.$$

Since

$$\int_0^t h(t-\tau)(f(u_m(\tau)),u_m(t))_{\Gamma_0}\,\mathrm{d}\tau \leqslant C\Big\{\frac{1}{2\eta}\|h\|_{L^1(0,\infty)}+\frac{1}{2\eta}\int_0^t h(t-\tau)\|\nabla u_m(\tau)\|^2\,\mathrm{d}\tau+\eta\|h\|_{L^1(0,\infty)}\|\nabla u_m(t)\|^2\Big\},$$

choosing η sufficiently small and employing Gronwall's lemma we conclude that

$$\|u_m'(t)\|^2 + \|u_m(t)\|_{\gamma+2}^{\gamma+2} + \|\nabla u_m(t)\|^2 + \int_0^t \|u_m'(s)\|_{\Gamma_0}^2\,\mathrm{d}s \leqslant L_1, \tag{3.6}$$

where L_1 is a positive constant independent of $m \in \mathbb{N}$. Moreover, from assumptions (A_2)–(A_3) and Eq. (2.1) we get

$$\int_0^t \|b^m(u_m(s))\|_{\Gamma_0}^2\,\mathrm{d}s + \int_0^t \|f(u_m(s))\|_{\Gamma_0}^2\,\mathrm{d}s \leqslant L_2, \tag{3.7}$$

where L_2 is a positive constant independent of $m \in \mathbb{N}$.

Next, taking into consideration that the injection $V \hookrightarrow L^{2(\gamma+1)}(\Omega)$ is continuous and using Eq. (2.1), we obtain from (3.1), (3.6) and (3.7) that

$$\int_0^t \|u_m''(s)\|_{V^*}^2\,\mathrm{d}s \leqslant L_3, \tag{3.8}$$

where L_3 is a positive constant independent of $m \in \mathbb{N}$.

Step 2: Passage to the limit.

From the a priori estimates (3.6)–(3.8) we have subsequences (in the sequel we denote subsequences by the same symbols as the original sequences) such that

$$u_m \to u \quad \text{weakly star in } L^\infty(0,T;V), \tag{3.9}$$

$$u_m' \to u' \quad \text{weakly star in } L^\infty(0,T;L^2(\Omega)), \tag{3.10}$$

$$u_m' \to u' \quad \text{weakly in } L^2(0,T;L^2(\Gamma_0)), \tag{3.11}$$

$$u_m'' \to u'' \quad \text{weakly in } L^2(0,T;V^*), \tag{3.12}$$

$$f(u_m) \to \chi_1 \quad \text{weakly in } L^2(0,T;L^2(\Gamma_0)), \tag{3.13}$$

$$b^m(u_m) \to \Xi \quad \text{weakly in } L^2(0,T;L^2(\Gamma_0)). \tag{3.14}$$

Using (3.9) and the fact that the imbedding $V \hookrightarrow L^{2(\gamma+1)}(\Omega)$ $(0 < \gamma \leqslant 2/(n-2)$ if $n \geqslant 3$ and $\gamma > 2$ if $n = 2)$ is continuous, we get

$$\||u_m|^\gamma u_m\|_{L^2(0,T;L^2(\Omega))}^2 = \int_0^T \int_\Omega |u_m(t)|^{2(\gamma+1)}\,\mathrm{d}x\,\mathrm{d}t \leqslant C.$$

This implies

$$|u_m|^\gamma u_m \to \chi \text{ weakly in } L^2(0,T;L^2(\Omega)). \tag{3.15}$$

On the other hand, considering that the imbedding $V \hookrightarrow L^2(\Omega)$ is compact and making use of the Aubin-Lions theorem [7], we arrive at

$$u_m \to u \quad \text{strongly in } L^2(0,T;L^2(\Omega)).$$

Then $u_m(x,t) \to u(x,t)$ a.e. in $\Omega \times (0,T)$ and thus $|u_m(x,t)|^\gamma u_m(x,t) \to |u(x,t)|^\gamma \times u(x,t)$ a.e. in $\Omega \times (0,T)$. Therefore we conclude from (3.15) that $\chi(x,t) = |u(x,t)|^\gamma \times u(x,t)$ a.e. in $\Omega \times (0,T)$. Now, we can take the limit $m \to \infty$ in Eq. (3.1). Therefore we obtain

$$\tag{3.16} \begin{aligned} \int_0^T \{\langle u'', v\rangle + a(u(t),v) + (|u(t)|^\gamma u(t), v) + (u'(t),v)_{\Gamma_0} + (\Xi, v)_{\Gamma_0}\}\,\mathrm{d}t \\ = \int_0^T \int_0^t h(t-\tau)(\chi_1, v)_{\Gamma_0}\,\mathrm{d}\tau\,\mathrm{d}t, \quad \forall v \in V. \end{aligned}$$

Step 3: (u, χ_1, Ξ) *is a solution of problem* (1.1)–(1.5).

