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Guoen Hu; Dachun Yang; Dongyong Yang

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A NEW CHARACTERIZATION OF RBMO(μ) BY JOHN-STRÖMBERG SHARP MAXIMAL FUNCTIONS

Guoen Hu, Henan, Dachun Yang, Beijing, Dongyong Yang, Beijing

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Abstract. Let μ be a nonnegative Radon measure on $\mathbb{R}^d$ which only satisfies $\mu(B(x,r)) \leqslant C_0 r^n$ for all $x \in \mathbb{R}^d$, $r > 0$, with some fixed constants $C_0 > 0$ and $n \in (0,d]$. In this paper, a new characterization for the space RBMO(μ) of Tolsa in terms of the John-Strömberg sharp maximal function is established.

Keywords: non-doubling measure, RBMO(μ), sharp maximal function

MSC 2010: 42B25, 42B35, 43A99

1. Introduction

Let μ be a nonnegative Radon measure on $\mathbb{R}^d$ which only satisfies the *growth condition* that there exist $C_0 > 0$ and $n \in (0,d]$ such that for all $x \in \mathbb{R}^d$ and $r > 0$,

$$\mu(B(x,r)) \leqslant C_0 r^n, \tag{1.1}$$

where $B(x,r)$ is the *open ball according to the usual Euclidean metric with the center at x and the radius r*. Such a measure μ in (1.1) is not necessarily doubling, which is a key assumption in the classical theory of harmonic analysis. Recall that μ is said to be *doubling* if there exists $C > 0$ such that for all $x \in \mathbb{R}^d$ and $r > 0$, $\mu(B(x,2r)) \leqslant C\mu(B(x,r))$. During the recent years, it was shown that many results on the Calderón-Zygmund theory remain valid for non-doubling measures. One of the main motivations for extending the classical theory to the non-doubling context was the solution of several questions related to analytic capacity, like Vitushkin's

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conjecture or Painlevé's problem; see [10], [11], [13] or survey papers [12], [14], [15], [16] for more details.

In [9], Tolsa found a suitable substitute for the classical BMO space in this setting, which is denoted by RBMO(μ). This space is small enough to posses the properties such as the John-Nirenberg inequality and big enough for Calderón-Zygmund operators which are bounded on $L^2(\mu)$ to be also bounded from $L^\infty(\mu)$ into RBMO(μ). It should be pointed out that BMO-type spaces with non-doubling measures were also considered by Mateu, Mattila, Nicolau and Orobitg in [5], as well as by Nazarov, Treil and Volberg in [7]. However, none of them can guarantee both the above mentioned properties at the same time.

The purpose of this paper is to establish a new characterization for RBMO(μ) in terms of the John-Strömberg sharp maximal function. Our result shows that as in the case that μ is the d-dimensional Lebesgue measure, a measurable function f belongs to RBMO(μ) if and only if its John-Strömberg sharp maximal function is in $L^\infty(\mu)$, and the *local integrability of f is superfluous in the definition of* $f \in \mathrm{RBMO}(\mu)$. To state this result more precisely, we first recall some definitions and notation.

By a *cube* $Q \subset \mathbb{R}^d$ we mean a closed cube whose sides are parallel to the axes and centered at some point of suppμ, and we denote its *side length* by $l(Q)$. If $\mu(\mathbb{R}^d) < \infty$, we also regard $\mathbb{R}^d$ as a cube. Let α, β be two positive constants. We say that a cube Q is (α, β)-*doubling* if it satisfies $\mu(\alpha Q) \leqslant \beta\mu(Q)$, where and in what follows, given $\lambda > 0$ and any cube Q, λQ denotes the *cube with the same center as Q whose radius is λ times that of Q*. It was pointed out by Tolsa (see [9, pp. 95–96]) that if $\beta > \alpha^n$, then for any $x \in \operatorname{supp}\mu$ and any $R > 0$ there exists an (α, β)-doubling cube Q centered at x with $l(Q) \geqslant R$, and that if $\beta > \alpha^d$, then for μ-almost every $x \in \mathbb{R}^d$ there exists a sequence of (α, β)-doubling cubes $\{Q_k\}_{k\in\mathbb{N}}$ centered at x with $l(Q_k) \to 0$ as $k \to \infty$. In the sequel, by a *doubling cube* we mean a $(2, \beta_d)$-doubling cube, where β_d is a constant such that $\beta_d > 2^d$.

For any cube Q, let $\widetilde{Q}$ be the *smallest doubling cube which has the form 2^kQ with* $k \in \mathbb{N} \cup \{0\}$. For two cubes $Q_1 \subset Q_2$, set

$$K_{Q_1,Q_2} = 1 + \sum_{k=1}^{N_{Q_1,Q_2}} \frac{\mu(2^kQ_1)}{[l(2^kQ_1)]^n},$$

where N_{Q_1,Q_2} is the first positive integer k such that $l(2^kQ_1) \geqslant l(Q_2)$.

