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Czechoslovak Mathematical Journal, Vol. 58 (2008), No. 4, 1257–1278

Persistent URL: http://dml.cz/dmlcz/140456

Czechoslovak Mathematical Journal, 58 (133) (2008), 1257–1278

UNCONDITIONAL IDEALS OF FINITE RANK OPERATORS

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(Received January 5, 2007)

Abstract. Let X be a Banach space. We give characterizations of when $\mathcal{F}(Y,X)$ is a u-ideal in $\mathcal{W}(Y,X)$ for every Banach space Y in terms of nets of finite rank operators approximating weakly compact operators. Similar characterizations are given for the cases when $\mathcal{F}(X,Y)$ is a u-ideal in $\mathcal{W}(X,Y)$ for every Banach space Y, when $\mathcal{F}(Y,X)$ is a u-ideal in $\mathcal{W}(Y,X^{**})$ for every Banach space Y, and when $\mathcal{F}(Y,X)$ is a u-ideal in $\mathcal{K}(Y,X^{**})$ for every Banach space Y.

Keywords: u-ideals, finite rank, compact, and weakly compact operators, Hahn-Banach extension operators

MSC 2010: 46B20, 46B28

1. Introduction

A closed subspace Z of a Banach space X is an *ideal* in X if the annihilator $Z^{\perp}$ is the kernel of a norm one projection on X^*. A linear operator $\varphi\colon Z^* \to X^*$ is called a *Hahn-Banach extension operator* if $\varphi(z^*)(z) = z^*(z)$ and $\|\varphi(z^*)\| = \|z^*\|$ for every $z \in Z$ and $z^* \in Z^*$. We write $\mathrm{HB}(Z,X)$ for the set of all Hahn-Banach extension operators from Z^* into X^*. It is not difficult to see that $\mathrm{HB}(Z,X) \neq \emptyset$ if and only if Z is an ideal in X. If Z is a subspace of a normed space X, we say that Z is an ideal in X if $\overline{Z}$ is an ideal in $\overline{X}$. The notion of an ideal was introduced and studied by Godefroy, Kalton and Saphar in [5].

The stronger notion of an *unconditional ideal* (u-*ideal* for short) was introduced and studied by Casazza and Kalton in [2]. If Z is an ideal in X such that the corresponding projection P on X^* satisfies $\|I - 2P\| = 1$, then Z is called a u-*ideal* in X. The projection is called a u-*projection* and the corresponding $\varphi \in \mathrm{HB}(Z,X)$ is called an *unconditional* Hahn-Banach extension operator. From Lemma 2.2 and Proposition 3.6 in [5] we can state the following result.

Theorem 1.1 (Godefroy, Kalton and Saphar)**.** *Let X be a Banach space and let Z be a subspace of X. The following statements are equivalent.*

(a) *Z is a u-ideal in X.*

(b) *There exists a Hahn-Banach extension operator $\varphi \in \mathrm{HB}(Z, X)$ such that whenever $\varepsilon > 0$, $x \in X$ and A is a convex subset of Z such that $\varphi^*(x)$ is in the weak*-closure of A then there exists $z \in A$ with $\|x - 2z\| < \|x\| + \varepsilon$.*

(c) *There exists a Hahn-Banach extension operator $\varphi \in \mathrm{HB}(Z, X)$ such that for every $x \in X$ there is a net (z_α) in Z such that $\varphi^*(x) = \lim_\alpha z_\alpha$ in the weak*-topology and $\limsup_\alpha \|x - 2z_\alpha\| \leqslant \|x\|$.*

(d) *For every finite dimensional subspace F of X and every $\varepsilon > 0$ there is a linear map $L\colon F \to Z$ such that*
   (1) *$L(x) = x$ for every $x \in F \cap Z$, and*
   (2) *$\|x - 2L(x)\| \leqslant (1 + \varepsilon)\|x\|$ for every $x \in F$.*

Note that (1) in Theorem 1.1 (d) can be substituted by the inequality $\|L(x) - x\| \leqslant \varepsilon\|x\|$ for every $x \in F \cap Z$. We will sometimes use this fact.

Let X and Y be Banach spaces. We denote by $\mathcal{L}(Y, X)$ the Banach space of bounded linear operators from Y to X, and by $\mathcal{F}(Y, X)$, $\mathcal{K}(Y, X)$ and $\mathcal{W}(Y, X)$ its subspaces of finite rank operators, compact operators and weakly compact operators, respectively.

In Section 2 we show that the set of Hahn-Banach extension operators $\mathrm{HB}(X, Y)$ is a face in the unit ball of $\mathcal{L}(X^*, Y^*)$. We show in Proposition 2.2 that an unconditional Hahn-Banach extension operator has to be a center of symmetry in $\mathrm{HB}(X, Y)$. If X contains a copy of ℓ_1 and is a u-ideal in its bidual, then we show that $\operatorname{diam} \mathrm{HB}(X, X^{**}) = 2$. We also show that in some important cases the set $\mathrm{HB}(X, Y)$ consists of a single element. The subspaces Z of X such that $\varphi^*|_{X^{**}}(Z^{\perp\perp}) \subset Z^{\perp\perp}$ where $\varphi \in \mathrm{HB}(X, X^{**})$ is unconditional are characterized.

In Section 3 we establish in Theorem 3.2 characterizations of the case when $\mathcal{F}(Y, X)$ is a u-ideal in $\mathcal{W}(Y, X)$ for every Banach space Y. The characterizations include a statement similar to Theorem 1.1 (b) involving a Hahn-Banach extension operator, a statement which is an approximation property for X and statements about approximating weakly compact operators by finite rank operators. In Theorem 3.5 we give similar characterizations of the case when $\mathcal{F}(X, Y)$ is a u-ideal in $\mathcal{W}(X, Y)$ for every Banach space Y.

In Section 4 we characterize the property that $\mathcal{F}(Y, X)$ is a u-ideal in $\mathcal{W}(Y, X^{**})$ for every Banach space Y in Theorem 4.3, and the property that $\mathcal{F}(Y, X)$ is a u-ideal in $\mathcal{K}(Y, X^{**})$ for every Banach space Y in Theorem 4.4 (by statements similar to those in Theorems 3.2 and 3.5). An example due to Oja [25, Example 3] shows that the latter property is strictly weaker than the first (see Remark 4.3 below).

We define an unconditional version of the weak metric approximation property. We show by giving an example that this property is strictly weaker than $\mathcal{F}(Y,X)$ being a u-ideal in $\mathcal{K}(Y,X^{**})$ for every Banach space Y.

We will frequently use the isometric version of the Davis-Figiel-Johnson-Pełczyński factorization lemma [3] due to Lima, Nygaard and Oja [16]. Let X be a Banach space and let K be a closed absolutely convex subset of the unit ball B_X of X. If Z is the Banach space constructed from K in the factorization lemma and J is the norm one identity embedding of Z into X (see [16, Lemma 1.1]), we will write

$$[Z,J] = \mathrm{DFJP}(K).$$

From the factorization lemma we know that Z is reflexive if and only if K is weakly compact. The factorization lemma also says that if K is compact, then Z is separable and J is compact.

From the isometric version of the factorization lemma proved by Lima, Nygaard and Oja [16, Theorem 2.3] we get that if $G \subset \mathcal{W}(Y,X)$ is a finite dimensional subspace, then there exist a reflexive Banach space Z, a norm one operator $J\colon Z \to X$ and a linear isometry $\Phi\colon G \to \mathcal{W}(Y,Z)$ such that $T = J \circ \Phi(T)$ for every $T \in G$. We will write

$$[Z,J,\Phi] = \mathrm{DFJP}(G), \tag{1}$$

for this construction. Similarly, using [16, Corollary 2.4], we get that if $G \subset \mathcal{W}(X,Y)$ is a finite dimensional subspace, then there exists a reflexive Banach space Z, a norm one operator $J\colon X \to Z$, and a linear isometry $\Phi\colon G \to \mathcal{W}(Z,Y)$ such that $T = \Phi(T) \circ J$ for every $T \in G$. We will write

$$[Z,\Phi,J] = \mathrm{DFJP}(G) \tag{2}$$

for this construction.

We use standard Banach space notation as used by Lindenstrauss and Tzafriri in [23]. Only real Banach spaces are considered unless otherwise stated. The closed unit ball of a Banach space X is denoted by B_X and the identity operator on X is denoted by I_X. We will write X^* for the dual space of X. If $Z \subset X$ is a subspace of X, then we will write $i_Z\colon Z \to X$ for the canonical embedding. We will write $k_X\colon X \to X^{**}$ for the natural embedding of X into its bidual. The symbol $\operatorname{ext} B_X$ denotes the set of extreme points in B_X. If $T\colon X \to Y$ is an operator and $x \in X$, then we will write Tx instead of $T(x)$ when there is no danger of confusion.

Let us start with a general result about the location and size of the set of Hahn-Banach extension operators.

Proposition 2.1. *Let Y be a Banach space. If X is an ideal in Y, then $\mathrm{HB}(X,Y)$ is a face in $B_{\mathcal{L}(X^*,Y^*)}$.*

Proof. Let φ_1, $\varphi_2 \in B_{\mathcal{L}(X^*,Y^*)}$ and suppose $\varphi = \frac{1}{2}(\varphi_1 + \varphi_2) \in \mathrm{HB}(X,Y)$. We then get that

$$\frac{i_X^*\varphi_1 + i_X^*\varphi_2}{2} = i_X^*\varphi = I_{X^*} \in \operatorname{ext} B_{\mathcal{L}(X^*,X^*)}.$$

Thus $i_X^*\varphi_i = I_{X^*}$ and $\varphi_i \in \mathrm{HB}(X,Y)$ for $i = 1, 2$. □

In Lemma 3.1 in [5] there is an algebraic proof of the fact that an unconditional Hahn-Banach extension operator is unique. Next we have a geometric proof. (Recall that x is a *center of symmetry* in a subset A of a linear space X if $2x - y \in A$ for every $y \in A$.)