First, we show that $f(u) = \chi_1$ in $L^2(0,T;L^2(\Gamma_0))$. Considering that the imbedding $V \hookrightarrow L^2(\Gamma_0)$ is continuous and compact and using the Aubin-Lions compactness lemma, we get from Eqs. (3.9) and (3.11) that

$$u_m \to u \quad \text{strongly in } L^2(0,T;L^2(\Gamma_0)).$$

Thus, $u_m(x,t) \to u(x,t)$ a.e. on $\Gamma_0 \times (0,T)$. Since f is continuous, we get

$$f(u_m(x,t)) \to f_1(u(x,t)) \quad \text{a.e. on } \Gamma_0 \times (0,T).$$

Combining this result and (3.13), we conclude that

$$f(u_m) \to f(u) = \chi_1 \quad \text{weakly in } L^2(0,T;L^2(\Gamma_0)).$$

It remains to prove that $\Xi \in \varphi(u(x,t))$ for a.e. $(x,t) \in \Gamma_0 \times (0,T)$. Since $u_m(x,t) \to u(x,t)$ a.e. on $\Sigma_0 := \Gamma_0 \times (0,T)$, using the theorems of Lusin and Egoroff, for a given $\eta > 0$ we can choose a subset $\omega \subset \Sigma_0$ such that $\operatorname{meas}(\omega) < \eta$ and $u_m \to u$ uniformly on $\Sigma_0 \setminus \omega$. Thus, for each $\varepsilon > 0$, there is an $N > 2/\varepsilon$ such that

$$\tag{3.17} |u_m(x,t) - u(x,t)| < \frac{\varepsilon}{2}, \quad \forall (x,t) \in \Sigma_0 \setminus \omega, \quad \forall m > N.$$

By the definition of b^m, we have

$$\operatorname{ess}\inf_{|s|\leqslant 1/m} b(t-s) \leqslant b^m(t) \leqslant \operatorname{ess}\sup_{|s|\leqslant 1/m} b(t-s).$$

So, we get from (3.17)

$$b^m(u_m(x,t)) \leqslant \operatorname{ess}\sup_{|u_m-s|\leqslant 1/m} b(s) \leqslant \operatorname{ess}\sup_{|u_m-s|<\varepsilon/2} b(s)$$
$$\leqslant \operatorname{ess}\sup_{|u-s|<\varepsilon} b(s) = \overline{b_\varepsilon}(u(x,t)),\ \ \forall m > N,\ \ \forall (x,t) \in \Sigma_0 \setminus \omega.$$

Similarly, we have

$$b^m(u_m(x,t)) \geqslant \underline{b_\varepsilon}(u(x,t)),\ \ \forall m > N,\ \ \forall (x,t) \in \Sigma_0 \setminus \omega.$$

Let $\varphi \in L^\infty(\Sigma_0), \varphi \geqslant 0$. Then

$$\int_{\Sigma_0\setminus\omega} \underline{b}_\varepsilon(u(x,t))\varphi(x,t)\,\mathrm{d}\Gamma\,\mathrm{d}t \leqslant \int_{\Sigma_0\setminus\omega} b^m(u_m(x,t))\varphi(x,t)\,\mathrm{d}\Gamma\,\mathrm{d}t$$
$$\leqslant \int_{\Sigma_0\setminus\omega} \overline{b}_\varepsilon(u(x,t))\varphi(x,t)\,\mathrm{d}\Gamma\,\mathrm{d}t.$$

Letting $m \to \infty$ in (3.18) and using (3.14), we obtain

$$\int_{\Sigma_0\setminus\omega} \underline{b}_\varepsilon(u(x,t))\varphi(x,t)\,\mathrm{d}\Gamma\,\mathrm{d}t \leqslant \int_{\Sigma_0\setminus\omega} \Xi(x,t)\varphi(x,t)\,\mathrm{d}\Gamma\,\mathrm{d}t$$
$$\leqslant \int_{\Sigma_0\setminus\omega} \overline{b}_\varepsilon(u(x,t))\varphi(x,t)\,\mathrm{d}\Gamma\,\mathrm{d}t. \tag{3.19}$$