As usual, $L^1_{\mathrm{loc}}(\mu)$ denotes the set of all locally integrable functions with respect to μ. We now recall the definition of RBMO(μ) given by Tolsa in [9].

Definition 1. Let $\varrho \in (1,\infty)$ be fixed. We say that $f \in L^1_{\rm loc}(\mu)$ is in the *space* $\mathrm{RBMO}(\mu)$ if there exists some constant $C_1 \geqslant 0$ such that

$$\sup_Q \frac{1}{\mu(\varrho Q)} \int_Q |f(x) - m_{\widetilde{Q}}(f)| \,\mathrm{d}\mu(x) \leqslant C_1, \tag{1.2}$$

and for any two doubling cubes $Q_1 \subset Q_2$,

$$|m_{Q_1}(f) - m_{Q_2}(f)| \leqslant C_1 K_{Q_1,Q_2}, \tag{1.3}$$

where the supremum is taken over all cubes centered at some point of $\operatorname{supp}\mu$, and $m_Q(f)$ denotes the *mean value of* f *on* Q, that is,

$$m_Q(f) = \frac{1}{\mu(Q)} \int_Q f(y) \,\mathrm{d}\mu(y).$$

The minimal constant C_1 in (1.2) and (1.3) is defined to be the $\mathrm{RBMO}(\mu)$ *norm of* f and denoted by $\|f\|_{\mathrm{RBMO}(\mu)}$.

For a cube Q with $\mu(Q) \neq 0$ and a real-valued μ-measurable function f, we define the *median value of* f *on the cube* Q, denoted by $m_f(Q)$, to be one of the numbers such that

$$\mu(\{y \in Q \colon\ f(y) > m_f(Q)\}) \leqslant \tfrac{1}{2}\mu(Q)$$

and

$$\mu(\{y \in Q \colon\ f(y) < m_f(Q)\}) \leqslant \tfrac{1}{2}\mu(Q).$$

For the case $\mu(Q) = 0$, we define $m_f(Q) = 0$. If f is complex-valued, the *median value* $m_f(Q)$ *of* f is defined by

$$m_f(Q) = m_{\mathrm{Re}(f)}(Q) + \mathrm{i} m_{\mathrm{Im}(f)}(Q),$$

where $\mathrm{i}^2 = -1$.

Let $0 < s < 1$. For any fixed cube Q and μ-measurable function f, we define the *quantity* $m_{0,s;Q}(f)$ by

$$m_{0,s;Q}(f) = \inf\{t > 0 \colon\ \mu(\{y \in Q \colon\ |f(y)| > t\}) < s\mu(\tfrac{3}{2}Q)\}$$

if $\mu(\frac{3}{2}Q) \neq 0$, and $m_{0,s;Q}(f) = 0$ if $\mu(\frac{3}{2}Q) = 0$. The *John-Strömberg sharp maximal function* $M^\sharp_{0,s} f$ for any μ-measurable function f is defined by

$$M^\sharp_{0,s} f(x) = \sup_{Q \ni x} m_{0,s;Q}(f - m_f(\widetilde{Q})) + \sup_{\substack{x \in Q \subset R \\ Q,R \text{ doubling}}} \frac{|m_f(Q) - m_f(R)|}{K_{Q,R}}.$$

For the case that μ is the d-dimensional Lebesgue measure, this sharp maximal operator was introduced by John [3] and then rediscovered by Strömberg [8].

Using $M^\sharp_{0,s}$, we introduce the function space $\mathrm{RBMO}_{0,s}(\mu)$ as follows.

Definition 2. Let $s \in (0,1)$. A μ-measurable function f is said to belong to the *space* $\mathrm{RBMO}_{0,s}(\mu)$ if $M^\sharp_{0,s}f \in L^\infty(\mu)$. Moreover, $\|M^\sharp_{0,s}f\|_{L^\infty(\mu)}$ is defined to be the $\mathrm{RBMO}_{0,s}(\mu)$ *norm of* f and denoted by $\|f\|_{\mathrm{RBMO}_{0,s}(\mu)}$.

The main purpose of this paper is to establish the coincidence between the space $\mathrm{RBMO}(\mu)$ and the space $\mathrm{RBMO}_{0,s}(\mu)$ in a certain range of s.

Theorem 1. *Let* $s \in (0, \beta_d^{-2}/2)$. *The space* $\mathrm{RBMO}(\mu)$ *and the space* $\mathrm{RBMO}_{0,s}(\mu)$ *coincide with equivalent norms.*

Remark 1. If μ is the d-dimensional Lebesgue measure, it was proved by Strömberg in [8] that $\mathrm{RBMO}(\mu) = \mathrm{RBMO}_{0,s}(\mu)$ if and only if $s \in (0, 1/2]$. A crucial ingredient in Strömberg's proof is Lemma 3.6 therein, which heavily depends on the doubling property of the considered measure μ. It is not clear so far if there is a proper substitution of Lemma 3.6 in [8] when μ is a nonnegative Radon measure only satisfying (1.1).

