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COMPACT IMAGES OF SPACES WITH A WEAKER METRIC TOPOLOGY

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Abstract. If X is a space that can be mapped onto a metric space by a one-to-one mapping, then X is said to have a weaker metric topology.

In this paper, we give characterizations of sequence-covering compact images and sequentially-quotient compact images of spaces with a weaker metric topology. The main results are that

(1) Y is a sequence-covering compact image of a space with a weaker metric topology if and only if Y has a sequence $\{\mathscr{F}_i\}_{i\in\mathbb{N}}$ of point-finite cs-covers such that $\bigcap\limits_{i\in\mathbb{N}} \operatorname{st}(y,\mathscr{F}_i)=\{y\}$ for each $y\in Y$.

(2) Y is a sequentially-quotient compact image of a space with a weaker metric topology if and only if Y has a sequence $\{\mathscr{F}_i\}_{i\in\mathbb{N}}$ of point-finite cs^*-covers such that $\bigcap\limits_{i\in\mathbb{N}} \operatorname{st}(y,\mathscr{F}_i)=\{y\}$ for each $y\in Y$.

Keywords: sequence-covering mappings, sequentially-quotient mappings, compact mappings, weaker metric topology

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1. Introduction

Since A. V. Arhangel'skii published the famous paper "Mappings and spaces" in 1966 ([1]), the behavior of certain images (including some compact images) on metric spaces has attracted considerable attention, and some noticeable results have been obtained ([4], [7], [16]). In recent years, a number of topologists use sequence-covering mappings to systematically study metric spaces and generalized metric spaces ([6], [8], [10], [11], [12], [13], [14], [15], [17]). Especially, J. Chaber investigated the class of spaces that can be mapped onto metric spaces by a mapping with fibers having

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a given property $\mathscr{P}$ in [2]. These inspire us to discuss spaces with a weaker metric topology and characterize sequence-covering compact images and sequentially-quotient compact images of the class of spaces.

Throughout this paper, all spaces are considered to be regular and T_1, and all mappings are continuous and onto. $\mathbb{N}$ denotes the set of all natural numbers. Let A be a subset of a space X, $x \in X$, and $\mathscr{U}$ be a family of subsets of X. We write $\operatorname{st}(x, \mathscr{U}) = \bigcup\{U \in \mathscr{U} \colon x \in U\}$ and $\operatorname{st}(A, \mathscr{U}) = \bigcup\{U \in \mathscr{U} \colon U \cap A \neq \emptyset\}$. For a product space $\prod\limits_{n\in\mathbb{N}} X_n$ and some $m \in \mathbb{N}$, the symbol $\pi_m \colon \prod\limits_{n\in\mathbb{N}} X_n \to X_m$ denotes the projection of $\prod\limits_{n\in\mathbb{N}} X_n$ onto its m-th coordinate.

First, recall some basic definitions. For terms which are not defined here, please refer to [3] and [9].

Definition 1 [5]**.** Let X be a space and $x \in P \subset X$. P is said to be a sequential neighborhood of x, if every sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x is eventually in P; i.e., there is $k \in \mathbb{N}$ such that $x_n \in P$ for $n > k$.

Definition 2 [9]**.** Let $f \colon X \to Y$ be a mapping.

(1) f is compact, if each $f^{-1}(y)$ is compact.

(2) f is sequence-covering, if for every convergent sequence S in Y, there is a convergent sequence L in X such that $f(L) = S$.

(3) f is sequentially-quotient, if for every convergent sequence S in Y, there is a convergent sequence L in X such that $f(L)$ is an infinite subsequence of S.

(4) f is 1-sequence-covering, if for each $y \in Y$, there is $x \in f^{-1}(y)$ such that whenever $\{y_n\}$ is a sequence converging to y in Y there is a sequence $\{x_n\}$ converging to x in X with each $x_n \in f^{-1}(y_n)$.

Definition 3 [9]**.** Let X be a space, and let $\mathscr{P}$ be a cover of X.

(1) $\mathscr{P}$ is a *cs*-cover of X, if for any convergent sequence S in X, there exists $P \in \mathscr{P}$ such that S is eventually in P.