Letting $\varepsilon \to 0^+$ in (3.19), we infer that

$$\Xi(x,t) \in \varphi(u(x,t)) \quad \text{a.e. in } \Sigma_0 \setminus \omega,$$

and letting $\eta \to 0^+$ we get

$$\Xi(x,t) \in \varphi(u(x,t)) \quad \text{a.e. } (x,t) \in \Sigma_0.$$

The proof of Theorem 2.1 is completed.

4. Energy decay of solutions

In this section we prove Theorem 2.2. The existence part of solutions in Theorem 2.2 is a consequence of the proof of Theorem 2.1. Thus, we prove the uniform decay for solutions of (1.1)–(1.5). For the rest of this section, let x_0 be a fixed point in $\mathbb{R}^n$. Then, consider

$$\beta(x) = x - x_0, \ \ R = \max_{x \in \overline{\Omega}} |x - x_0|$$

and a partition of the boundary Γ into two pieces

$$\Gamma_0 = \{x \in \Gamma\colon \ \beta(x) \cdot \nu(x) \geqslant \delta > 0\} \quad \text{and} \quad \Gamma_1 = \{x \in \Gamma\colon \ \beta(x) \cdot \nu(x) \leqslant 0\}.$$

Furthermore, we assume that $\|\nabla a\|_\infty / a_0 \leqslant \mu$, where μ is the constant satisfying $(\mathrm{A}_2)^*$. We define the energy $E(t)$ of the problem (1.1)–(1.5) by

$$E(t) = \frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\int_\Omega a(x)|\nabla u(x,t)|^2 \,\mathrm{d}x + \frac{1}{\gamma+2}\|u(t)\|_{\gamma+2}^{\gamma+2}. \tag{4.1}$$

To prove the decay property, we first establish uniform estimates for the approximated energy

$$E_m(t) = \frac{1}{2}\|u_m'(t)\|^2 + \frac{1}{2}\int_\Omega a(x)|\nabla u_m(x,t)|^2 \,\mathrm{d}x + \frac{1}{\gamma+2}\|u_m(t)\|_{\gamma+2}^{\gamma+2} \tag{4.2}$$

and then pass to the limit.

Direct computation and the fact $h(0) = 0$ show that

$$\begin{aligned} E_m'(t) = & -\|u_m'(t)\|_{\Gamma_0}^2 - (b^m(u_m(t)), u_m'(t))_{\Gamma_0} \\ & - \frac{1}{2}(h \,\Box\, u_m)'(t) + \frac{1}{2}(h' \,\Box\, u_m)(t) \\ & + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\Big\{\Big(\int_0^t h(s)\,\mathrm{d}s\Big)\|u_m(t)\|_{\Gamma_0}^2\Big\} - \frac{1}{2}h(t)\|u_m(t)\|_{\Gamma_0}^2 \end{aligned} \tag{4.3}$$

where

$$(h \,\Box\, u_m)(t) = \int_0^t h(t-\tau)\|f(u_m(\tau)) - u_m(t)\|_{\Gamma_0}^2 \,\mathrm{d}\tau.$$

Define the modified energy by

$$\begin{aligned} e_m(t) = & \frac{1}{2}\|u_m'(t)\|^2 + \frac{1}{2}\int_\Omega a(x)|\nabla u_m(x,t)|^2 \,\mathrm{d}x + \frac{1}{\gamma+2}\|u_m(t)\|_{\gamma+2}^{\gamma+2} \\ & - \frac{1}{2}\Big(\int_0^t h(s)\,\mathrm{d}s\Big)\|u_m(t)\|_{\Gamma_0}^2. \end{aligned} \tag{4.4}$$

Then, it is easily shown that

$$E_m(t) \leqslant l^{-1} e_m(t), \ \ \forall t \geqslant 0. \tag{4.5}$$

Indeed, by virtue of Eq. (2.1), for $0 < l < 1$ and $a(x) \geqslant a_0$, we have

$$\begin{aligned} e_m(t) \geqslant \frac{1}{2}\|u_m'(t)\|^2 + \frac{1}{2}\left(1 - \frac{\lambda}{a_0}\int_0^\infty h(s)\,\mathrm{d}s\right)\int_\Omega a(x)|\nabla u_m(x,t)|^2\,\mathrm{d}x \\ + \frac{1}{\gamma+2}\|u_m(t)\|_{\gamma+2}^{\gamma+2} \geqslant lE_m(t). \end{aligned}$$

