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Applications of Mathematics, Vol. 54 (2009), No. 6, 527–549

Persistent URL: `http://dml.cz/dmlcz/140382`

A SURVEY AND SOME NEW RESULTS ON THE EXISTENCE OF SOLUTIONS OF IPBVPs FOR FIRST ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS*

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(Received October 22, 2007)

Abstract. This paper deals with the periodic boundary value problem for nonlinear impulsive functional differential equation

$$\begin{cases} x'(t) = f(t, x(t), x(\alpha_1(t)), \ldots, x(\alpha_n(t))) \text{ for a.e. } t \in [0,T], \\ \Delta x(t_k) = I_k(x(t_k)),\ k = 1, \ldots, m, \\ x(0) = x(T). \end{cases}$$

We first present a survey and then obtain new sufficient conditions for the existence of at least one solution by using Mawhin's continuation theorem. Examples are presented to illustrate the main results.

Keywords: periodic boundary value problem, impulsive differential equation, fixed-point theorem, growth condition

MSC 2010: 34B10, 34B15

1. Introduction

In the past twenty years, there has been many papers concerned with the solvability of periodic boundary value problems for first order impulsive differential equations (IPBVPs for short) [1]–[4], [6]–[23]. We address some of the related ones.

Using fixed point theorems and the lower and upper solution methods, in [16], a pioneering paper concerning the solvability of periodic boundary value problems,

* The author is supported by the Science Foundation of Hunan Province (06JJ5008) and the Natural Sciences Foundation of Guangdong province (No:7004569).

Nieto studied the following IPBVP

$$
(1) \qquad \begin{cases} x'(t) + \lambda x(t) = F(t, x(t)) \text{ for a.e. } t \in [0, T], \\ x(t_k^+) - x(t_k) = I_k(x(t_k)), \ k = 1, \dots, p, \\ x(0) = x(T), \end{cases}
$$

where $\lambda \neq 0$, $J = [0, T]$, $0 = t_0 < t_1 < \ldots < t_p < t_{p+1} = T$. He transformed (1) into the integral equation

$$
x(t) = \int_0^T g(t, s) F(s, x(s)) \, \mathrm{d}s + \sum_{k=1}^{p} g(t, t_k) I_k(x(t_k)),
$$

where

$$
g(t, s) = \frac{1}{1 - \mathrm{e}^{-\lambda T}} \begin{cases} \mathrm{e}^{-\lambda(t-s)}, \ 0 \leqslant s < t \leqslant T, \\ \mathrm{e}^{-\lambda(T+t-s)}, \ 0 \leqslant t \leqslant s \leqslant T. \end{cases}
$$

Then it was showed that IPBVP(1) has at least one solution. The main assumptions in [16] are one of the following:

(H_1) F is bounded and I_k $(k = 1, \dots, p)$ are bounded;

(H_2) There exists $l_k > 0$ such that $|I_k(x) - I_k(y)| \leqslant l_k |x - y|$ and there is $l > 0$ such that $|F(t, x) - F(t, y)| \leqslant l|x - y|$ holds for all $t \in J$ and $(x, y) \in \mathbb{R}^2$;

(H_3) There exist $\alpha \in [0, 1)$, $\alpha_k \in [0, 1)$ $(k = 1, \dots, p)$ and $a_k, b_k, b \in \mathbb{R}$, $a \in PC(J)$ such that

$$
|F(t, x)| \leqslant a(t) + b|x|^{\alpha}, \quad |I_k(x)| \leqslant a_k + b_k |x|^{\alpha_k}, \ k = 1, \dots, p,
$$

hold for all $t \in J$ and $x \in \mathbb{R}$.

In [17], Nieto considered the following IPBVP

$$
(2) \qquad \begin{cases} x'(t) + F(t, x(t)) = 0 \text{ for a.e. } t \in [0, 1], \\ x(t_k^+) - x(t_k) = I_k(x(t_k)), \ k = 1, 2, \dots, p, \\ x(0) = x(T), \end{cases}
$$

where $0 = t_0 < t_1 < \ldots < t_p < t_{p+1} = T$, F is an impulsive Caratheodory function, I_k is continuous. At this time, IPBVP(2) cannot be transformed into an integral equation. He proved the following theorem.