Proposition 2.2. *Let X be a u-ideal in Y with unconditional $\varphi \in \mathrm{HB}(X,Y)$. For $x^* \in X^*$, let $\mathrm{HB}(x^*) \subset Y^*$ be the set of norm preserving extensions of x^* to Y. Then $\varphi(x^*)$ is the center of symmetry in $\mathrm{HB}(x^*)$ for every $x^* \in X^*$. In particular, the unconditional Hahn-Banach extension operator φ is unique, and φ is a center of symmetry in $\mathrm{HB}(X,Y)$.*

Proof. Let $y^* \in \mathrm{HB}(x^*)$ and let $P_\varphi = \varphi i_X^*$ be the u-projection. Then $\|x^*\| = \|y^*\| = \|(I - 2P_\varphi)y^*\| = \|y^* - 2\varphi(x^*)\|$ so that $2\varphi(x^*) - y^* \in \mathrm{HB}(x^*)$. Hence $\varphi(x^*)$ is a center of symmetry in $\mathrm{HB}(x^*)$. Since a center of symmetry in a convex bounded set is unique, it follows that there is at most one unconditional extension operator in $\mathrm{HB}(X,Y)$.

If $\psi \in \mathrm{HB}(X,Y)$ and $x^* \in X^*$, then $\psi(x^*) \in \mathrm{HB}(x^*)$. Using the fact that $\varphi(x^*)$ is a center of symmetry in $\mathrm{HB}(x^*)$ we get $2\varphi(x^*) - \psi(x^*) \in \mathrm{HB}(x^*)$. Hence we get $2\varphi - \psi \in \mathrm{HB}(X,Y)$ and φ is a center of symmetry in $\mathrm{HB}(X,Y)$. □

The following result shows that if a Banach space X contains a subspace isomorphic to ℓ_1 and is a u-ideal in its bidual, then the diameter of $\mathrm{HB}(X,X^{**})$ is as large as possible.

Proposition 2.3. *Let X be a Banach space which contains a subspace isomorphic to ℓ_1. If X is a u-ideal in its bidual, then* $\operatorname{diam} \mathrm{HB}(X, X^{**}) = 2$.

P r o o f. Let $\pi = k_{X^*} k_X^*$ and $P_\varphi = \varphi k_X^*$ respectively be the canonical projection and the u-projection on X^{***}. By Proposition 2.2 the unconditional Hahn-Banach extension operator φ is a center of symmetry in $\mathrm{HB}(X, X^{**})$, i.e. $\psi = 2\varphi - k_{X^*} \in \mathrm{HB}(X, X^{**})$. Let $P_\psi = \psi k_X^*$ and note that P_ψ is an ideal projection on X^{***}. By Proposition 2.6 in [5] we have $\|I - 2\pi\| = 3$, so

$$2 \geqslant \|P_\psi - \pi\| = \|2P_\varphi - 2\pi\| \geqslant \|I - 2\pi\| - \|I - 2P\varphi\| = 2.$$

Hence $\|\psi - k_{X^*}\| = \|P_\psi - \pi\| = 2$, so $\operatorname{diam} \mathrm{HB}(X, X^{**}) = 2$. □

Note that the proof of Proposition 1 in [1] shows that if a non-reflexive Banach space X is 1 -complemented in its bidual by a projection P, then $\mathrm{HB}(X, X^{**})$ consists of at least two elements.

One direction of the following theorem was proved for separable h-ideals in [5, Theorem 6.7]. Our argument, just as the proof of Theorem 6.7 in [5], is based on an application of Theorem 1.1 (b).

Theorem 2.4. *Let X be a Banach space. Assume that X is a u-ideal in X^{**} with unconditional $\varphi \in \mathrm{HB}(X, X^{**})$. Let Z be a closed subspace of X. Then $\varphi^*(Z^{\perp\perp}) \subset Z^{\perp\perp}$ if and only if Z is a u-ideal in Z^{**} with an unconditional Hahn-Banach extension operator $\psi \in \mathrm{HB}(Z, Z^{**})$ such that $i_Z^{**}\psi^*|_{Z^{**}} = \varphi^* i_Z^{**}$.*

P r o o f. Suppose $\varphi^*(Z^{\perp\perp}) \subset Z^{\perp\perp}$. $i_Z \colon Z \to X$ is the natural embedding, so i_Z^* is the restriction and i_Z^{**} is weak*-weak* continuous, isometric, and onto $Z^{\perp\perp}$.

Define $\psi \colon Z^* \to Z^{***}$ by

$$\psi(z^*) = \psi(x^* + Z^\perp) = i_Z^{***}\varphi(x^*)$$

for $z^* = x^* + Z^\perp \in Z^*$. Since $i_Z^{**}(Z^{**}) \subset Z^{\perp\perp}$ we get that ψ is well-defined:

$$\langle \psi(z^*), z^{**}\rangle = \langle x^* + Z^\perp, \varphi^*(i_Z^{**}(z^{**}))\rangle = \langle x^*, \varphi^*(i_Z^{**}(z^{**}))\rangle = \langle i_Z^{***}\varphi(x^*), z^{**}\rangle$$

for $z^{**} \in Z^{**}$. Thus we have $\psi(i_Z^*(x^*)) = i_Z^{***}\varphi(x^*)$ for all $x^* \in X^*$. Taking adjoints we get $i_Z^{**}\psi^*|_{Z^{**}} = \varphi^* i_Z^{**}$.

Let us show that ψ is an unconditional Hahn-Banach extension operator. Clearly ψ is linear with norm at most one. For $z \in Z$ and $z^* = x^* + Z^\perp \in Z^*$ we have

$$\psi(z^*)(z) = \langle \varphi(x^*), i_Z(z)\rangle = \langle x^*, i_Z(z)\rangle = z^*(z).$$

Let $z^{**} \in B_{Z^{**}}$ and $\varepsilon > 0$. Since X is a u-ideal in X^{**} and $\varphi^*(i_Z^{**}(z^{**}))$ is in the w*-closure of B_Z in X^{**} there exists a $z \in B_Z$ such that

$$\|z^{**} - 2z\| = \|i_Z^{**}(z^{**}) - 2i_Z(z)\| < \|z^{**}\| + \varepsilon$$

by Theorem 1.1 (b). Thus there is a net $(z_\alpha) \subset B_Z$ with $\limsup_\alpha \|z^{**} - 2z_\alpha\| \leqslant \|z^{**}\|$ such that $z_\alpha \to \psi^*(z^{**})$ weak* in Z^{**} (here we have used $i_Z^{**}\psi^*|_{Z^{**}} = \varphi^* i_Z^{**}$). Hence $\|z^{**} - 2k_Z^{**}(\psi(z^{**}))\| \leqslant \|z^{**}\|$ and ψ is unconditional.

For the converse assume that Z is a u-ideal in Z^{**} with an unconditional $\psi \in \mathrm{HB}(Z, Z^{**})$ such that $i_Z^{**}\psi^*|_{Z^{**}} = \varphi^* i_Z^{**}$. Let $x^{**} \in Z^{\perp\perp}$ in X^{**} and choose $z^{**} \in Z^{**}$ such that $i_Z^{**}(z^{**}) = x^{**}$; then $\varphi^*(x^{**}) = i_Z^{**}(\psi^* z^{**}) \in Z^{\perp\perp}$. $\square$

Recall that a Banach space X is said to have the *approximation property* (AP) if there exists a net $(S_\alpha) \subset \mathcal{F}(X, X)$ such that $S_\alpha \to I_X$ uniformly on compact sets in X. Lima, Nygaard and Oja have proved [16, Theorem 3.3] that a Banach space X has the AP if and only if the set $\mathrm{HB}(\mathcal{F}(Y, X), \mathcal{W}(Y, X))$ of Hahn-Banach extension operators is non-empty for every Banach space Y.

In some cases the set of Hahn-Banach extension operators consists of a single element. For example, if X is an M-ideal in a Banach space Y, then $\mathrm{HB}(X, Y)$ contains a single element (see [7, Proposition 1.2]; cf. [7, p. 1] for definition of an M-ideal). A Banach space X such that $\mathrm{HB}(X, X^{**})$ consists of a single element is said to have the *unique extension property (UEP)*. This notion was introduced and studied by Godefroy and Saphar in [6]. They proved in [6, Corollary 5.4] that if X and Y are Banach spaces such that X is reflexive and Y^* has the Radon-Nikodým property (RNP) and contains no proper norming subspace, then $X \otimes_\varepsilon Y$ and $\mathcal{K}(X, Y)$ have the UEP. (Recall that a subspace Z of Y^* is *norming* if $\|y\| = \sup\{y^*(y) \colon y^* \in Z, \|y^*\| \leqslant 1\}$ for $y \in Y$.)

From [24] we also know that $\mathrm{HB}(\mathcal{F}(Y, X), \mathcal{L}(Y, X))$ contains a single element for every Banach space Y whenever X is either ℓ_p or the Lorentz sequence space $d(\omega, p)$ for $1 < p < \infty$ (see also [7, Example 4.1] for the case $X = \ell_p$ and $Y = \ell_q$ where $1 < q \leqslant p < \infty$). Dually we also have that $\mathrm{HB}(\mathcal{F}(X, Y), \mathcal{L}(X, Y))$ contains a single element for every Y whenever X is either ℓ_p or $d(\omega, p)^*$ for $1 < p < \infty$. From [26, Theorem 3] we have in addition that the above holds if X is a closed subspace of either ℓ_p, $d(\omega, p)$ or $d(\omega, p)^*$ with the AP. Also the set $\mathrm{HB}(\mathcal{F}(Y, c_0), \mathcal{L}(Y, c_0))$ consists of a single element for every Banach space Y ($\mathcal{F}(Y, c_0)$ is an M-ideal in $\mathcal{L}(Y, c_0)$, see [7, Example 4.1]). The next results tell us that in many more cases the set of Hahn-Banach extension operators consists of a single element.