Therefore it is enough to obtain the desired exponential decay for the modified energy $e_m(t)$, which will be done below. On the other hand, considering assumptions $(\mathrm{A}_1), (\mathrm{A}_2)^*, (\mathrm{A}_3)$ and (A_4) it follows from (4.3)–(4.5) that

$$\begin{aligned} e_m'(t) = &- \|u_m'(t)\|_{\Gamma_0}^2 - (b^m(u_m(t)), u_m'(t))_{\Gamma_0} + \frac{1}{2}(h' \,\square\, u_m)(t) - \frac{1}{2}h(t)\|u_m(t)\|_{\Gamma_0}^2 \\ \leqslant &- (1 - \frac{\mu}{2})\|u_m'(t)\|_{\Gamma_0}^2 + \frac{\mu\lambda}{2}\|\nabla u_m(t)\|^2 - \frac{\xi_2}{2}(h \,\square\, u_m)(t) - \frac{1}{2}h(t)\|u_m(t)\|_{\Gamma_0}^2 \\ \leqslant &- \frac{1}{2}\|u_m'(t)\|_{\Gamma_0}^2 + \frac{\mu\lambda}{2a_0}\int_\Omega a(x)|\nabla u_m(x,t)|^2\,\mathrm{d}x - \frac{\xi_2}{2}(h \,\square\, u_m)(t) \\ \leqslant &\, C(\mu)l^{-1}e_m(t) - \frac{1}{2}\|u_m'(t)\|_{\Gamma_0}^2 - \frac{\xi_2}{2}(h \,\square\, u_m)(t), \ \ \forall t \geqslant t_0, \end{aligned}$$

where $C(\mu) = (\lambda/a_0)\mu$. For every $\varepsilon > 0$ let us define the perturbed modified energy by

$$e_{m\varepsilon}(t) = e_m(t) + \varepsilon\psi_m(t),$$

where $\psi_m(t) = 2(u_m'(t), (\beta\cdot\nabla u_m)(t)) + (n-1)(u_m'(t), u_m(t))$.

Proposition 4.1. *There exists $C_1 > 0$ such that for each $\varepsilon > 0$,*

$$|e_{m\varepsilon}(t) - e_m(t)| \leqslant \varepsilon C_1 e_m(t), \ \ \forall t \geqslant 0.$$

P r o o f. Applying Eq. (2.1), Cauchy-Schwarz's inequality and inequality (4.5), we have

$$\begin{aligned} |\psi_m(t)| &\leqslant 2R\|u_m'(t)\|\|\nabla u_m(t)\| + (n-1)\|u_m'(t)\|\|u_m(t)\| \\ &\leqslant C\|u_m'(t)\|\|\nabla u_m(t)\| \leqslant Cl^{-1}e_m(t). \end{aligned}$$

Taking $C_1 = Cl^{-1}$, we have

$$|e_{m\varepsilon}(t) - e_m(t)| = \varepsilon|\psi_m(t)| \leqslant \varepsilon C_1 e_m(t).$$

□

Proposition 4.2. *There exist $C_2 > 0$ and $\varepsilon_1 > 0$ such that for each $\varepsilon \in (0, \varepsilon_1]$,*

$$e'_{m\varepsilon}(t) \leqslant -C_2 e_m(t), \quad \forall t \geqslant t_0.$$

Proof. Using problem (1.1)–(1.5) and Eq. (2.1), we calculate

$$\begin{aligned}\psi_m(t) &= 2(u''_m(t), (\beta\cdot\nabla u_m)(t)) + 2(u'_m(t), (\beta\cdot\nabla u'_m)(t)) \\ &\quad + (n-1)(u''_m(t), u_m(t)) + (n-1)\|u'_m(t)\|^2 \\ &= 2(\operatorname{div}(a\nabla u_m(t)), (\beta\cdot\nabla u_m)(t)) - 2(|u_m(t)|^\gamma u_m(t), (\beta\cdot\nabla u_m)(t)) \\ &\quad + 2(u'_m(t), (\beta\cdot\nabla u'_m)(t)) + (n-1)(\operatorname{div}(a\nabla u_m(t)), u_m(t)) \\ &\quad - (n-1)(|u_m(t)|^\gamma u_m(t), u_m(t)) + (n-1)\|u'_m(t)\|^2.\end{aligned}$$

Now, we analyze the terms on the right hand side of (4.7).