Remark 2. Let μ be an absolutely continuous measure on $\mathbb{R}^d$, namely, such that there exists a weight ω such that $d\mu = \omega\,\mathrm{d}x$. Lerner [4] also established the John-Strömberg characterization of $\mathrm{BMO}(\omega)$ in [5].

We now give some applications of Theorem 1.

Corollary 1. *Let f be a measurable function with respect to μ. If f satisfies* (1.3) *for doubling cubes, then* $f \in \mathrm{RBMO}(\mu)$ *if and only if*

$$\lim_{t\to\infty} \sup_{Q\subset\mathbb{R}^d} \frac{1}{\mu(\frac{3}{2}Q)} \mu(\{y \in Q\colon\ |f(x) - m_f(\widetilde{Q})| > t\}) = 0. \tag{1.4}$$

Let φ be a strictly increasing and nonnegative function on $[0,\infty)$ such that

$$\lim_{t\to\infty} \varphi(t) = \infty.$$

Denote by φ^{-1} the *inverse function of* φ. Notice that for any cube Q,

$$m_{0,s;Q}(f - m_f(\widetilde{Q})) \leqslant \varphi^{-1}\biggl(\frac{1}{s\mu(\frac{3}{2}Q)} \int_Q \varphi(|f(x) - m_f(\widetilde{Q})|)\,\mathrm{d}\mu(x)\biggr).$$

From this and Theorem 1, we immediately deduce the following conclusion.

Corollary 2. *Let f be a measurable function with respect to μ. If f satisfies* (1.3) *for doubling cubes, $\varphi(|f|)$ is μ-locally integrable and*

$$\sup_{Q\subset\mathbb{R}^d}\frac{1}{\mu(\frac{3}{2}Q)}\int_Q \varphi(|f(x)-m_f(\widetilde{Q})|)\,\mathrm{d}\mu(x)<\infty,$$

then $f\in \mathrm{RBMO}(\mu)$.

We remark that Corollary 2 when $\varphi(r)=r^p$ with $p\in(0,1)$ was obtained in [1], which was used to obtain the boundedness of some operators in $\mathrm{RBMO}(\mu)$ and Lebesgue spaces with non-doubling measures; see [1] and [6]. Other typical examples of φ satisfying Corollary 2 are

$$\varphi(t)=\underbrace{\log(\dots\log}_{k}(e^k+t)\dots)$$

with $k\in\mathbb{N}$.

Throughout the paper, we always denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. A constant with subscript such as C_1 does not change in different occurrences. The *symbol* $A\lesssim B$ means that $A\leqslant CB$, and the *symbol* $A\sim B$ means that $A\lesssim B$ and $B\lesssim A$. For a μ-measurable set $E\subset\mathbb{R}^d$, we denote by χ_E the *characteristic function of* E.

2. Proofs of Theorem 1 and Corollary 1

We begin with some preliminary lemmas. The following Lemma 1 and Lemma 2 are special cases of Lemma 2.5 and Lemma 2.3 in [2], respectively. For reader's convenience, we still present some details here.

Lemma 1. *Let $s\in(0,\beta_d^{-1}/2]$ and let Q be a doubling cube. If f is real-valued, then*

$$|m_f(Q)|\leqslant m_{0,s;Q}(f).$$

P r o o f. If f is real-valued and $m_f(Q)\geqslant 0$, we have

$$\{y\in Q\colon\ |f(y)|\geqslant|m_f(Q)|\}=\{y\in Q\colon\ f(y)\geqslant m_f(Q)\}\cup\{y\in Q\colon\ f(y)\leqslant -m_f(Q)\};$$

and if $m_f(Q)<0$, then

$$\{y\in Q\colon\ |f(y)|\geqslant|m_f(Q)|\}=\{y\in Q\colon\ f(y)\geqslant -m_f(Q)\}\cup\{y\in Q\colon\ f(y)\leqslant m_f(Q)\}.$$

Therefore, by the definition of $m_f(Q)$,

$$\mu(\{y \in Q\colon\ |f(y)| \geqslant |m_f(Q)|\}) \geqslant \max\{\mu(\{y \in Q\colon\ f(y) \geqslant m_f(Q)\}), \\ \mu(\{y \in Q\colon\ f(y) \leqslant m_f(Q)\})\} \geqslant \mu(Q)/2.$$

This fact implies that for any $t > 0$ satisfying

$$\mu(\{y \in Q\colon\ |f(y)| > t\}) < s\mu(\tfrac{3}{2}Q),$$

we have that $t \geqslant |m_f(Q)|$; otherwise we have a contradiction

$$\mu(\{y \in Q\colon\ |f(y)| \geqslant |m_f(Q)|\}) < s\mu(2Q) \leqslant \tfrac{1}{2}\mu(Q).$$