Proposition 2.5. *Let X and Y be Banach spaces. If X has the AP and Y is reflexive, then $\mathrm{HB}(\mathcal{F}(Y,X),\mathcal{W}(Y,X))$ consists of one element only.*

P r o o f. Let $\Phi \in \mathrm{HB}(\mathcal{F}(Y,X),\mathcal{W}(Y,X))$, let $x^* \in X^*$ and $y \in B_Y$. Assume that y is a strongly exposed point. Then by Lemma 3.4 in [15] $x^* \otimes y$ has a unique norm-preserving extension from $\mathcal{F}(Y,X)$ to $\mathcal{W}(Y,X)$ and hence $\Phi(x^* \otimes y) = x^* \otimes y$. Since Y has the RNP we get $\Phi(x^* \otimes y)$ for every $x^* \in X^*$ and $y \in Y$ by linearity and continuity. By a theorem of Feder and Saphar [4, Theorem 1] $\mathcal{F}(Y,X)^*$ is a quotient of $X^* \hat{\otimes}_\pi Y$ and it follows that Φ is just the identity and hence unique. $\square$

A Banach space X has the AP if and only if $\mathcal{F}(Y,X)$ is dense in $\mathcal{K}(Y,X)$ for every Banach space Y (cf. e.g. [23, Theorem 1.e.4]). By [17, Theorem 5.1] X has the AP if and only if $\mathcal{F}(Y,X)$ is a (trivially unconditional) ideal in $\mathcal{K}(Y,X)$ for every Banach space Y.

For Y reflexive, we can combine Proposition 2.5 with the isometries $\mathcal{F}(X,Y) = \mathcal{F}(Y^*,X^*)$ and $\mathcal{W}(X,Y) = \mathcal{W}(Y^*,X^*)$ obtaining the following corollary.

Corollary 2.6. *Let X and Y be Banach spaces. If X^* has the AP and Y is reflexive, then $\mathrm{HB}(\mathcal{F}(X,Y),\mathcal{W}(X,Y))$ consists of one element only.*

The dual of a Banach space X has the AP if and only if $\mathcal{F}(X,Y)$ is dense in $K(X,Y)$ for every Banach space Y (cf. e.g. [23, Theorem 1.e.5]). By [17, Theorem 5.2] X^* has the AP if and only if $\mathcal{F}(X,Y)$ is a (trivially unconditional) ideal in $\mathcal{K}(X,Y)$ for every Banach space Y.

3. $\mathcal{F}(Y,X)$ as a u-ideal in $\mathcal{W}(Y,X)$

From [17, Theorem 5.1] and [19, Theorem 4.4] (resp. [19, Theorem 4.3]) we have the following result.

Theorem 3.1 (Lima and Oja). *Let X be a closed subspace of a Banach space Z. Then $\mathcal{F}(Y,X)$ is a u-ideal in $\mathcal{W}(Y,Z)$ (resp. $\mathcal{K}(Y,Z)$) for all Banach spaces Y if and only if $\mathcal{F}(Y,X)$ is a u-ideal in $\mathcal{W}(Y,Z)$ (resp. $\mathcal{K}(Y,Z)$) for all separable, or, respectively, reflexive separable Banach spaces Y.*

The next theorem characterizes the property that $\mathcal{F}(Y,X)$ is a u-ideal in $\mathcal{W}(Y,X)$ for every Banach space Y in terms of convergence of nets of finite rank operators. The statements should be compared with their prototypes in similar results on ideals (see [12, Theorem 5.2] and [20, Theorem 2.3]).

Theorem 3.2. *Let X be a Banach space. The following statements are equivalent.*

(a) *$\mathcal{F}(Y,X)$ is a u-ideal in $\mathcal{W}(Y,X)$ for every Banach space Y.*

(b) *$\mathcal{F}(Y,X)$ is a u-ideal in span$(\mathcal{F}(Y,X),\{T\})$ for every $T\in\mathcal{W}(Y,X)$ and for every reflexive Banach space Y.*

(c) *For every reflexive Banach space Y there exists a Hahn-Banach extension operator $\Psi\in \mathrm{HB}(\mathcal{F}(Y,X),\mathcal{W}(Y,X))$ such that for every $T\in\mathcal{W}(Y,X)$ there is a net $(T_\alpha)\subset\mathcal{F}(Y,X)$ with $\limsup\limits_\alpha\|T-2T_\alpha\|\leqslant\|T\|$ such that $T_\alpha\to\Psi^*(T)=T$ weak* in $\mathcal{F}(Y,X)^{**}$.*

(d) *For every weakly compact set $K\subset X$ there is a net $(S_\alpha)\subset\mathcal{F}(X,X)$ with $\limsup\limits_\alpha\sup\limits_{x\in K}\|x-2S_\alpha x\|\leqslant\sup\limits_{x\in K}\|x\|$ such that $S_\alpha\to I_X$ uniformly on compact subsets of K.*

(e) *For every Banach space Y and $T\in\mathcal{W}(Y,X)$ there is a net $(S_\alpha)\subset\mathcal{F}(X,X)$ with $\limsup\limits_\alpha\|T-2S_\alpha T\|\leqslant\|T\|$ such that $S_\alpha\to I_X$ uniformly on compact sets in X.*

(f) *For every Banach space Y and $T\in\mathcal{W}(Y,X)$ there is a net $(S_\alpha)\subset\mathcal{F}(X,X)$ with $\limsup\limits_\alpha\|T-2S_\alpha T\|\leqslant\|T\|$ such that $S_\alpha\to I_X$ in the strong operator topology.*

(g) *For every reflexive Banach space Y and $T\in\mathcal{W}(Y,X)$ there is a net $(S_\alpha)\subset\mathcal{F}(X,X)$ with $\limsup\limits_\alpha\|T-2S_\alpha T\|\leqslant\|T\|$ such that $S_\alpha T\to T$ in the strong operator topology.*

Proof. (a) $\Rightarrow$ (b) is immediate from the local characterization of u-ideals, Theorem 1.1.

(b) $\Rightarrow$ (c). Assume that Y is reflexive and let $T\in\mathcal{W}(Y,X)$. Since $\mathcal{F}(Y,X)$ is a u-ideal in $\mathcal{B}=\mathrm{span}(\mathcal{F}(Y,X),\{T\})$ we can, using the local characterization of u-ideals in Theorem 1.1, find a net $(T_\alpha)\subset\mathcal{F}(Y,X)$ with $\limsup\limits_\alpha\|T-2T_\alpha\|\leqslant\|T\|$ such that $T_\alpha\to\Phi_T^*(T)$ weak*, where $\Phi_T\in\mathrm{HB}(\mathcal{F}(Y,X),\mathcal{B})$ is the unconditional extension operator. From the argument in the proof of Proposition 2.5 Φ_T is unique and of the form $\Phi_T=I_{X^*}\otimes I_Y$. A straightforward calculation shows that $\Phi_T^*(T)=T$. Thus the operator $\Psi=I_{X^*}\otimes I_Y\in\mathrm{HB}(\mathcal{F}(Y,X),\mathcal{W}(Y,X))$ satisfies (c) in Theorem 1.1.

(c) $\Rightarrow$ (d). Let $K\subset X$ be weakly compact, $\varepsilon>0$, and $u=\sum\limits_{n=1}^\infty x_n^*\otimes x_n\in X^*\hat{\otimes}_\pi X$. Assume that K is a symmetric subset of B_X. Assume also that $1\geqslant\|x_n\|\to 0$ and that $\sum\limits_{n=}^\infty\|x_n^*\|<\infty$. Put $[Z,J]=\mathrm{DFJP}(\overline{\mathrm{conv}}(K\cup(\pm x_n)_{n=1}^\infty))$. Now Z is reflexive, $J\in\mathcal{W}(Z,X)$ and $\|J\|\leqslant 1$. Find $z_n\in B_Z$ such that $x_n=Jz_n$. Choose a net $(J_\alpha)\subset\mathcal{F}(Z,X)$ with $\limsup\limits_\alpha\|J-2J_\alpha\|\leqslant\|J\|$ such that $J_\alpha\to J$ weak* in $\mathcal{F}(Z,X)^{**}$. Since J^*X^* is norm-dense in Z^* [16, Lemma 1.1] we can write $J_\alpha=S_\alpha J$

where $(S_\alpha) \subset \mathcal{F}(X,X)$ (see the proof of [21, Theorem 3.2]). Now we can find an S among the S_α's such that

$$\varepsilon > \left| \sum_{n=1}^{\infty} \langle SJz_n, x_n^* \rangle - \sum_{n=1}^{\infty} \langle Jz_n, x_n^* \rangle \right| = \left| \sum_{n=1}^{\infty} \langle Sx_n, x_n^* \rangle - \sum_{n=1}^{\infty} \langle x_n, x_n^* \rangle \right|$$

and $\sup_{x \in K} \|x - 2Sx\| \leqslant \sup_{z \in B_Z} \|Jz - 2SJz\| \leqslant \|J - 2SJ\| < 1 + \varepsilon$.

(d) $\Rightarrow$ (e). Let Y be a Banach space and let $T \in \mathcal{W}(Y,X)$ be of norm one. Let $C \subset B_X$ be compact and let $\varepsilon > 0$. Define $K = \overline{\mathrm{conv}}(\pm(C \cup T(B_Y)))$ and note that $K \subset B_X$ and is weakly compact. By assumption there is $S \in \mathcal{F}(X,X)$ with $\sup_{x \in K} \|x - 2Sx\| < 1 + \varepsilon$ and $\sup_{x \in C} \|x - Sx\| < \varepsilon$. From this (e) follows.