We have

$$\begin{aligned}(4.8)\quad & 2(\operatorname{div}(a\nabla u_m), (\beta\cdot\nabla u_m)) \\ &= 2\int_\Gamma \nu\cdot(a\nabla u_m)(\beta\cdot\nabla u_m)\,\mathrm{d}\Gamma - \int_\Gamma a(\beta\cdot\nu)|\nabla u_m|^2\,\mathrm{d}\Gamma \\ &\quad + (n-2)\int_\Omega a|\nabla u_m|^2\,\mathrm{d}x + \int_\Omega(\beta\cdot\nabla a)|\nabla u_m|^2\,\mathrm{d}x,\end{aligned}$$

$$\begin{aligned}(4.9)\quad -2(|u_m|^\gamma u_m, (\beta\cdot\nabla u_m)) &= \frac{2n}{\gamma+2}\|u_m\|_{\gamma+2}^{\gamma+2} - \frac{2}{\gamma+2}\int_{\Gamma_0}(\beta\cdot\nu)|u_m|^{\gamma+2}\,\mathrm{d}\Gamma \\ &\leqslant \frac{2n}{\gamma+2}\|u_m\|_{\gamma+2}^{\gamma+2},\end{aligned}$$

where we have used that $\beta\cdot\nu > 0$ on Γ_0,

$$(4.10)\qquad 2(u'_m, (\beta\cdot\nabla u'_m)) = -n\|u'_m\|^2 + \int_{\Gamma_0}(\beta\cdot\nu)|u'_m|^2\,\mathrm{d}\Gamma$$

and

$$\begin{aligned}(4.11)\qquad & (n-1)(\operatorname{div}(a\nabla u_m), u_m) \\ &\leqslant (n-1)\int_0^t h(t-\tau)(f(u_m(\tau)), u_m)_{\Gamma_0}\,\mathrm{d}\tau \\ &\quad - (n-1)\int_\Omega a|\nabla u_m|^2\,\mathrm{d}x - (n-1)(u'_m, u_m)_{\Gamma_0},\end{aligned}$$

where we have used assumption $(A_2)^*$. Combining (4.7)–(4.11), we obtain

(4.12)

$$\begin{aligned}\psi_m'(t) \leqslant & -\int_\Omega a|\nabla u_m(t)|^2\,\mathrm{d}x - \|u_m'(t)\|^2 - \Big(n-1-\frac{2n}{\gamma+2}\Big)\|u_m(t)\|_{\gamma+2}^{\gamma+2} \\ & + \int_\Omega (\beta\cdot\nabla a)|\nabla u_m(t)|^2\,\mathrm{d}x + 2\int_\Gamma \nu\cdot(a\nabla u_m)(t)(\beta\cdot\nabla u_m)(t)\,\mathrm{d}\Gamma \\ & - \int_\Gamma a(\beta\cdot\nu)|\nabla u_m(t)|^2\,\mathrm{d}\Gamma + \int_{\Gamma_0}(\beta\cdot\nu)|u_m'(t)|^2\,\mathrm{d}\Gamma \\ & - (n-1)(u_m'(t),u_m(t))_{\Gamma_0} + (n-1)\int_0^t h(t-\tau)(f(u_m(\tau)),u_m(t))_{\Gamma_0}\,\mathrm{d}\tau.\end{aligned}$$

Since $\beta\cdot\nabla u_m = (\beta\cdot\nu)\partial u_m/\partial\nu$, $|\nabla u_m|^2 = (\partial u_m/\partial\nu)^2$ and $\beta\cdot\nu\leqslant 0$ on Γ_1, we have

$$\begin{aligned} & 2\int_\Gamma \nu\cdot(a\nabla u_m)(\beta\cdot\nabla u_m)\,\mathrm{d}\Gamma - \int_\Gamma a(\beta\cdot\nu)|u_m|^2\,\mathrm{d}\Gamma \\ & = -2\int_{\Gamma_0} u_m'(\beta\cdot\nabla u_m)\,\mathrm{d}\Gamma - 2\int_{\Gamma_0} b^m(u_m)(\beta\cdot\nabla u_m)\,\mathrm{d}\Gamma \\ & \quad + 2\int_0^t h(t-\tau)(f(u_m(\tau)),(\beta\cdot\nabla u_m))_{\Gamma_0}\,\mathrm{d}\tau - \int_{\Gamma_0} a(\beta\cdot\nu)|\nabla u_m|^2\,\mathrm{d}\Gamma.\end{aligned}$$