Theorem A ([17]). *Suppose there exist constants $r > 0$ and $k > 0$ such that*

$$\frac{F(t,u)}{u} \geqslant k > 0 \quad \text{for a.e. } t \in J \text{ and for every } |u| \geqslant r;$$
$$\lim_{u\to 0} \frac{I_k(u)}{u} = 0 \quad \text{for } k = 1,\dots,p.$$

Then IPBVP(2) *has at least one solution.*

In the paper [13], the author proved that if there is $r > 0$, $k > 0$, $c_j, k_j \in \mathbb{R}$, and $\xi \in L^1(J)$ such that

$$\begin{gathered}\frac{F(t,u)}{u} \geqslant k + \frac{\xi(t)}{u} \quad \text{for a.e. } t \in J,\ |u| > r,\\ |I_k(x)| \leqslant c_k + k_k|x|,\ |x| > r,\ k = 1,\dots,p,\\ \sum_{k=1}^{p} k_j < 1 - \mathrm{e}^{-kT},\end{gathered}$$

then IPBVP(2) has at least one solution.

In [4], Franco and Nieto studied the IPBVP

$$\tag{3} \begin{cases} x'(t) = f(t,x(t)) \quad \text{for a.e. } t \in J,\\ x(t_k^+) - x(t_k) = I_k(x(t_k)),\ k = 1,2,\dots,p,\\ x(0) = x(T). \end{cases}$$

Using upper and lower solutions method and the monotone technique, they proved IPBVP(3) has at least one solution under the existence assumptions of lower solution α and upper solution β and the following condition:

(H_4) I_k are continuous and nondecreasing and f satisfies

$$f(t,u) - f(t,v) \geqslant -M(u-v)$$

for a.e. $t \in J$ and all $(u,v) \in \mathbb{R}^2$ with $\alpha(t) \leqslant v \leqslant u \leqslant \beta(t)$, where $M = \min\{M_\alpha, M_\beta\}$ and M_α and M_β satisfy

$$-\int_{t_p}^{T} \mathrm{e}^{-M_\beta(T-s)}[f(s,\beta(s)) - \beta'(s)]\,\mathrm{d}s \geqslant \beta(T) - \beta(0)$$

and

$$\int_{t_p}^{T} \mathrm{e}^{-M_\alpha(T-s)}[f(s,\alpha(s)) - \alpha'(s)]\,\mathrm{d}s \geqslant \alpha(0) - \beta(T).$$

In recent papers, Liu and Ge [15], Tang and Chen [21], Li, Lin, Jiang and Zhang [9], and Liu, Bai and Ge [14] studied the existence of periodic solutions of the following IPBVP with linear impulse effects

$$\begin{cases} x'(t)+a(t)x(t)+F(t,x(t-\tau(t)))=0 \text{ for a.e. } t\in\mathbb{R},\\ x(t_k^+)-x(t_k)=b_kx(t_k),\ \ k=1,2,\ldots. \end{cases} \tag{4}$$

Using a fixed point theorem in cones in Banach spaces, they proved that the equation (4) has at least three positive periodic solutions under some assumptions imposed on F and b_k, and at least one periodic solution under some other assumption.

In a recent paper [10], Li and Shen studied the problem

$$\begin{cases} x'(t)=f(t,x(t),x(\theta(t))) \text{ for a.e. } t\in[0,T],\\ \Delta x(t_k)=I_k(x(t_k)),\ \ k=1,\ldots,m,\\ x(0)=x(T). \end{cases} \tag{5}$$

They proposed the following definition.