Then the desired conclusion follows by taking the infimum over t, which completes the proof of Lemma 1. □

Now for any μ-measurable function f, we define the *doubling local maximal function* $M^d_{0,s}f$ by

$$M^d_{0,s}f(x) = \sup_{Q \ni x,\ Q \text{ doubling}} m_{0,s;Q}(f).$$

Lemma 2. *Let* $s \in (0, \beta_d^{-1})$. *Then for any* $\lambda > 0$,

$$\mu(\{x \in \mathbb{R}^d\colon\ |f(x)| > \lambda\} \leqslant \mu(\{x \in \mathbb{R}^d\colon\ M^d_{0,s}f(x) \geqslant \lambda\}).$$

P r o o f. We first claim that

$$\mu\left(\left\{x \in \mathbb{R}^d\colon\ \chi_{\{y \in \mathbb{R}^d\colon\ |f(y)| > \lambda\}}(x) > \beta_d s\right\}\right) \leqslant \mu\left(\left\{x \in \mathbb{R}^d\colon\ M^d_{0,s}f(x) \geqslant \lambda\right\}\right).$$

In fact, the Lebesgue differentiation theorem tells that for μ-a.e. x such that $|f(x)| > \lambda$, there is a doubling cube Q containing x such that

$$\mu(\{y \in Q\colon\ |f(y)| > \lambda\}) > s\mu(\tfrac{3}{2}Q);$$

while for any $t > m_{0,s;Q}(f)$ we have

$$\mu(\{y \in Q\colon\ |f(y)| > t\}) < s\mu(\tfrac{3}{2}Q).$$

These facts indicate that $m_{0,s;Q}(f) \geqslant \lambda$ and hence $M^d_{0,s}f(x) \geqslant \lambda$.

Observe that for $s \in (0, \beta_d^{-1})$,

$$\{x \in \mathbb{R}^d\colon\ |f(x)| > \lambda\}) \subset \left\{x \in \mathbb{R}^d\colon\ \chi_{\{y \in \mathbb{R}^d\colon\ |f(y)| > \lambda\}}(x) > \beta_d s\right\}.$$

The desired conclusion of Lemma 2 then follows directly. □

Lemma 3. *For any fixed* $q > 0$ *and a real-valued function* $f \in \mathrm{RBMO}_{0,s}(\mu)$*, let* $f_q(x) = f(x)$ *when* $|f(x)| \leqslant q$*, and* $f_q(x) = qf(x)/|f(x)|$ *when* $|f(x)| > q$*. Moreover, for each* $Q \subset \mathbb{R}^d$*, let*

$$m_f^q(Q) = \min(m_f^+(Q), q) - \min(m_f^-(Q), q),$$

where $m_f^+(Q) = \max(m_f(Q), 0)$ *and* $m_f^-(Q) = -\min(m_f(Q), 0)$*. Then for any cubes* Q *and* R*,*

$$|m_f^q(Q) - m_f^q(R)| \leqslant |m_f(Q) - m_f(R)|$$

and

$$|f_q - m_f^q(Q)| \leqslant |f - m_f(Q)|.$$

P r o o f. We only prove the first conclusion of this lemma by similarity. Without loss of generality, we may assume $m_f(Q) < m_f(R)$. We then have the following three cases.

Case 1. $m_f(Q) > 0$ and $m_f(R) > 0$. In this case, $m_f^q(Q) = \min(m_f(Q), q)$ and $m_f^q(R) = \min(m_f(R), q)$. A trivial computation yields to that if $m_f(Q) \geqslant q$ and $m_f(R) \geqslant q$, then $|m_f^q(Q) - m_f^q(R)| = 0$; if $m_f(Q) < q$ and $m_f(R) < q$, then

$$|m_f^q(Q) - m_f^q(R)| = |m_f(Q) - m_f(R)|;$$

and if $m_f(Q) < q \leqslant m_f(R)$, then

$$|m_f^q(Q) - m_f^q(R)| = q - m_f(Q) \leqslant |m_f(Q) - m_f(R)|.$$

Case 2. $m_f(Q) \leqslant 0$ and $m_f(R) \leqslant 0$. In this case, $m_f^q(Q) = -\min(-m_f(Q), q)$ and $m_f^q(R) = -\min(-m_f(R), q)$. Exactly as in Case 1, we also have

$$|m_f^q(Q) - m_f^q(R)| \leqslant |m_f(Q) - m_f(R)|.$$

Case 3. $m_f(Q) \leqslant 0 < m_f(R)$. In this case, $m_f^q(Q) = -\min(-m_f(Q), q)$ and

$$m_f^q(R) = \min(m_f(R), q).$$

Thus, if $-m_f(Q) \geqslant q$ and $m_f(R) \geqslant q$, then $|m_f^q(Q) - m_f^q(R)| = 0$; if $-m_f(Q) < q$ and $m_f(R) < q$, then