Thus we get

$$\begin{aligned}(4.13)\qquad \psi_m'(t) \leqslant & -rl^{-1}e_m(t) + \int_\Omega(\beta\cdot\nabla a)|\nabla u_m(t)|^2\,\mathrm{d}x \\ & -2\int_{\Gamma_0} u_m'(t)(\beta\cdot\nabla u_m)(t)\,\mathrm{d}\Gamma - 2\int_{\Gamma_0} b^m(u_m(t))(\beta\cdot\nabla u_m)(t)\,\mathrm{d}\Gamma \\ & + 2\int_0^t h(t-\tau)(f(u_m(\tau)),(\beta\cdot\nabla u_m)(t))_{\Gamma_0}\,\mathrm{d}\tau \\ & - \int_{\Gamma_0} a(x)(\beta\cdot\nu)|\nabla u_m(t)|^2\,\mathrm{d}\Gamma + \int_{\Gamma_0}(\beta\cdot\nu)|\nabla u_m'(t)|^2\,\mathrm{d}\Gamma \\ & - (n-1)(u_m'(t),u_m(t))_{\Gamma_0} + (n-1)\int_0^t h(t-\tau)(f(u_m(\tau)),u_m(t))_{\Gamma_0}\,\mathrm{d}\tau,\end{aligned}$$

where $r=\min\{2,(\gamma+2)(n-1)-2n\}>0$. Next, we are going to analyze the terms on the right hand side of (4.13).

Estimate for $I_1 := \int_\Omega(\beta\cdot\nabla a)|\nabla u_m(t)|^2\,\mathrm{d}x$

Since $a(x)\geqslant a_0>0$ on Ω, we have

$$|I_1| \leqslant \frac{R}{a_0}\|\nabla a\|_\infty \int_\Omega a(x)|\nabla u_m(t)|^2\,\mathrm{d}x \leqslant 2R\mu l^{-1}e_m(t),$$

where we used our assumption $\|\nabla a\|_\infty/a_0\leqslant\mu$.

Estimate for $I_2 := -2\int_{\Gamma_0} u_m'(t)(\beta\cdot\nabla u_m)(t)\,\mathrm{d}\Gamma$
Using the inequality $ab \leqslant \eta a^2 + b^2/4\eta$, we have

$$|I_2| \leqslant \frac{R^2}{\eta}\|u_m'(t)\|_{\Gamma_0}^2 + \eta\|\nabla u_m(t)\|_{\Gamma_0}^2.$$

Estimate for $I_3 := 2\int_0^t h(t-\tau)(f(u_m(\tau)),\beta\cdot\nabla u_m(t))_{\Gamma_0}\,\mathrm{d}\tau$
Analogously, we have

$$|I_3| \leqslant \frac{R^2}{\eta}\left(\int_0^t h(t-\tau)\|f(u_m(\tau))\|_{\Gamma_0}\,\mathrm{d}\tau\right)^2 + \eta\|\nabla u_m(t)\|_{\Gamma_0}^2.$$

Estimate for $I_4 := -2\int_{\Gamma_0} b^m(u_m(t))(\beta\cdot\nabla u_m)(t)\,\mathrm{d}\Gamma$
Using assumption $(\mathrm{A}_2)^*$, we get

$$|I_4| \leqslant 2R\mu\int_{\Gamma_0}|u_m(t)||\nabla u_m(t)|\,\mathrm{d}\Gamma \leqslant \frac{R^2\mu^2}{\eta}\|u_m(t)\|_{\Gamma_0}^2 + \eta\|\nabla u_m(t)\|_{\Gamma_0}^2.$$