(e) $\Rightarrow$ (f) and (f) $\Rightarrow$ (g) are trivial.

(g) $\Rightarrow$ (a). Let Y be a Banach space, let $\varepsilon > 0$ and choose a finite dimensional subspace $F \subset \mathcal{W}(Y,X)$. Put $[Z,J,\Phi] = \mathrm{DFJP}(F)$ (see (1), page 3) and let $G = F \cap \mathcal{F}(Y,X)$. Then

$$K = \overline{\bigcup_{T \in B_G} T(B_Y)}$$

is a compact subset of X and of Z. It follows from the assumptions that we can find an $S \in \mathcal{F}(X,X)$ with $\|J - 2SJ\| \leqslant 1 + \varepsilon$ such that $\|z - Sz\| \leqslant \varepsilon$ for every $z \in K$. Define $L\colon F \to \mathcal{F}(Y,X)$ by $L(T) = ST$. Then $\|T - L(T)\| \leqslant \|\Phi(T)\|\|J - SJ\| \leqslant \varepsilon\|T\|$ for every $T \in G$ and $\|T - 2L(T)\| = \|T - 2ST\| \leqslant \|\Phi(T)\|\|J - 2SJ\| \leqslant (1+\varepsilon)\|T\|$ for $T \in F$. The result now follows from local characterization of u-ideals in Theorem 1.1. □

Remark 3.1. Let $\hat{\ell}_2$ be the equivalently re-normed version of ℓ_2 defined by Oja and denoted by F in Example 3 in [25]. The space $\mathcal{F}(\ell_1, \hat{\ell}_2)$ is not a u-ideal in $\mathcal{W}(\ell_1, \hat{\ell}_2)$ (by [25, Example 3] and [27, Theorem 1.2] or [28, Proposition 1]). Since $\hat{\ell}_2$ has the AP, $\mathcal{F}(Y, \hat{\ell}_2)$ is an ideal in $\mathcal{W}(Y, \hat{\ell}_2)$ for all Banach spaces Y (see [25, Example 3] or [16, Theorem 3.3]). Thus statement (a) in Theorem 3.2 is strictly stronger than statement (a) in Proposition 3.3 below. Note that this implies that the bound $\limsup_\alpha \|T - 2S_\alpha T\| \leqslant \|T\|$ in statement (f) in Theorem 3.2 is strictly stronger than the bound $\limsup_\alpha \|T_\alpha\| \leqslant \|T\|$ in (iii) in Corollary 1.5 in [16].

Since $\hat{\ell}_2$ is reflexive, we also get that $\mathcal{F}(\hat{\ell}_2^*, \ell_\infty)$ is not a u-ideal in $\mathcal{W}(\hat{\ell}_2^*, \ell_\infty)$. Hence, also ℓ_∞ is an example of a Banach space X such that $\mathcal{F}(Y,X)$ is an ideal in $\mathcal{W}(Y,X)$ for all Banach spaces Y, without being a u-ideal for all Y. Also, if for $0 < r < 1$, Y_r are the equivalently re-normed versions of c_0 defined in [8], then $\mathcal{F}(\ell_1, Y_r)$ is not a u-ideal in $\mathcal{W}(\ell_1, Y_r)$ for any $0 < r < 1$, even though $\mathcal{F}(Y, Y_r)$ is an ideal in $\mathcal{W}(Y, Y_r)$ for all Banach spaces Y and $0 < r < 1$ (see the last paragraph in [25]).

Remark 3.2. Let X be a Banach space and let $K \subset B_X$ be a weakly compact subset. If X has the AP, then there is a net $(S_\alpha) \subset \mathcal{F}(X,X)$ with $\sup\limits_{x\in K} \|S_\alpha x\| \leqslant 1$ such that $S_\alpha \to I_X$ uniformly on compact sets in X. Indeed, put $[Z,J] = \mathrm{DFJP}(\overline{\mathrm{conv}}(\pm K))$. Using [4, Theorem 1] we get that $B_{\mathcal{F}(Z,X)}$ cannot be strongly separated from $\mathrm{conv}(S_\alpha J)$. This should be compared with statement (d) in Theorem 3.2.

A Banach space X is said to have the *unconditional metric approximation property* (UMAP) if there is a net $(T_\alpha) \subset \mathcal{F}(X,X)$ with $\limsup\limits_\alpha \|I_X - 2T_\alpha\| \leqslant 1$ such that $T_\alpha(x) \to x$ for all $x \in X$. Like u-ideals, also the notion of the UMAP (for separable spaces using sequences) was introduced by Casazza and Kalton in [2].

In Theorem 5.2 in [12] it was proved that X has the UMAP if and only if $\mathcal{F}(Y,X)$ is a u-ideal in $\mathcal{L}(Y,X)$ for every Banach space Y.

If X is reflexive, then (d) in Theorem 3.2 says that X has the UMAP. By [2, Theorem 3.9] it follows that in this case $\mathcal{F}(Y,X)$ is a u-ideal in $\mathcal{W}(Y,X)$ for all Banach spaces Y if and only if $\mathcal{F}(X,X)$ is a u-ideal in $\mathcal{W}(X,X)$.

From [16, Theorem 3.3] and [14, Corollary 2] (see also [9, Theorem 5.1], [30, Proposition 2.1]) we get the following proposition.

Proposition 3.3. *Let X be a Banach space. The following statements are equivalent.*

(a) *$\mathcal{F}(Y,X)$ is an ideal in $\mathcal{W}(Y,X)$ for every Banach space Y.*
(b) *X has the AP.*
(c) *Every separable ideal Z in X has the AP.*
(d) *$\mathcal{F}(Y,Z)$ is an ideal in $\mathcal{W}(Y,Z)$ for every Banach space Y and a separable ideal Z in X.*

For u-ideals we have the following result.

Proposition 3.4. *Let X be a Banach space and assume $\mathcal{F}(Y,X)$ is a u-ideal in $\mathcal{W}(Y,X)$ for every Banach space Y. Then a closed subspace Z of X has the AP if and only if $\mathcal{F}(Y,Z)$ is a u-ideal in $\mathcal{W}(Y,Z)$ for every Banach space Y.*

P r o o f. One direction is immediate from Proposition 3.3.

For the reverse direction let Y be a reflexive Banach space, Z a subspace of X with the AP, and $T \in \mathcal{W}(Y,Z)$. Put $\hat{T} = i_Z \circ T$, choose a compact subset K of Z, and let $\varepsilon > 0$. By Theorem 3.2 there is a net $(S_\alpha) \subset \mathcal{F}(X,X)$ with $\limsup\limits_\alpha \|\hat{T} - 2S_\alpha \hat{T}\| \leqslant \|\hat{T}\| = \|T\|$ such that $S_\alpha \to I_X$ uniformly on compact sets. Since Z has the AP, there is a net $(U_\beta) \subset \mathcal{F}(Z,Z)$ such that $U_\beta \to I_Z$ uniformly on compact sets. After switching to the product index set we may suppose that (U_β) is indexed by the same set as (S_α). Hence we shall write (U_α) from now on.

Now let $u \in \mathcal{F}(Y,X)^*$. Since Y is reflexive and X has the AP, $\mathcal{F}(Y,X)^*$ is isometrically isomorphic to a quotient of $X^*\hat{\otimes}_\pi Y$ by a theorem of Feder and Saphar [4, Theorem 1]. Choose a representation $\sum_{n=1}^{\infty} x_n^* \otimes y_n$ for u. For the net $T_\alpha = S_\alpha i_Z T - i_Z U_\alpha T$ we have

$$\begin{aligned}\langle u, T_\alpha\rangle &= \sum_{n=1}^{\infty} \langle x_n^*, (S_\alpha i_Z T - i_Z U_\alpha T)(y_n)\rangle \\ &\to \sum_{n=1}^{\infty} \langle i_Z^* x_n^*, T y_n\rangle - \sum_{n=1}^{\infty} \langle i_Z^* x_n^*, T y_n\rangle = 0.\end{aligned}$$

Hence $T_\alpha \to 0$ weakly in $\mathcal{F}(Y,X)$. Consequently, a suitable net of convex combinations of T_α converges in norm to 0. Thus there exist α_0, $\hat{S}_{\alpha_0} \in \operatorname{co}\{S_\alpha\colon\ \alpha > \alpha_0\}$ and $\hat{U}_{\alpha_0} \in \operatorname{co}\{U_\alpha\colon\ \alpha > \alpha_0\}$ such that $\|\hat{S}_{\alpha_0} i_Z T - i_Z \hat{U}_{\alpha_0} T\| \leqslant \varepsilon$, $\sup_{z\in K} \|\hat{U}_{\alpha_0}(z) - z\| \leqslant \varepsilon$ and $\|\hat{T} - 2\hat{S}_{\alpha_0}\hat{T}\| \leqslant \|\hat{T}\| + \varepsilon$. We get that

$$\|i_Z T - 2 i_Z \hat{U}_{\alpha_0} T\| \leqslant \|i_Z T - 2\hat{S}_{\alpha_0} i_Z T\| + 2\|\hat{S}_{\alpha_0} i_Z T - i_Z \hat{U}_{\alpha_0} T\| \leqslant \|\hat{T}\| + 3\varepsilon.$$

Hence $\|T - 2\hat{U}_{\alpha_0} T\| \leqslant \|T\| + 3\varepsilon$, and the result follows from the local characterization of u-ideals in Theorem 1.1. □

Remark 3.3. If $\mathcal{F}(Y,Z)$ is a u-ideal in $\mathcal{W}(Y,Z)$ for every Banach space Y and every subspace Z of X with the AP, then $\mathcal{F}(Y,X)$ is not necessarily a u-ideal in $\mathcal{W}(Y,X)$ for every Banach space Y. Indeed, for $1 < p < \infty$, choose a subspace X of ℓ_p such that X does not have the AP (cf. e.g. [23, p. 91]). X cannot be complemented and hence it is not an ideal in ℓ_p. It is probably well known that $\mathcal{F}(Y,\ell_p)$ is a u-ideal in $\mathcal{W}(Y,\ell_p)$ for all Banach spaces Y. (This can be proved by using that the standard basis of ℓ_p is 1-unconditional and then Theorem 3.2 (g).) By Proposition 3.4, $\mathcal{F}(Y,Z)$ is a u-ideal in $\mathcal{W}(Y,Z)$ for every subspace Z of X with the AP. But X does not have the AP so $\mathcal{F}(Y_0,X)$ is not even an ideal in $\mathcal{W}(Y_0,X)$ for some Banach space Y_0 by [16, Theorem 3.3].