(2) $\mathscr{P}$ is a *cs**-cover of X, if for any convergent sequence S in X, there exists $P \in \mathscr{P}$ such that some subsequence of S is eventually in P.

(3) $\mathscr{P}$ is an *sn*-cover of X, if each element of $\mathscr{P}$ is a sequential neighborhood of some point of X and for each $x \in X$, there exists $P \in \mathscr{P}$ such that P is the sequential neighborhood of x.

Definition 4 [2]**.** If X is a space that can be mapped onto a metric space by a one-to-one mapping, then X has a weaker metric topology.

2. Main results

Lemma 1. *Let X be a space with a weaker metric topology. Then there is a sequence $\{\mathscr{P}_i\}_{i\in\mathbb{N}}$ of locally finite open covers of X such that $\bigcap_{i\in\mathbb{N}} \operatorname{st}(K,\mathscr{P}_i) = K$ for each compact subset $K \subset X$.*

Proof. Suppose $f\colon X \to M$ is a one-to-one mapping, M being a metric space. There is a sequence $\{\mathscr{U}_i\}_{i\in\mathbb{N}}$ of locally finite open covers in M such that $\{\operatorname{st}(L,\mathscr{U}_i)\}_{i\in\mathbb{N}}$ is a neighborhood base of L for each compact subset $L \subset M$. For each $i \in \mathbb{N}$, put $\mathscr{P}_i = f^{-1}(\mathscr{U}_i)$ in X. Then $\mathscr{P}_i$ is a locally finite open cover. Notice that any compact subset $K \subset X$ is a compact set of M. Thus, $\bigcap_{i\in\mathbb{N}} \operatorname{st}(K,\mathscr{P}_i) = \bigcap_{i\in\mathbb{N}} \operatorname{st}(K,\mathscr{U}_i) = K$. The lemma holds. □

Theorem 2. *The following conditions are equivalent for a space Y:*
1. *Y is a 1-sequence-covering compact image of a space with a weaker metric topology.*
2. *Y is a sequence-covering compact image of a space with a weaker metric topology.*
3. *Y has a sequence $\{\mathscr{F}_i\}_{i\in\mathbb{N}}$ of point-finite sn-covers such that $\bigcap_{i\in\mathbb{N}} \operatorname{st}(y,\mathscr{F}_i) = \{y\}$ for each $y \in Y$.*
4. *Y has a sequence $\{\mathscr{F}_i\}_{i\in\mathbb{N}}$ of point-finite cs-covers such that $\bigcap_{i\in\mathbb{N}} \operatorname{st}(y,\mathscr{F}_i) = \{y\}$ for each $y \in Y$.*

Proof. (1) ⇒ (2), (3) ⇒ (4) Obvious.

(2) ⇒ (4) Suppose $f\colon X \to Y$ is a sequence-covering compact mapping, here X being a space with a weaker metric topology. There is a sequence $\{\mathscr{P}_i\}_{i\in\mathbb{N}}$ of locally finite open covers of X such that $\bigcap_{i\in\mathbb{N}} \operatorname{st}(K,\mathscr{P}_i) = K$ for each compact subset $K \subset X$ by Lemma 1. For each $i \in \mathbb{N}$, put $\mathscr{F}_i = f(\mathscr{P}_i)$. f is compact, so each $\mathscr{F}_i$ is a point-finite cover of Y. Let S be a convergent sequence in Y containing its limit point y_0. f is sequence-covering, so there is a convergent sequence L in X containing its limit point x_0 such that $f(L) = S$. Each $\mathscr{P}_i$ is an open cover of X; there is $P \in \mathscr{P}_i$ such that $x_0 \in P$, so L is eventually in P. Thus, $S = f(L)$ is eventually in $F = f(P) \in \mathscr{F}_i$. Hence each $\mathscr{F}_i$ is a cs-cover of Y. For each $y \in Y$, $f^{-1}(y)$ is a compact subset of X and $\bigcap_{i\in\mathbb{N}} \operatorname{st}(f^{-1}(y),\mathscr{P}_i) = f^{-1}(y)$. Thus $\bigcap_{i\in\mathbb{N}} \operatorname{st}(y,\mathscr{F}_i) = \{y\}$.