Estimate for $I_5 := -\int_{\Gamma_0} a(x)(\beta\cdot\nu)|\nabla u_m(t)|^2\,\mathrm{d}\Gamma$
Using $a(x) \geqslant a_0 > 0$ on Ω and $\beta\cdot\nu \geqslant \delta > 0$ on Γ_0, we have

$$I_5 \leqslant -a_0\delta\|\nabla u_m(t)\|_{\Gamma_0}^2.$$

Estimate for $I_6 := \int_{\Gamma_0}(\beta\cdot\nu)|u_m'(t)|^2\,\mathrm{d}\Gamma$

$$I_6 \leqslant R\|u_m'(t)\|_{\Gamma_0}^2.$$

Estimate for $I_7 := -(n-1)(u_m'(t),u_m(t))_{\Gamma_0}$
Using $a(x) \geqslant a_0 > 0$ on Ω and Eqs. (2.1) and (4.5), we obtain

$$\begin{aligned}|I_7| &\leqslant \frac{(n-1)^2\lambda}{4\eta a_0}\|u_m'(t)\|_{\Gamma_0}^2 + \eta\int_\Omega a(x)|\nabla u_m(t)|^2\,\mathrm{d}x\\ &\leqslant \frac{(n-1)^2\lambda}{4\eta a_0}\|u_m'(t)\|_{\Gamma_0}^2 + 2\eta l^{-1}e_m(t).\end{aligned}$$

Estimate for $I_8 := (n-1)\int_0^t h(t-\tau)(f(u_m(\tau)),u_m(t))_{\Gamma_0}\,\mathrm{d}\tau$
Similarly, we obtain

$$\begin{aligned}|I_8| &\leqslant \frac{(n-1)^2\lambda}{4\eta a_0}\left(\int_0^t h(t-\tau)\|f(u_m(\tau))\|_{\Gamma_0}\,\mathrm{d}\tau\right)^2 + \eta\int_\Omega a(x)|\nabla u_m(t)|^2\,\mathrm{d}x\\ &\leqslant \frac{(n-1)^2\lambda}{4\eta a_0}\left(\int_0^t h(t-\tau)\|f(u_m(\tau))\|_{\Gamma_0}\,\mathrm{d}\tau\right)^2 + 2\eta l^{-1}e_m(t).\end{aligned}$$

Combining (4.13) and the estimates for $I_1 - I_8$, we obtain

(4.14)

$$\begin{aligned}\psi_m'(t) \leqslant &-l^{-1}(r-2R\mu-4\eta)e_m(t)-(a_0\delta-3\eta)\|\nabla u_m(t)\|_{\Gamma_0}^2+M_1(\eta)\|u_m'(t)\|_{\Gamma_0}^2\\ &+\frac{R^2\mu^2}{\eta}\|u_m(t)\|_{\Gamma_0}^2+M_2(\eta)\Big(\int_0^t h(t-\tau)\|f(u_m(\tau))\|_{\Gamma_0}\,\mathrm{d}\tau\Big)^2,\end{aligned}$$

where

$$M_1(\eta)=\frac{(n-1)^2\lambda}{4\eta a_0}+R+\frac{R^2}{\eta}\quad\text{and}\quad M_2(\eta)=\frac{(n-1)^2\lambda}{4\eta a_0}+\frac{R^2}{\eta}.$$

We use the estimate

$$\begin{aligned}&\Big(\int_0^t h(t-\tau)\|f(u_m(\tau))\|_{\Gamma_0}\,\mathrm{d}\tau\Big)^2\\ &\qquad\leqslant 2\|h\|_{L^1(0,\infty)}\Big\{(h\,\Box\,u_m)(t)+\Big(\int_0^t h(t-\tau)\,\mathrm{d}\tau\Big)\|u_m(t)\|_{\Gamma_0}^2\Big\}\end{aligned}$$

to get

$$\begin{aligned}\psi_m'(t) \leqslant &-l^{-1}(r-2R\mu-4\eta)e_m(t)-(a_0\delta-3\eta)\|\nabla u_m(t)\|_{\Gamma_0}^2+M_1(\eta)\|u_m'(t)\|_{\Gamma_0}^2\\ &+\Big(\frac{R^2\mu^2}{\eta}+2\|h\|_{L^1(0,\infty)}^2M_2(\eta)\Big)\|u_m(t)\|_{\Gamma_0}^2+2\|h\|_{L^1(0,\infty)}M_2(\eta)(h\,\Box\,u_m)(t).\end{aligned}$$