Let X be a Banach space. In the next theorem we want to study the problem when $\mathcal{F}(X,Y)$ is a u-ideal in $\mathcal{W}(X,Y)$ for all Banach spaces Y. In Theorem 6.5 in [12] it was proved that (a) $\mathcal{K}(X,Y)$ is a u-ideal in $\mathcal{L}(X,Y)$ for all Banach spaces Y is equivalent to (c) there is a net $(T_\alpha) \subset \mathcal{K}(X,X)$ with $\limsup_\alpha \|I - 2T_\alpha\| \leqslant 1$ such that $T_\alpha x \to x$ for all $x \in X$ and $T_\alpha^* x^* \to x^*$ for all $x^* \in X^*$, which in turn is equivalent to (e) X has the metric compact approximation property and X has the property

(wM^*). Note that the equivalence of (c) and (e) follows from the equivalence of $(3^{\rm o})$ and $(2^{\rm o})$ in Corollary 4.5 in [29] by taking $a = 1$ and $B = \{-2\}$. In all these statements $\mathcal{K}(X, X)$ (resp. $\mathcal{K}(X, Y)$) may be replaced by $\mathcal{F}(X, X)$ (resp. $\mathcal{F}(X, Y)$) (see the text after Corollary 4.6 in [29]).

Theorem 3.5. *Let X be a Banach space. The following statements are equivalent.*

(a) *$\mathcal{F}(X, Y)$ is a u-ideal in $\mathcal{W}(X, Y)$ for every Banach space Y.*
(b) *$\mathcal{F}(X, Y)$ is a u-ideal in $\mathcal{W}(X, Y)$ for every reflexive Banach space Y.*
(c) *$\mathcal{F}(X, Y)$ is a u-ideal in span$(\mathcal{F}(X, Y), \{T\})$ for every $T \in \mathcal{W}(X, Y)$ and for every reflexive Banach space Y.*
(d) *For every reflexive Banach space Y there exists a Hahn-Banach extension operator $\Psi \in \mathrm{I\!H\!B}(\mathcal{F}(X, Y), \mathcal{W}(X, Y))$ such that for every $T \in \mathcal{W}(X, Y)$ there is a net $(T_\alpha) \subset \mathcal{F}(X, Y)$ with $\limsup_\alpha \|T - 2T_\alpha\| \leqslant \|T\|$ such that $T_\alpha \to \Psi^*(T) = T$ weak* in $\mathcal{F}(X, Y)^{**}$.*
(e) *For every weakly compact set $K \subset X^*$ there is a net $(S_\alpha) \subset \mathcal{F}(X, X)$ with $\lim_\alpha \sup_{x^* \in K} \|x^* - 2S_\alpha^* x^*\| \leqslant \sup_{x^* \in K} \|x^*\|$ such that $S_\alpha^* \to I_{X^*}$ uniformly on compact subsets of K.*
(f) *For every Banach space Y and $T \in \mathcal{W}(X, Y)$ there is a net $(S_\alpha) \subset \mathcal{F}(X, X)$ such that $\limsup_\alpha \|T - 2TS_\alpha\| \leqslant \|T\|$ and $S_\alpha^* \to I_{X^*}$ uniformly on compact sets in X^*.*
(g) *For every Banach space Y and $T \in \mathcal{W}(X, Y)$ there is a net $(S_\alpha) \subset \mathcal{F}(X, X)$ such that $\limsup_\alpha \|T - 2TS_\alpha\| \leqslant \|T\|$ and $S_\alpha^* \to I_{X^*}$ in the strong operator topology.*
(h) *For every reflexive Banach space Y and $T \in \mathcal{W}(X, Y)$ there is a net $(S_\alpha) \subset \mathcal{F}(X, X)$ such that $\limsup_\alpha \|T - 2TS_\alpha\| \leqslant \|T\|$ and $S_\alpha^* T^* \to T^*$ in the strong operator topology.*

Proof. If Y is a reflexive Banach space, we have isometries $\mathcal{F}(X, Y) = \mathcal{F}(Y^*, X^*)$ and $\mathcal{W}(X, Y) = \mathcal{W}(Y^*, X^*)$. Using this observation, Theorem 3.5, for reflexive spaces Y, follows from Theorem 3.2.

It now suffices to show that the statements in (a) and (f) hold whenever they hold for reflexive spaces Y. Indeed, to see that (a) holds we can use the local characterization of u-ideals in Theorem 1.1 and an argument similar to (g) $\Rightarrow$ (a) in Theorem 3.2 (use (2) on page 3 instead of (1)).

To see that (f) holds we put $[Z, \Phi, J] = \mathrm{DFJP}(\mathrm{span}(\{T\}))$ where Y is a Banach space and $T \in \mathcal{W}(X, Y)$. Since Z is reflexive and $J \in \mathcal{W}(X, Z)$ there is a net $(S_\alpha) \subset \mathcal{F}(X, X)$ with $\limsup_\alpha \|J - 2JS_\alpha\| \leqslant \|J\| = 1$ such that $S_\alpha^* \to I_{X^*}$ uniformly on compact sets in X^*. Finally, write $\limsup_\alpha \|T - 2TS_\alpha\| \leqslant \limsup_\alpha \|\Phi(T)\| \|J - 2JS_\alpha\| \leqslant \|T\|$ and we are done. $\square$

Remark 3.4. By [16, Theorem 3.4] we get that $\mathcal{F}(\ell_1, Y)$ is an ideal in $\mathcal{W}(\ell_1, Y)$ for every Banach space Y. In Remark 3.1 we noticed that $\mathcal{F}(\ell_1, \hat{\ell}_2)$ is not a u-ideal in $\mathcal{W}(\ell_1, \hat{\ell}_2)$ where $\hat{\ell}_2$ is the equivalent re-norming of ℓ_2 constructed by Oja in [25]. Thus ℓ_1 does not fulfil statement (a) in Theorem 3.5.

Note that Proposition 2.3 in [22] for M-ideals also holds for u-ideals by using the local characterization of u-ideals in Theorem 1.1 instead of the 3-ball-property used in [22, Proposition 2.3] (see [13, Theorem 6.17], [7, Theorem I.2.2] or [22, Theorem 2.1]). Thus if a dual space X^* contains a copy of c_0, then $\mathcal{F}(\ell_1, Y)$ is a u-ideal in $\mathcal{W}(\ell_1, Y)$ whenever $\mathcal{F}(X, Y)$ is a u-ideal in $\mathcal{W}(X, Y)$. If $\hat{\ell}_2$ is the equivalently re-normed version of ℓ_2 constructed by Oja, it follows from Remark 3.4 that $\mathcal{F}(X, \hat{\ell}_2)$ fails to be a u-ideal in $\mathcal{W}(X, \hat{\ell}_2)$ whenever X^* contains a copy of c_0.

Remark 3.5. Recall that a u-ideal Z in X is *strict* if the u-complement of $Z^\perp$ in X^* is a norming subspace for X, i.e. if $\varphi(Z^*)$ is a norming subspace of X^* where $\varphi \in \mathrm{HB}(Z, X)$ is the unconditional Hahn-Banach extension operator.

If Y is a reflexive Banach space and $\mathcal{F}(Y, X)$ is a u-ideal in $\mathcal{W}(Y, X)$ then it is in fact a strict u-ideal. This is easily seen from the proof of Proposition 2.5. Indeed, in this case there is a unique Hahn-Banach extension operator $\Phi \in \mathrm{HB}(\mathcal{F}(Y, X), \mathcal{W}(Y, X))$ which is of the form $\Phi = I_{X^*} \otimes I_Y$. Since $B_{X^*} \otimes B_Y \subset \mathcal{W}(Y, X)^*$ is norming for $\mathcal{W}(Y, X)$ the claim follows. Similarly by Corollary 2.6, if Y is reflexive, then $\mathcal{F}(X, Y)$ is a strict u-ideal in $\mathcal{W}(X, Y)$ whenever it is a u-ideal.

If X is a Banach space it follows from [16, Theorem 3.4] and [11, Proposition 2.5] that $\mathcal{F}(X, Y)$ is an ideal in $\mathcal{W}(X, Y)$ for every Banach space Y if and only if $\mathcal{F}(Z, Y)$ is an ideal in $\mathcal{W}(Z, Y)$ for every Banach space Y and for every separable ideal Z in X. For u-ideals we have the following result.