$$|m_f^q(Q) - m_f^q(R)| = |m_f(Q) - m_f(R)|;$$

if $m_f(R) < q \leqslant -m_f(Q)$, then

$$|m_f^q(Q) - m_f^q(R)| = q + m_f(R) \leqslant |m_f(Q) - m_f(R)|;$$

and if $-m_f(Q) < q \leqslant m_f(R)$, then

$$|m_f^q(Q) - m_f^q(R)| = q - m_f(Q) \leqslant |m_f(Q) - m_f(R)|.$$

Combining these estimates then leads to the first conclusion of Lemma 3. □

Lemma 4. *For any given $f \in L^1_{\rm loc}(\mu)$, let $\|f\|_\circ$ be defined to be the* minimal constant $C_2 \geqslant 0$ *such that*

$$\sup_{Q\ni x} \frac{1}{\mu(2Q)} \int_Q |f(y) - m_f(\widetilde{Q})| \, \mathrm{d}\mu(y) \leqslant C_2,$$

and for any two doubling cubes $Q \subset R$,

$$|m_f(Q) - m_f(R)| \leqslant C_2 K_{Q,R}.$$

Then $\|\cdot\|_\circ$ is a norm of $\mathrm{RBMO}(\mu)$, *which is equivalent to $\|\cdot\|_{\mathrm{RBMO}(\mu)}$.*

Lemma 4 was established by Tolsa in [9, p. 116]. Based on this, we identify $\|f\|_\circ$ with $\|f\|_{\mathrm{RBMO}(\mu)}$. Moreover, by Remark 2.7 of [9], we can also replace $\mu(2Q)$ by $\mu(\varrho Q)$ with any fixed $\varrho > 1$ in Lemma 4 to obtain other equivalent norms of $\mathrm{RBMO}(\mu)$.

P r o o f o f T h e o r e m 1. It is easy to verify that if $f \in \mathrm{RBMO}(\mu)$, then for any cube Q,

$$m_{0,s;Q}(f - m_f(\widetilde{Q})) \leqslant \frac{s^{-1}}{\mu(\frac{3}{2}Q)} \int_Q |f(x) - m_f(\widetilde{Q})| \, \mathrm{d}\mu(x),$$

and so

$$\|f\|_{\mathrm{RBMO}_{0,s}(\mu)} \lesssim s^{-1} \|f\|_{\mathrm{RBMO}(\mu)}.$$

To see the inverse, if we can prove that for all real-valued functions $f \in \mathrm{RBMO}_{0,s}(\mu)$,

$$\|f\|_{\mathrm{RBMO}(\mu)} \lesssim \|f\|_{\mathrm{RBMO}_{0,s}(\mu)}, \tag{2.1}$$

then for any function $f \in \mathrm{RBMO}_{0,s}(\mu)$ with $f = f_1 + i f_2$, where f_1 and f_2 are the real and the imaginary part of f respectively, since $\|f_1\|_{\mathrm{RBMO}_{0,s}(\mu)}$ and $\|f_2\|_{\mathrm{RBMO}_{0,s}(\mu)}$ are both no more than $\|f\|_{\mathrm{RBMO}_{0,s}(\mu)}$, we then also have that (2.1) holds for any $f \in \mathrm{RBMO}_{0,s}(\mu)$ and this would complete the proof of Theorem 1.

To prove (2.1), if $\|f\|_{\mathrm{RBMO}_{0,s}(\mu)} = 0$, the definition of $\|f\|_{\mathrm{RBMO}_{0,s}(\mu)}$ tells us on the one hand that for any doubling Q and R, $m_f(Q) = m_f(R)$, and on the other hand that $\sup_Q m_{0,s;Q}(f - m_f(\widetilde{Q})) = 0$. Therefore there is a constant Z such that for any doubling cube Q, $m_f(Q) = Z$ and so $m_{0,s;Q}(f - Z) = 0$ for any doubling cube Q. This via Lemma 2 shows that $f(x) = Z$ for μ-almost every $x \in \mathbb{R}^d$, and then $\|f\|_{\mathrm{RBMO}(\mu)} = 0$.

Now we assume $\|f\|_{\mathrm{RBMO}_{0,s}(\mu)} > 0$. For each fixed cube $Q \subset \mathbb{R}^d$, set $Q' = \frac{4}{3}Q$. Let B be a positive constant which will be determined later. Recalling that $s \in (0, \beta_d^{-2}/2)$, we can take $\gamma > \beta_d$ such that $\gamma s < \beta_d^{-1}/2$. For μ-a.e. $x \in Q$ such that

$$|f(x) - m_f(\widetilde{Q})| > B\|f\|_{\mathrm{RBMO}_{0,s}(\mu)},$$

by the Lebesgue differentiation theorem there is a doubling cube Q_x centered at x such that

$$m_{Q_x}(\chi_{\{y\in\mathbb{R}^d\colon |f(y)-m_f(\widetilde{Q})|>B\|f\|_{\mathrm{RBMO}_{0,s}(\mu)}\}}) > \gamma s. \tag{2.2}$$