Applying the relation

$$\|u_m(t)\|_{\Gamma_0}^2\leqslant\frac{\lambda}{a_0}\int_\Omega a(x)|\nabla u_m(t)|^2\,\mathrm{d}x\leqslant\frac{2\lambda}{a_0}l^{-1}e_m(t)$$

to (4.15), we obtain

$$\begin{aligned}\psi_m'(t)\leqslant &-l^{-1}\Big\{r-2R\mu-4\eta-\frac{2\lambda}{a_0}\Big(\frac{R^2\mu^2}{\eta}+2\|h\|_{L^1(0,\infty)}^2M_2(\eta)\Big)\Big\}e_m(t)\\ &-(a_0\delta-3\eta)\|\nabla u_m(t)\|_{\Gamma_0}^2+M_1(\eta)\|u_m'(t)\|_{\Gamma_0}^2\\ &+2\|h\|_{L^1(0,\infty)}M_2(\eta))(h\,\Box\,u_m)(t).\end{aligned}$$

Choose $\eta, \|h\|_{L^1(0,\infty)}$ and μ sufficiently small such that $a_0\delta-3\eta>0$ and

$$L=r-2R\mu-4\eta-\frac{2\lambda}{a_0}\Big(\frac{R^2\mu^2}{\eta}+2\|h\|_{L^1(0,\infty)}^2M_2(\eta)\Big)>0.$$

From (4.6) and (4.16), we have for all $t \geqslant t_0$

$$e'_{m\varepsilon}(t) = e'_m(t) + \varepsilon\psi'_m(t) \leqslant -l^{-1}(\varepsilon L - C(\mu))e_m(t) - \Big(\frac{1}{2} - \varepsilon M_1(\eta)\Big)\|u'_m(t)\|^2_{\Gamma_0}$$
$$- \Big(\frac{\xi_2}{2} - 2\varepsilon\|h\|_{L^1(0,\infty)}M_2(\eta)\Big)(h \,\Box\, u_m)(t).$$

Define $\varepsilon_1 = \min\{1/(2M_1(\eta)), \xi_2/(4\|h\|_{L^1(0,\infty)}M_2(\eta))\}$ and choose μ sufficiently small such that $C_2 := l^{-1}(\varepsilon L - C(\mu)) > 0$. Then for each $\varepsilon \in (0, \varepsilon_1]$ we have

$$e'_{m\varepsilon}(t) \leqslant -C_2 e_m(t), \quad \forall t \geqslant t_0.$$

□

P r o o f of Theorem 2.2 continued..

Let $\varepsilon_0 = \min\{1/(2C_1), \varepsilon_1\}$ and let us consider $\varepsilon \in (0, \varepsilon_0]$. As we have $\varepsilon < 1/(2C_1)$, we conclude from Proposition 4.1

$$\frac{1}{2}e_m(t) \leqslant e_{m\varepsilon}(t) \leqslant \frac{3}{2}e_m(t). \tag{4.17}$$

By virtue of Proposition 4.2 we get

$$e'_{m\varepsilon}(t) \leqslant -C_2 e_m(t) \leqslant -\frac{2}{3}C_2 e_{m\varepsilon}(t), \quad \forall t \geqslant t_0$$

and

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big[e_{m\varepsilon}(t)\exp\Big(\frac{2}{3}C_2 t\Big)\Big] \leqslant 0, \quad \forall t \geqslant t_0. \tag{4.18}$$

Integrating (4.18) we obtain from inequality (4.17) that

$$e_m(t) \leqslant 3e_m(0)\exp\Big(-\frac{2}{3}C_2 t\Big), \quad \forall t \geqslant t_0. \tag{4.19}$$

Hence (4.5), (4.19) and the fact that $e_m(0) = E_m(0)$ yield

$$E_m(t) \leqslant l^{-1}e_m(t) \leqslant 3E_m(0)l^{-1}\exp\Big(-\frac{2}{3}C_2 t\Big), \quad \forall t \geqslant t_0.$$

On the other hand, from (3.9)–(3.11) it is easy to obtain

$$u_m(t) \to u(t) \quad \text{weakly in } V \text{ for a.e. } t \geqslant 0,$$

and

$$u'_m(t) \to u'(t) \quad \text{weakly in } L^2(\Omega) \text{ for a.e. } t \geqslant 0.$$

Thus, we finally conclude that

$$E(t) \leqslant \liminf_{m\to\infty} E_m(t) \leqslant C_3\exp\Big(-\frac{2}{3}C_2 t\Big) \text{ a.e. } t \geqslant t_0.$$

This completes the proof of Theorem 2.2. □

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