Proposition 3.6. *Let X be a Banach space. If $\mathcal{F}(X, Y)$ is a u-ideal in $\mathcal{W}(X, Y)$ for every Banach space Y, then $\mathcal{F}(Z, Y)$ is a u-ideal in $\mathcal{W}(Z, Y)$ for every ideal Z in X and every Banach space Y.*

P r o o f. Let Y be a Banach space and let Z be an ideal in X with the corresponding Hahn-Banach extension operator $\varphi \in \mathrm{HB}(Z, X)$. Let G be a finite dimensional subspace of $\mathcal{W}(Z, Y)$ and define a map $L\colon G \to \mathcal{W}(X, Y)$ by

$$L(T) = T^{**} \circ \varphi^*|_X, \; T \in G.$$

Let $\varepsilon > 0$. By the local characterization of u-ideals, Theorem 1.1, there is an operator $M\colon L(G) \to \mathcal{F}(X,Y)$ such that $M(S) = S$ for every $S \in \mathcal{F}(X,Y) \cap L(G)$ and $\|S - 2M(S)\| \leqslant (1+\varepsilon)\|S\|$ for every $S \in L(G)$. Now define an operator $N\colon G \to \mathcal{F}(Z,Y)$ by

$$N(T) = M(L(T)) \circ i_Z.$$

It is straightforward to verify that the operator N fulfils (d) in Theorem 1.1 and the result follows. □

4. $\mathcal{F}(Y,X)$ as a u-ideal in $\mathcal{K}(Y,X^{**})$ and $\mathcal{W}(Y,X^{**})$

From [17, Theorem 5.1] and [19, Proposition 2.10] we have the following result.

Proposition 4.1 (Lima and Oja). *Let X be a closed subspace of a Banach space Y. If $\mathcal{F}(Z,X)$ is a u-ideal in $\mathcal{K}(Z,Y)$ for every reflexive Banach space Z, then X is a u-ideal in Y.*

The next result tells us more.

Proposition 4.2. *Let X be a closed subspace of a Banach space Y and let Z be a reflexive Banach space. Assume $\mathcal{F}(Z,X)$ is a u-ideal in $\mathcal{K}(Z,Y)$ with an unconditional extension operator Ψ. Then X is a u-ideal in Y with an unconditional extension operator ψ satisfying*

$$\Psi(x^* \otimes z) = (\psi x^*) \otimes z$$

for all $z \in Z$ and $x^* \in X^*$.

Moreover, if the above assumption holds for every separable reflexive Banach space Z, then $\psi^*|_Y$ is in the w^*-closure of $\mathcal{F}(Y,X)$ in $\mathcal{L}(Y,X^{**})$.

Proof. We proceed as in the proof of [18, Theorem 2.3]. Let $\Psi \in \mathrm{HB}(\mathcal{F}(Z,X), \mathcal{K}(Z,Y))$ be the unconditional Hahn-Banach extension operator and denote the corresponding ideal projection on $\mathcal{K}(Z,Y)^*$ by P_Ψ. Since Z is reflexive, it follows from [18, Theorem 1.3] that there exist $\{\psi_i\colon\ i = 1,\ldots,n\} \subset \mathrm{HB}(X,Y)$ such that

$$Z = \sum_{i=1}^{n} \oplus_1 Z_{\Psi\psi_i}, \qquad Z_{\Psi\psi_i} \neq \{0\} \text{ for all } 1 \leqslant i \leqslant n,$$

where

$$Z_{\Psi\psi_i} = \{z \in Z\colon\ \Psi(x^* \otimes z) = (\psi_i x^*) \otimes z, \forall x^* \in X^*\}.$$

Let (P_{ψ_i}) be the corresponding ideal projections on Y^*. It now follows that for $z \in Z_{\Psi\psi_i}$ and $y^* \in Y^*$

$$\begin{aligned}\|z\|\|y^*\| &= \|y^* \otimes z\| \geqslant \|(I - 2P_\Psi)(y^* \otimes z)\| = \|y^* \otimes z - 2P_\Psi(y^* \otimes z)\| \\ &= \|y^* \otimes z - 2(P_{\psi_i}y^*) \otimes z\| = \|(y^* - 2P_{\psi_i}y^*) \otimes z\| = \|z\|\|y^* - 2P_{\psi_i}y^*\|.\end{aligned}$$

Hence every ψ_i is unconditional and by uniqueness, see Proposition 2.2, they all coincide. With $\psi = \psi_i$ we have $Z = Z_{\Psi\psi}$.

Furthermore, if $\mathcal{F}(Z, X)$ is a u-ideal in $\mathcal{K}(Z, Y)$ for all separable reflexive Z, then by Lemma 2.1 in [20] there is for every such Z and $T \in \mathcal{K}(Z, Y)$ a net (T_α) in $\mathcal{F}(Z, X)$ with $\sup_\alpha \|T_\alpha\| \leqslant \|T\|$ such that $T_\alpha^* \to T^*\psi$ in the strong operator topology. By boundedness we may also assume that $\langle u, T_\alpha\rangle \to \langle u, T\rangle$ for all $u \in X^*\hat{\otimes}_\pi Z$.

Choose $u = \sum_n x_n^* \otimes y_n \in X^*\hat{\otimes}_\pi Y$ and assume that $\sum_n \|x_n^*\| = 1$ and $1 \geqslant \|y_n\| \to 0$ and put $[Z, J] = \mathrm{DFJP}(\overline{\mathrm{conv}}((\pm y_n)_{n=1}^\infty)$. Then Z is a separable reflexive Banach space and $J \in \mathcal{K}(Z, Y)$ with $\|J\| \leqslant 1$. Pick a net $(J_\alpha) \subset \mathcal{F}(Z, X)$ with $\sup_\alpha \|J_\alpha\| \leqslant \|J\|$ such that $J_\alpha^* \to J^*\psi$ uniformly on compact sets. As in the proof of (c)$\Rightarrow$ (d) in Theorem 3.2 we may assume that each $J_\alpha^* = J^*S_\alpha^*$ for some $S_\alpha \in \mathcal{F}(Y, X)$. Now choose $\varepsilon > 0$ and let $z_n \in B_Z$ be such that $y_n = Jz_n$. Since $J_\alpha^* \to J^*\psi$ uniformly on compact sets, it follows from [23, Proposition 1.e.3] that there is an operator $S \in \mathcal{F}(Y, X)$ such that

$$\varepsilon > \left| \sum_{n=1}^\infty \langle J^*S^*x_n^*, z_n\rangle - \sum_{n=1} \langle J^*\psi x_n^*, z_n\rangle \right| = \left| \sum_{n=1} \langle x_n^*, Sy_n\rangle - \sum_{n=1} \langle x_n^*, \psi^* y_n\rangle \right|.$$

Hence $\psi^*|_Y$ is in the w^*-closure of the $\mathcal{F}(Y, X)$ in $\mathcal{L}(Y, X^{**})$. $\square$

Remark 4.1. If $Y = X^{**}$ in Proposition 4.2 we actually have that $\psi^*|_{X^{**}}$ is in the weak*-closure of set $\mathcal{F}(X, X)$ in $\mathcal{L}(X^{**}, X^{**})$. In this case $J^*(X^*)$ and not just $J^*(X^{***})$ is norm-dense in Z^* (see the proof of [10, Proposition 2.1]). Thus for each J_α^* we can write $J_\alpha^* = J^*S_\alpha^*$ for some S_α in $\mathcal{F}(X, X)$ (and not only in $\mathcal{F}(X^{**}, X)$).

Let X be a Banach space. From Theorem 3.1 we have that $\mathcal{F}(Y, X)$ is a u-ideal in $\mathcal{W}(Y, X^{**})$ for every Banach space Y if and only if $\mathcal{F}(Y, X)$ is a u-ideal in $\mathcal{W}(Y, X^{**})$ for every reflexive Banach space Y. The next results contain other characterizations of these statements.

Theorem 4.3. *Let X be a Banach space. The following statements are equivalent.*

(a) *$\mathcal{F}(Y, X)$ is a u-ideal in $\mathcal{W}(Y, X^{**})$ for every Banach space Y.*

(b) *X is a u-ideal in its bidual with an unconditional Hahn-Banach extension operator $\psi \in \mathrm{HB}(X, X^{**})$ such that for every Banach space Y and $T \in \mathcal{W}(Y, X^{**})$ there is a net $(S_\alpha) \subset \mathcal{F}(X, X)$ with $\limsup_\alpha \|T - 2S_\alpha^{**}T\| \leqslant \|T\|$ such that $S_\alpha^{**}T \to \psi^* T$ weak* in $\mathcal{L}(Y, X^{**})$.*

(c) *There exists a Hahn-Banach extension operator $\psi \in \mathrm{HB}(X, X^{**})$ such that for every Banach space Y and $T \in \mathcal{W}(Y, X^{**})$ there is a net $(S_\alpha) \subset \mathcal{F}(X, X)$ with $\limsup_\alpha \|T - 2S_\alpha^{**}T\| \leqslant \|T\|$ such that $S_\alpha^{**}T \to \psi^* T$ weak* in $\mathcal{L}(Y, X^{**})$.*

(d) *For every weakly compact set $K \subset X^{**}$ there is a net $(S_\alpha) \subset \mathcal{F}(X, X)$ with $\lim_\alpha \sup_{x^{**} \in K} \|x^{**} - 2S_\alpha^{**}x^{**}\| \leqslant \sup_{x^{**} \in K} \|x^{**}\|$ such that $S_\alpha \to I_X$ uniformly on compact subsets of $K \cap X$.*

(e) *For every Banach space Y and $T \in \mathcal{W}(Y, X^{**})$ there is a net $(S_\alpha) \subset \mathcal{F}(X, X)$ with $\limsup_\alpha \|T - 2S_\alpha^{**}T\| \leqslant \|T\|$ such that $S_\alpha \to I_X$ uniformly on compact sets in X.*

(f) *For every reflexive Banach space Y and $T \in \mathcal{W}(Y, X^{**})$ there is a net $(S_\alpha) \subset \mathcal{F}(X, X)$ with $\limsup_\alpha \|T - 2S_\alpha^{**}T\| \leqslant \|T\|$ such that $S_\alpha \to I_X$ uniformly on compact sets in X.*