Moreover, we can suppose that Q_x is the biggest doubling cube satisfying (2.2) with side length $2^{-k}l(Q)$ for some $k \in \mathbb{N}$ and $l(Q_x) \leqslant l(Q)/10$. By Besicovitch's covering theorem, there exists an almost disjoint subfamily $\{Q_i\}_i$ of $\{Q_x\}_x$ such that

$$\{x \in Q\colon |f(x) - m_f(\widetilde{Q})| > B\|f\|_{\mathrm{RBMO}_{0,s}(\mu)}\} \subset \bigcup_i Q_i.$$

Because Q_i satisfies (2.2) and $Q_i \subset Q'$, it is easy to see that

$$\begin{aligned}\sum_i \mu(Q_i) &< \gamma^{-1}s^{-1}\sum_i \mu(\{x \in Q_i\colon |f(x) - m_f(\widetilde{Q})| > B\|f\|_{\mathrm{RBMO}_{0,s}(\mu)}\})\\ &\leqslant \gamma^{-1}s^{-1}\mu(\{x \in Q'\colon |f(x) - m_f(\widetilde{Q})| > B\|f\|_{\mathrm{RBMO}_{0,s}(\mu)}\}).\end{aligned}$$

Notice that the definition of $M^\sharp_{0,s}f(x)$ implies that for any $\varepsilon > 0$,

$$\mu(\{y \in Q'\colon |f(y) - m_f(\widetilde{Q'})| > \|f\|_{\mathrm{RBMO}_{0,s}(\mu)} + \varepsilon\}) < s\mu(\tfrac{3}{2}Q').$$

Therefore, if we can show that there exists a constant $C_3 > 0$ such that

$$|m_f(\widetilde{Q'}) - m_f(\widetilde{Q})| \leqslant C_3\|f\|_{\mathrm{RBMO}_{0,s}(\mu)}, \tag{2.3}$$

by taking $\varepsilon = \|f\|_{\mathrm{RBMO}_{0,s}(\mu)}$ and $B > C_3 + 2$ we then have

$$\begin{aligned}\sum_i \mu(Q_i) &\leqslant \gamma^{-1}s^{-1}\mu(\{x \in Q'\colon |f(x) - m_f(\widetilde{Q'})| > 2\|f\|_{\mathrm{RBMO}_{0,s}(\mu)}\})\\ &< \gamma^{-1}\mu(2Q).\end{aligned} \tag{2.4}$$

We now prove (2.3). In fact, if $l(\widetilde{Q}) \leqslant l(\widetilde{Q'})$, then $\widetilde{Q} \subset 4\widetilde{Q'}$. Setting $Q'' = \widetilde{4\widetilde{Q'}}$, by Lemma 2.1 in [9], we obtain

$$\begin{aligned} |m_f(\widetilde{Q'}) - m_f(Q'')| &\leqslant K_{\widetilde{Q'},Q''} \|f\|_{\mathrm{RBMO}_{0,s}(\mu)} \\ &\lesssim (K_{\widetilde{Q'},4\widetilde{Q'}} + K_{4\widetilde{Q'}Q''}) \|f\|_{\mathrm{RBMO}_{0,s}(\mu)} \\ &\lesssim \|f\|_{\mathrm{RBMO}_{0,s}(\mu)} \end{aligned}$$

and

$$\begin{aligned} |m_f(Q'') - m_f(\widetilde{Q})| \leqslant K_{\widetilde{Q},Q''} \|f\|_{\mathrm{RBMO}_{0,s}(\mu)} &\lesssim K_{Q,Q''} \|f\|_{\mathrm{RBMO}_{0,s}(\mu)} \\ &\lesssim \|f\|_{\mathrm{RBMO}_{0,s}(\mu)}. \end{aligned}$$

We then see that

$$|m_f(\widetilde{Q'}) - m_f(\widetilde{Q})| \leqslant |m_f(\widetilde{Q'}) - m_f(Q'')| + |m_f(Q'') - m_f(\widetilde{Q})| \lesssim \|f\|_{\mathrm{RBMO}_{0,s}(\mu)}.$$

Assume now that $l(\widetilde{Q'}) \leqslant l(\widetilde{Q})$, then $\widetilde{Q'} \subset 4\widetilde{Q}$. There is an integer $m \geqslant 1$ such that $l(\widetilde{Q'}) \geqslant l(2^m Q)/10$, $\widetilde{Q'} \subset 2^m Q \subset 4\widetilde{Q}$. Because $l(\widetilde{Q'}) \sim l(2^m Q)$, another application of Lemma 2.1 in [9] leads to $K_{\widetilde{Q'},2^m Q} \lesssim 1$. Setting $Q_{0,1} = \widetilde{4\widetilde{Q}}$, we then have that

$$K_{\widetilde{Q'},Q_{0,1}} \lesssim K_{\widetilde{Q'},2^m Q} + K_{2^m Q,4\widetilde{Q}} + K_{4\widetilde{Q},Q_{0,1}} \lesssim 1$$

and

$$K_{\widetilde{Q},Q_{0,1}} \lesssim K_{\widetilde{Q},4\widetilde{Q}} + K_{4\widetilde{Q},Q_{0,1}} \lesssim 1.$$