(4) ⇒ (3) It suffices to show that whenever $\mathscr{F}$ is a cs-cover of Y, there exists $\mathscr{F}' \subset \mathscr{F}$ which is an sn-cover of Y. Notice that $\mathscr{F}$ is point-finite. For each $y \in Y$, put $(\mathscr{F})_y = \{F\colon y \in F, F \in \mathscr{F}\} = \{F_j\colon j \leqslant k\}$. If each element of $(\mathscr{F})_y$ is not the sequential neighborhood of y, then there is a sequence $\{y_{jn}\}$ converging to y in $Y \setminus F_j$

for each $j \leqslant k$. For each $n \in \mathbb{N}$, $j \in K$, put $z_{j+(n-1)k} = y_{jn}$. Then the sequence $\{z_m\}$ is still converging to y, but not eventually in F_j for each $j \leqslant k$, contradicting that $\mathscr{F}$ is a cs-cover of Y. Thus there exists $F_y \in \mathscr{F}$ which is a sequential neighborhood of y in Y. Then $\mathscr{F}' = \{F_y\colon y \in Y\} \subset \mathscr{F}$ is a point-finite sn-cover of Y.

(3) $\Rightarrow$ (1) For each $i \in \mathbb{N}$, put $\mathscr{F}_i = \{F_\alpha\colon \alpha \in \Lambda_i\}$. Each Λ_i is endowed with discrete topology. Let $M = \{\{\alpha_i\} \in \prod\limits_{i\in\mathbb{N}} \Lambda_i$: there is $y \in Y$ such that $\bigcap\limits_{i\in\mathbb{N}} F_{\alpha_i} = \{y\}\}$ and give M the subspace topology induced from the usual product topology. Then M is a metric space. Let $X = \{(y, \{\alpha_i\}) \in Y \times M\colon y \in \bigcap\limits_{i\in\mathbb{N}} F_{\alpha_i}\}$. Let f and p be the restrictions to X of the projections of $Y \times M$ onto Y and M. For each $\{\alpha_i\} \in M$, there is $y \in Y$ such that $\bigcap\limits_{i\in\mathbb{N}} F_{\alpha_i} = \{y\}$. Then $p^{-1}(\{\alpha_i\}) = (y, \{\alpha_i\})$, and p is a one-to-one mapping. Thus X is a space with a weaker metric topology. As $\mathscr{F}_i$ is a point-finite cover of Y for each $i \in \mathbb{N}$, it is easy to show that f is a compact mapping.

Next we prove that f is a 1-sequence-covering mapping.

Take $y_0 \in Y$. For each $i \in \mathbb{N}$, choose $\alpha_i \in \Lambda_i$ such that F_{α_i} is a sequential neighborhood of y_0. Let $\beta_0 = (y_0, \{\alpha_i\}) \in Y \times \prod\limits_{i\in\mathbb{N}} \Lambda_i$. Then $\beta_0 \in f^{-1}(y_0) \subset Y \times M$. If $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in Y converging to y_0, then $\{y_n\}_{n\in\mathbb{N}}$ is eventually in F_{α_i} for each $i \in \mathbb{N}$. For each $n \in \mathbb{N}$, if $y_n \in F_{\alpha_i}$, define $\alpha_{in} = \alpha_i$; if $y_n \notin F_{\alpha_i}$, take $\alpha_{in} \in \Lambda_i$ such that $y_n \in F_{\alpha_{in}}$. Thereby, there exists $n_i \in \mathbb{N}$ such that $\alpha_{in} = \alpha_i$ when $n \geqslant n_i$. Thus the sequence $\{\alpha_{in}\}_{n\in\mathbb{N}}$ is converging to α_i in Λ_i. Put $\beta_n = (y_n, \{\alpha_{in}\})$ for each $n \in \mathbb{N}$. Then $f(\beta_n) = y_n$ and the sequence $\{\beta_n\}_{n\in\mathbb{N}}$ is converging to β_0 in X. So f is a 1-sequence-covering mapping. □