Proof. (a) $\Rightarrow$ (b). Let Y be a Banach space and let $T \in \mathcal{W}(Y, X^{**})$. Put $G = \operatorname{span}(\{T\})$ and let $[Z, J, \Phi] = \mathrm{DFJP}(G)$. Now Z is reflexive and $J \in \mathcal{W}(Z, X^{**})$ is of norm 1. Let $\Psi\colon \mathcal{F}(Z, X)^* \to \mathcal{W}(Z, X^{**})^*$ be the unconditional Hahn-Banach extension operator. As in the proof of Proposition 4.2 we can show that X is a u-ideal in X^{**} with $\psi \in \mathrm{HB}(X, X^{**})$ unconditional such that $\Psi(x^* \otimes z) = \psi(x^*) \otimes z$ for every $x^* \in X^*$ and $z \in Z$. By Theorem 1.1 there is a net $(J_\alpha) \subset \mathcal{F}(Z, X)$ such that $\limsup_\alpha \|J - 2J_\alpha\| \leqslant 1$ and $J_\alpha \to \Psi^*(J)$ weak*. Since $J^*(X^*)$ is norm dense in Z^* we can assume that for each α, $J_\alpha = S_\alpha^{**}J$ where $(S_\alpha) \subset \mathcal{F}(X, X)$. Since $\|T - 2S_\alpha^{**}T\| = \|J\Phi(T) - 2S_\alpha^{**}J\Phi(T)\| \leqslant \|T\|\|J - 2S_\alpha^{**}J\|$ we get $\limsup_\alpha \|T - 2S_\alpha^{**}T\| \leqslant \|T\|$.

Let $u = \sum_n x_n^* \otimes y_n \in X^* \hat{\otimes}_\pi Y$. Then $v = \sum_n x_n^* \otimes (\Phi(T)y_n) \in X^* \hat{\otimes}_\pi Z$. We get that

$$
\begin{aligned}
\langle u, \psi^* T\rangle &= \sum_n \langle \psi x_n^*, J\Phi(T)y_n\rangle = \langle \Psi(v), J\rangle = \langle v, \Psi^*(J)\rangle \\
&= \lim_\alpha \langle v, S_\alpha^{**}J\rangle = \lim_\alpha \sum_n \langle x_n^*, S_\alpha^{**}Ty_n\rangle = \lim_\alpha \langle u, S_\alpha^{**}T\rangle.
\end{aligned}
$$

This shows that $S_\alpha^{**}T \to \psi^* T$ weak* in $\mathcal{L}(Y, X^{**})$.

(b) $\Rightarrow$ (c) is trivial.

(c) $\Rightarrow$ (d) is similar to the proof of (c) $\Rightarrow$ (d) in Theorem 3.2.

(d) $\Rightarrow$ (e) is similar to the proof of (d) $\Rightarrow$ (e) in Theorem 3.2.

(e) $\Rightarrow$ (f) is trivial.

(f) $\Rightarrow$ (a) is similar to the proof of (f) $\Rightarrow$ (a) in Theorem 3.2. $\square$

Remark 4.2. Note that $X = c_0$ fulfils Theorem 4.3 since c_0 is an M_∞ space (see [7] p. 306) and [7, Proposition 5.6].

Theorem 4.4. *Let X be a Banach space. The following statements are equivalent.*

(a) *$\mathcal{F}(Y,X)$ is a u-ideal in $\mathcal{K}(Y,X^{**})$ for every Banach space Y.*

(b) *X is a u-ideal in X^{**} with an unconditional Hahn-Banach extension ψ such that $\psi^*|_{X^{**}}$ is in the weak*-closure of the $\mathcal{F}(X,X)$ in $\mathcal{L}(X^{**},X^{**})$.*

(c) *X is a u-ideal in its bidual with an unconditional Hahn-Banach extension operator $\psi \in \mathrm{HB}(X,X^{**})$ such that for every Banach space Y and $T \in \mathcal{K}(Y,X^{**})$ there is a net $(S_\alpha) \subset \mathcal{F}(X,X)$ with $\limsup_\alpha \|T - 2S_\alpha^{**}T\| \leqslant \|T\|$ such that $S_\alpha^{**}T \to \psi^*T$ weak* in $\mathcal{L}(Y,X^{**})$.*

(d) *For every Banach space Y and $T \in \mathcal{K}(Y,X^{**})$ there is a net $(S_\alpha) \subset \mathcal{F}(X,X)$ with $\limsup_\alpha \|T - 2S_\alpha^{**}T\| \leqslant \|T\|$ such that $S_\alpha \to I_X$ uniformly on compact sets in X.*

(e) *For every separable reflexive Banach space Y and $T \in \mathcal{K}(Y,X^{**})$ there is a net $(S_\alpha) \subset \mathcal{F}(X,X)$ with $\limsup_\alpha \|T - 2S_\alpha^{**}T\| \leqslant \|T\|$ such that $S_\alpha \to I_X$ uniformly on compact sets in X.*

Proof. (a) $\Rightarrow$ (b) follows from Proposition 4.2.

(b) $\Rightarrow$ (c). Let Y be a Banach space and let $T \in \mathcal{K}(Y,X^{**})$. Put $G = \operatorname{span}(\{T\})$ and write $[Z,J,\Phi] = \mathrm{DFJP}(G)$. Now Z is reflexive and $J \in \mathcal{K}(Z,X^{**})$ has norm one. Let $\psi \in \mathrm{HB}(X,X^{**})$ be the unconditional Hahn-Banach extension operator and choose a net $(S_\alpha) \subset \mathcal{F}(X,X)$ such that $S_\alpha^{**} \to \psi^*|_{X^{**}}$ weak* in $\mathcal{L}(X^{**},X^{**})$. Since Z is reflexive, $\mathcal{K}(Z,X^{**})^*$ is a quotient of $X^{***}\hat{\otimes}_\pi Z$ by [4, Theorem 1] of Feder and Saphar. Now let $\varepsilon > 0$ and let $u \in X^{***}\hat{\otimes}_\pi Z$. Choose a representation $\sum_{n=1}^{\infty} x_n^{***} \otimes z_n$ for u such that $\sum_{n=1}^{\infty} \|x_n^{***}\|\|z_n\| \leqslant \|u\|_\pi + \varepsilon$ and write $x_n^* = x_n^{***}|_X$. We get that

$$\begin{aligned} |\langle u, J - 2S_\alpha^{**}J\rangle| &= \left|\sum_{n=1}^{\infty} \langle x_n^{***}, (J - 2S_\alpha^{**}J)z_n\rangle\right| = \left|\sum_{n=1}^{\infty} \langle x_n^{***} - 2S_\alpha^{*}x_n^{*}, Jz_n\rangle\right| \\ &\to \left|\sum_{n=1}^{\infty} \langle x_n^{***} - 2\psi x_n^{*}, Jz_n\rangle\right| \leqslant \sum_{n=1}^{\infty} \|x_n^{***}\|\|Jz_n\| \leqslant \|u\|_\pi + \varepsilon. \end{aligned}$$

Hence $\operatorname{conv}(J - 2S_\alpha^{**}J)$ cannot be strongly separated from $B_{\mathcal{K}(Z,X^{**})}$. By taking successive convex combinations we get a new net, also denoted by (S_α), such that

$\limsup\limits_{\alpha} \|J - 2S_\alpha^{**}J\| \leqslant 1$. Thus

$$\limsup_{\alpha} \|T - 2S_\alpha^{**}T\| \leqslant \limsup_{\alpha} \|\Phi(T)\| \|J - 2S_\alpha^{**}J\| \leqslant \|T\|.$$

Obviously $S_\alpha^{**}T \to \psi^* T$ weak* in $\mathcal{L}(Y, X^{**})$.

(c) $\Rightarrow$ (d). Argue as in the proof of (d) $\Rightarrow$ (e) in Theorem 4.3.

(d) $\Rightarrow$ (e) is trivial.

(e) $\Rightarrow$ (a). Argue as in the proof of (g) $\Rightarrow$ (a) in Theorem 3.2. □

Remark 4.3. In [10, Proposition 2.1] it is proved that $\mathcal{F}(Y, X)$ is an ideal in $\mathcal{W}(Y, X^{**})$ for every Banach space Y if and only if $\mathcal{F}(Y, X)$ is an ideal in $\mathcal{K}(Y, X^{**})$ for every Banach space Y. This fails if we replace "ideal" with "u-ideal". Indeed, if we let $X = \hat{\ell}_2$, the equivalent re-norming of ℓ_2 obtained by Oja (see Remark 3.1), then we have a counterexample. This proves that the statements in Theorem 4.4 are strictly weaker than those in Theorem 4.3.

The next result shows that $\mathcal{F}(Y, X)$ being a u-ideal in $\mathcal{W}(Y, X^{**})$ for all Banach spaces Y is inherited by some subspaces of X.

Proposition 4.5. *Suppose $\mathcal{F}(Y, X)$ is a u-ideal in $\mathcal{W}(Y, X^{**})$ for every Banach space Y and let $\varphi \in \mathrm{HB}(X, X^{**})$ be the unconditional Hahn-Banach extension operator. Then $\mathcal{F}(Y, Z)$ is a u-ideal in $\mathcal{W}(Y, Z^{**})$ for every Banach space Y and every ideal Z in X such that $\varphi^*(Z^{\perp\perp}) \subset Z^{\perp\perp}$.*

Proof. Let Y be a reflexive Banach space and let Z be an ideal in X such that $\varphi^*(Z^{\perp\perp}) \subset Z^{\perp\perp}$. Denote by $i_Z\colon Z \to X$ the natural embedding. Since $\varphi^*(Z^{\perp\perp}) \subset Z^{\perp\perp}$, it follows from Theorem 2.4 that Z is a u-ideal in its bidual with an unconditional extension operator $\psi \in \mathrm{HB}(Z, Z^{**})$ such that $i_Z^{**}\psi^*|_{Z^{**}} = \varphi^* i_Z^{**}$. From Theorem 4.4 we have $\varphi^*|_{X^{**}}$ in the weak*-closure of $\mathcal{F}(X, X)$ in $\mathcal{L}(X^{**}, X^{**})$. By the Principle of Local Reflexivity it is routine to check that $\psi^*|_{Z^{**}}$ is in the weak*-closure of $\mathcal{L}(Z^{**}, Z^{**})$.