As a consequence,

$$\begin{aligned} |m_f(\widetilde{Q'}) - m_f(\widetilde{Q})| &\leqslant |m_f(\widetilde{Q'}) - m_f(Q_{0,1})| + |m_f(\widetilde{Q}) - m_f(Q_{0,1})| \\ &\leqslant (K_{\widetilde{Q'},Q_{0,1}} + K_{\widetilde{Q},Q_{0,1}}) \|f\|_{\mathrm{RBMO}_{0,s}(\mu)} \\ &\lesssim \|f\|_{\mathrm{RBMO}_{0,s}(\mu)}. \end{aligned}$$

Thus (2.3) holds.

Our next goal is to show that there exists a constant $C_4 > 0$ such that for all i and f,

$$|m_f(Q_i) - m_f(\widetilde{Q})| \leqslant C_4 \|f\|_{\mathrm{RBMO}_{0,s}(\mu)}. \tag{2.5}$$

To prove this, we consider the following three cases.

Case I. If $l(\widetilde{2Q_i}) > 10l(\widetilde{Q})$, then there exists an integer $m \geqslant 1$ such that $\widetilde{Q} \subset 2^m Q_i$ and $l(\widetilde{Q}) \sim l(2^m Q_i) \leqslant l(\widetilde{2Q_i})$. It follows from Lemma 2.1 in [9] that

$$\begin{aligned}|m_f(Q_i) - m_f(\widetilde{Q})| &\leqslant |m_f(Q_i) - m_f(\widetilde{2Q_i})| + |m_f(\widetilde{2Q_i}) - m_f(\widetilde{Q})| \\ &\leqslant (K_{Q_i,\widetilde{2Q_i}} + K_{\widetilde{Q},\widetilde{2Q_i}})\|f\|_{\mathrm{RBMO}_{0,s}(\mu)} \\ &\lesssim (K_{Q_i,\widetilde{2Q_i}} + K_{\widetilde{Q},2^m Q_i} + K_{2^m Q_i,\widetilde{2Q_i}})\|f\|_{\mathrm{RBMO}_{0,s}(\mu)} \\ &\lesssim \|f\|_{\mathrm{RBMO}_{0,s}(\mu)}.\end{aligned}$$

Case II. If $l(Q)/10 < l(\widetilde{2Q_i}) \leqslant 10l(\widetilde{Q})$, we see that $Q \subset 40\widetilde{2Q_i} \subset \widetilde{1600\widetilde{Q}}$. Notice that

$$K_{Q_i,\widetilde{1600\widetilde{Q}}} \lesssim K_{Q_i,40\widetilde{2Q_i}} + K_{40\widetilde{2Q_i},\widetilde{1600\widetilde{Q}}} \lesssim 1 + K_{Q,\widetilde{1600\widetilde{Q}}} \lesssim 1.$$

It consequently follows that

$$|m_f(Q_i) - m_f(\widetilde{Q})| \lesssim (K_{Q_i,\widetilde{1600\widetilde{Q}}} + K_{\widetilde{Q},\widetilde{1600\widetilde{Q}}})\|f\|_{\mathrm{RBMO}_{0,s}(\mu)} \lesssim \|f\|_{\mathrm{RBMO}_{0,s}(\mu)}.$$

Case III. If $l(\widetilde{2Q_i}) \leqslant l(Q)/10$, then for any $\delta > 0$ such that $\gamma s + \delta < \beta_d^{-1}/2$, we have

$$m_{\widetilde{2Q_i}}(\chi_{|f(x)-m_f(\widetilde{Q})|>B\|f\|_{\mathrm{RBMO}_{0,s}(\mu)}}) < \gamma s + \delta$$

by the choice of Q_i, which implies

$$m_{0,\gamma s+\delta;\widetilde{2Q_i}}(f - m_f(\widetilde{Q})) \leqslant B\|f\|_{\mathrm{RBMO}_{0,s}(\mu)}.$$

This fact together with Lemma 1 yields

$$|m_f(\widetilde{2Q_i}) - m_f(\widetilde{Q})| = |m_{f-m_f(\widetilde{Q})}(\widetilde{2Q_i})| \leqslant B\|f\|_{\mathrm{RBMO}_{0,s}(\mu)}.$$