Theorem 3. *The following conditions are equivalent for a space Y:*

1. *Y is a sequentially-quotient compact image of a space with a weaker metric topology.*
2. *Y has a sequence $\{\mathscr{F}_i\}_{i\in\mathbb{N}}$ of point-finite cs^*-covers such that $\bigcap\limits_{i\in\mathbb{N}} \operatorname{st}(y, \mathscr{F}_i) = \{y\}$ for each $y \in Y$.*

Proof. (1) $\Rightarrow$ (2) Suppose $f\colon X \to Y$ is a sequentially-quotient compact mapping, here X being a space with a weaker metric topology. There is a sequence $\{\mathscr{P}_i\}_{i\in\mathbb{N}}$ of locally finite open covers of X and $\{\mathscr{F}_i\}_{i\in\mathbb{N}} = \{f(\mathscr{P}_i)\}_{i\in\mathbb{N}}$ is a sequence of point-finite covers such that $\bigcap\limits_{i\in\mathbb{N}} \operatorname{st}(y, \mathscr{F}_i) = \{y\}$ for each $y \in Y$ (see the proof of (2) $\Rightarrow$ (4) in Theorem 2). We show that each $\mathscr{F}_i$ is a cs^*-cover of Y.

Let S be a convergent sequence in Y containing its limit point y_0. f is sequentially-quotient, so there is a convergent sequence L in X containing its limit point x_0 such that $f(L)$ is an infinite subsequence of S. As each $\mathscr{P}_i$ is an open cover of X, there is $P \in \mathscr{P}_i$ such that $x_0 \in P$. So L is eventually in P and $f(L)$ is eventually in $F = f(P) \in \mathscr{F}_i$. Hence each $\mathscr{F}_i$ is a cs^*-cover of Y.

(2) $\Rightarrow$ (1) For each $i \in \mathbb{N}$, put $\mathscr{F}_i = \{F_\alpha\colon \alpha \in \Lambda_i\}$. Each Λ_i is endowed with discrete topology. Let $M = \{\{\alpha_i\} \in \prod\limits_{i\in\mathbb{N}} \Lambda_i\colon$ there is $y \in Y$ such that $\bigcap\limits_{i\in\mathbb{N}} F_{\alpha_i} = \{y\}\}$ and give M the subspace topology induced from the usual product topology. Then M is a metric space. Let $X = \{(y, \{\alpha_i\}) \in Y \times M\colon\ y \in \bigcap\limits_{i\in\mathbb{N}} F_{\alpha_i}\}$. Let f and p be the restrictions to X of the projections of $Y \times M$ onto Y and M. From the proof of Theorem 2, X is a space with a weaker metric topology and f is a compact mapping.

It is sufficient to show that f is a sequentially-quotient mapping.

Let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence converging to y_0 in Y. Without loss of generality, suppose $y_n \neq y_0$ for each $n \in \mathbb{N}$. As $\mathscr{F}_1$ is a cs^*-cover of Y, there exists a subsequence T_1 of $\{y_n\}_{n\in\mathbb{N}}$ and $\alpha_1 \in \Lambda_1$ such that T_1 is eventually in F_{α_1}. Inductively, for each $i \in \mathbb{N}$ we can choose T_i and $\alpha_i \in \Lambda_i$ such that T_{i+1} is a subsequence of T_i and T_i is eventually in F_{α_i}. Thus $T_i \subset \bigcap\limits_{k\leqslant i} F_{\alpha_k}$. Take $y_{n_i} \in T_i$ and $\beta_i \in f^{-1}(y_{n_i})$ such that $n_i < n_{i+1}$ and that $\pi_k(\beta_i) = \alpha_{k-1}$ when $1 < k \leqslant i+1$. Thus $\lim\limits_{i\to\infty} \pi_k(\beta_i) = \alpha_{k-1}$. Put $\beta_0 = (y_0, \{\alpha_i\})$. Then the sequence $\{\beta_i\}_{i\in\mathbb{N}}$ is converging to β_0 in X. Thus f is a sequentially-quotient mapping. $\square$

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