Choose a compact subset K of Z and an operator $T \in \mathcal{W}(Y, Z^{**})$. Put $\hat{T} = i_Z^{**} \circ T \in \mathcal{W}(Y, X^{**})$. By Theorem 4.3 there is a net $(S_\alpha) \subset \mathcal{F}(X, X)$ with $\limsup\limits_{\alpha} \|\hat{T} - 2S_\alpha^{**}\hat{T}\| \leqslant \|\hat{T}\| = \|T\|$ such that $S_\alpha^{**}\hat{T} \to \varphi^*|_{X^{**}}\hat{T}$ weak* in $\mathcal{L}(X^{**}, X^{**})$. From the first paragraph there is a net $(U_i) \subset \mathcal{F}(Z, Z)$ such that $U_i^{**} \to \psi^*|_{Z^{**}}$ weak* in $\mathcal{L}(Z^{**}, Z^{**})$. Assume (S_α) and (U_i) have the same index set. Thus we will write (U_α) for the net in $\mathcal{F}(Z, Z)$. Note that $U_\alpha \to I_Z$ uniformly on compact sets in Z. Now let $u = \sum\limits_n x_n^* \otimes y_n \in \mathcal{F}(Y, X)^*$ and $T_\alpha = S_\alpha^{**} i_Z^{**} T - i_Z^{**} U_\alpha^{**} T$. From this we get that

$$
\begin{aligned}
\langle u, T_\alpha \rangle &= \sum_n \langle x_n^*, (S_\alpha^{**} i_Z^{**} - i_Z^{**} U_\alpha^{**})(Ty_n)\rangle \\
&= \sum_n \langle x_n^*, S_\alpha^{**}(i_Z^{**} Ty_n)\rangle - \sum_n \langle i_Z^* x_n^*, U_\alpha^{**}(Ty_n)\rangle \\
&\to \sum_n \langle x_n^*, \varphi^*(i_Z^{**} Ty_n)\rangle - \sum_n \langle i_Z^* x_n^*, \psi^*(Ty_n)\rangle = 0.
\end{aligned}
$$

Hence $T_\alpha \to 0$ weakly in $\mathcal{F}(Y, X)$. Consequently, a suitable net of convex combinations of T_α converges in norm to 0. Thus there exist α_0, $\hat{S}_{\alpha_0} \in \mathrm{co}\{S_\alpha^{**}\colon\ \alpha > \alpha_0\}$ and $\hat{U}_{\alpha_0} \in \mathrm{co}\{U_\alpha^{**}\colon\ \alpha > \alpha_0\}$ such that $\|\hat{T} - 2\hat{S}_{\alpha_0}\hat{T}\| \leqslant \|\hat{T}\| + \varepsilon$, $\sup_{z \in K} \|\hat{U}_{\alpha_0} z - z\| \leqslant \varepsilon$ and $\|\hat{S}_{\alpha_0} i_Z^{**} T - i_Z^{**}\hat{U}_{\alpha_0} T)\| \leqslant \varepsilon$. We get

$$
\|i_Z^{**}T - 2 i_Z^{**}\hat{U}_{\alpha_0} T\| \leqslant \|i_Z^{**}T - 2\hat{S}_{\alpha_0} i_Z^{**} T\| + 2\|\hat{S}_{\alpha_0} i_Z^{**} T - i_Z^{**}\hat{U}_{\alpha_0} T\| \leqslant \|\hat{T}\| + 3\varepsilon.
$$

Hence $\|T - 2\hat{U}_{\alpha_0} T\| \leqslant \|T\| + 3\varepsilon$, and the result follows. □

In [21] Lima and Oja introduced and studied the weak metric approximation property. Following Lima and Oja a Banach space X is said to have the *weak metric approximation property (weak MAP)* if for every Banach space Y and every operator $T \in \mathcal{W}(X, Y)$ there is a net $(S_\alpha) \subset \mathcal{F}(X, X)$ with $\sup_\alpha \|TS_\alpha\| \leqslant \|T\|$ such that $S_\alpha \to I_X$ uniformly on compact subsets in X. It is easy to see that the MAP implies the weak MAP. In [31, Corollary 1] it is shown that the weak MAP and the MAP are indeed equivalent for a Banach space for which either its dual or its bidual has the RNP.

Lima proved in [10] that X has the weak MAP if and only if $\mathcal{F}(Y, X)$ is an ideal in $\mathcal{K}(Y, X^{**})$ for every Banach space Y. Based on this, it is natural to guess that an "unconditional version" of the weak MAP could be the property that for every Banach space Y and every operator $T \in \mathcal{K}(X, Y)$ there is a net $(S_\alpha) \subset \mathcal{F}(X, X)$ with $\limsup_\alpha \|T - 2TS_\alpha\| \leqslant \|T\|$ such that $S_\alpha \to I_X$ uniformly on compact sets in X. As remarked below, this property is strictly weaker than the statements in Theorem 4.4.

Proposition 4.6. *Let X be a Banach space. The following statements are equivalent.*

(a) *For every Banach space Y and every operator $T \in \mathcal{K}(X, Y)$ there is a net $(S_\alpha) \subset \mathcal{F}(X, X)$ such that $\limsup_\alpha \|T - 2TS_\alpha\| \leqslant \|T\|$ and $S_\alpha \to I_X$ uniformly on compact sets.*

(b) *For every reflexive Banach space Y and every operator $T \in \mathcal{K}(X, Y)$ there is a net $(S_\alpha) \subset \mathcal{F}(X, X)$ such that $\limsup_\alpha \|T - 2TS_\alpha\| \leqslant \|T\|$ and $TS_\alpha \to T$ uniformly on compact sets.*

(c) *There is a Hahn-Banach extension operator $\psi \in \mathrm{HB}(X, X^{**})$ with $\|I_{X^{**}} - 2\psi^*{}_{|X^{**}}\| = 1$ such that $\psi^*|_{X^{**}}$ is in the weak*-closure of $\mathcal{F}(X, X)$ in $\mathcal{L}(X^{**}, X^{**})$.*

Proof. (a) $\Rightarrow$ (b) is trivial.

(b) $\Rightarrow$ (c). The proof is essentially that of [10, Proposition 2.5].

(c) $\Rightarrow$ (a) is similar to Theorem 4.4 (c) $\Rightarrow$ (d). □

Remark 4.4. If $\psi \in \mathrm{HB}(X, X^{**})$ is an unconditional extension operator then $\|I_{X^{**}} - 2\psi^*|_{X^{**}}\| = \|I_{X^{***}} - 2\psi k_X^*\| = 1$. To see this, first note that $1 = \|I_{X^{***}} - 2\psi k_X^*\| = \|I_{X^{****}} - 2k_X^{**}\psi^*\|$. Write the identity operator on the dual X^* as $I_{X^*} = k_X^* k_{X^*}$ and the identity operator on the bidual X^{**} as $I_{X^{**}} = k_{X^*}^* k_{X^{**}}$. By taking adjoints we obtain from the first equality that $I_{X^{**}} = (I_{X^*})^* = k_{X^*}^* k_X^{**}$. It follows that

$$\begin{aligned}\|I_{X^{**}} - 2\psi^* k_{X^{**}}\| &= \|I_{X^{**}} - 2I_{X^{**}}\psi^* k_{X^{**}}\| \\ &= \|k_{X^*}^* k_{X^{**}} - 2k_{X^*}^* k_X^{**}\psi^* k_{X^{**}}\| \leqslant 1.\end{aligned}$$

Proposition 4.7. *Let X be a Banach space. If every equivalent re-norming of X is a u-ideal in its bidual, then X is a strict u-ideal in its bidual.*

Proof. Let $x^{***} \in X^{***}$, $x^* = k_X^*(x^{***})$, and let $\varepsilon > 0$. By [12, Lemma 2.4] there is an equivalent re-norming X_1 of X which is locally uniformly rotund at x^* such that $B_X \subseteq B_{X_1} \subseteq B_X(0, 1+\varepsilon)$. Let $|\cdot|$ be the norm on X_1 and let $P\colon X_1^{***} \to X_1^{***}$ be the u-ideal projection. Then $P(x^{***}) = x^*$ and

$$\|x^{***} - 2x^*\| \leqslant |x^{***} - 2x^*| = |x^{***} - 2P(x^{***})| \leqslant |x^{***}| \leqslant (1+\varepsilon)\|x^{***}\|,$$

which shows that $\|I - 2\pi\| = 1$ where $\pi = k_{X^*}k_X^*$, so X is a strict u-ideal in its bidual. □

Remark 4.5. The statements in Proposition 4.6 are strictly weaker than those in Theorem 4.4. Indeed, as noted in [5] (see p. 29), ℓ_1 is not a strict u-ideal in its bidual. Thus it follows from Proposition 4.7 that there exists an equivalent re-norming $\hat{\ell_1}$ of ℓ_1 for which $\hat{\ell_1}$ is not a u-ideal in its bidual. Since $\hat{\ell_1}$ has the AP, Proposition 4.6 (c) is fulfilled with $\psi = k_{\hat{\ell}_1^*}$.

Acknowledgement. We would like to thank Eve Oja for many helpful comments.

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