Therefore,

$$\begin{aligned}|m_f(Q_i) - m_f(\widetilde{Q})| &\leqslant |m_f(Q_i) - m_f(\widetilde{2Q_i})| + |m_f(\widetilde{2Q_i}) - m_f(\widetilde{Q})| \\ &\leqslant K_{Q_i,\widetilde{2Q_i}}\|f\|_{\mathrm{RBMO}_{0,s}(\mu)} + B\|f\|_{\mathrm{RBMO}_{0,s}(\mu)} \\ &\lesssim \|f\|_{\mathrm{RBMO}_{0,s}(\mu)}.\end{aligned}$$

We can now conclude the proof of Theorem 1. Let

$$X = \sup_Q \frac{1}{\mu(2Q)} \int_Q |f(x) - m_f(\widetilde{Q})| \,\mathrm{d}\mu(x),$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^d$. The estimates (2.4) and (2.5) now give that due to the fact that Q_i's are doubling, we have

$$\begin{aligned}
&\frac{1}{\mu(2Q)} \int_Q |f(x) - m_f(\widetilde{Q})| \,\mathrm{d}\mu(x) \\
&\quad\leqslant \frac{1}{\mu(2Q)} \int_{Q \setminus \cup_i Q_i} |f(x) - m_f(\widetilde{Q})| \,\mathrm{d}\mu(x) \\
&\qquad + \frac{1}{\mu(2Q)} \sum_i \int_{Q_i} |f(x) - m_f(Q_i)| \,\mathrm{d}\mu(x) + C_4 \|f\|_{\mathrm{RBMO}_{0,s}(\mu)} \\
&\quad\leqslant C\|f\|_{\mathrm{RBMO}_{0,s}(\mu)} + \frac{X}{\mu(2Q)} \sum_i \mu(2Q_i) \\
&\quad\leqslant C\|f\|_{\mathrm{RBMO}_{0,s}(\mu)} + \frac{\beta_d}{\gamma} X,
\end{aligned}$$

where $C > 0$ is independent of f. If $f \in L^\infty(\mu)$, then $X < \infty$ and the last inequality together with $\gamma > \beta_d$ implies that

$$\|f\|_{\mathrm{RBMO}(\mu)} \lesssim \|f\|_{\mathrm{RBMO}_{0,s}(\mu)}.$$

For a general $f \in \mathrm{RBMO}_{0,s}(\mu)$ we consider the function f_q with $q > 0$ in Lemma 3. By repeating the foregoing proof we arrive at

$$\sup_Q \frac{1}{\mu(2Q)} \int_Q |f_q(x) - m_f^q(\widetilde{Q})| \,\mathrm{d}\mu(x) \lesssim \|f\|_{\mathrm{RBMO}_{0,s}(\mu)},$$

which together with Lemma 3 and the Fatou lemma leads to the desired conclusion of Theorem 1. □

P r o o f o f C o r o l l a r y 1. If $f \in \mathrm{RBMO}(\mu)$, then for any cube Q and $t > 0$,

$$\frac{1}{\mu(\frac{3}{2}Q)} \mu(\{y \in Q\colon\ |f - m_f(\widetilde{Q})| > t\}) \leqslant \frac{1}{\mu(\frac{3}{2}Q)t} \int_Q |f(x) - m_f(\widetilde{Q})| \,\mathrm{d}\mu(x) \lesssim \frac{1}{t},$$

and the inequality (1.4) follows directly. To prove sufficiency, we choose $s \in (0, \beta_d^{-2}/2)$. If $f \notin \mathrm{RBMO}(\mu)$, then $f \notin \mathrm{RBMO}_{0,s}(\mu)$ by Theorem 1. Therefore, by (1.3), there exists a sequences of cubes $\{Q_j\}$ such that

$$\lim_{j \to \infty} m_{0,s;Q_j}(f - m_f(\widetilde{Q}_j)) = \infty.$$

Let $A_j = m_{0,s;Q_j}(f - m_f(\widetilde{Q}_j))$. We then have that

$$\mu(\{y \in Q_j\colon\ |f - m_f(\widetilde{Q}_j)| > A_j/2\}) \geqslant s\mu(\tfrac{3}{2}Q_j),$$

which in turn implies that

$$\sup_{Q\subset\mathbb{R}^d}\frac{1}{\mu(\frac{3}{2}Q)}\mu(\{y\in Q\colon\ |f-m_f(\widetilde{Q})|>A_j/2\})\geqslant s.$$

This contradicts with (1.4) and hence, completes the proof of Corollary 1. □

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Authors' addresses: Guoen Hu, Department of Applied Mathematics, University of Information Engineering, P.O. Box 1001-747, Zhengzhou 450002, People's Republic of China, e-mail: `huguoen@eyou.com`; Dachun Yang (Corresponding author), School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, People's Republic of China, e-mail: `dcyang@bnu.edu.cn`; Dongyong Yang, School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, People's Republic of China, e-mail: `dyyang@mail.bnu.edu.cn`.