Definition 1. We say that the functions $\alpha,\beta\in X$ are lower and upper solution of IPBVP(5) if there exist $M,N\geqslant 0$ and $0\leqslant L_k<1$ such that

$$\begin{cases} \alpha'(t)\leqslant f(t,\alpha(t),\alpha(\theta(t)))-a(t),\ \ t\in[0,T],\\ \Delta\alpha(t_k)\leqslant I_k(\alpha(t_k))-L_ka_k,\ \ k=1,\ldots,p, \end{cases}$$

where

$$a(t)=\begin{cases} 0, & \alpha(0)\leqslant\alpha(T),\\ \dfrac{Mt+N\theta(t)+1}{T}(\alpha(0)-\alpha(T)), & \alpha(0)>\alpha(T), \end{cases}$$

$$a_k=\begin{cases} 0, & \alpha(0)\leqslant\alpha(T),\\ \dfrac{t_k}{T}(\alpha(0)-\alpha(T)), & \alpha(0)>\alpha(T), \end{cases}$$

and

$$\begin{cases} \beta'(t)\geqslant f(t,\beta(t),\beta(\theta(t)))-b(t),\ \ t\in[0,T],\\ \Delta\beta(t_k)\leqslant I_k(\beta(t_k))-L_kb_k,\ \ k=1,\ldots,p, \end{cases}$$

where

$$b(t)=\begin{cases} 0, & \beta(0)\leqslant\beta(T),\\ \dfrac{Mt+N\theta(t)+1}{T}(\beta(T)-\beta(0)), & \beta(0)>\beta(T), \end{cases}$$

$$b_k=\begin{cases} 0, & \beta(0)\leqslant\beta(T),\\ \dfrac{t_k}{T}(\beta(0)-\beta(T)), & \beta(0)>\beta(T), \end{cases}$$

respectively. Then they proved the following theorem.

Theorem B ([10]). *Suppose that the following conditions hold:*

(H$_5$) *α and β are lower and upper solutions for* (5) *with $\alpha \leqslant \beta$;*

(H$_6$) *$g(t,x,y)-g(t,u,v)-M(x-u)-N(y-v)$ for every $t \in [0,T]$, $\alpha \leqslant u \leqslant x \leqslant \beta$, $\alpha(\theta(t)) \leqslant v(\theta(t)) \leqslant y(\theta(t)) \leqslant \beta(\theta(t))$;*

(H$_7$) *$I_k \in C(\mathbb{R},\mathbb{R})$ satisfies $I_k(x) - I_k(y) \geqslant -L_k(x-y)$ for $\beta(t_k) \leqslant y(t_k) \leqslant x(t_k) \leqslant \alpha(t_k)$, $0 \leqslant L_k < 1$, $k = 1,\dots,p$;*

(H$_8$) $N \int_0^T \prod\limits_{t<t_k<T} (1-L_k)\mathrm{e}^{M(t-\theta(t))}\,\mathrm{d}t \leqslant \prod\limits_{k=1}^{p}(1-L_k)$.

Then there exist monotone sequences $\{\overline{\alpha_n}(t)\}$ and $\{\overline{\beta_n}(t)\}$ with $\overline{\alpha_0}(t) = \overline{\alpha}(t)$ and $\overline{\beta_0}(t) = (\overline{\beta})(t)$, where $\overline{\alpha}(t)$ and $(\overline{\beta})(t)$ are as follows

$$\overline{\alpha}(t) = \begin{cases} \alpha(t), & \alpha(0) \leqslant \alpha(T), \\ \alpha(t) + \dfrac{t}{T}(\alpha(0)-\alpha(T)), & \alpha(0) > \alpha(T), \end{cases}$$

and

$$\overline{\beta}(t) = \begin{cases} \beta(t), & \beta(0) \leqslant \beta(T), \\ \beta(t) - \dfrac{t}{T}(\beta(T)-\beta(0)), & \beta(0) > \beta(T), \end{cases}$$

such that $\lim\limits_{n\to\infty} \overline{\alpha_n}(t) = \gamma(t)$ and $\lim\limits_{n\to\infty} \overline{\beta_n}(t) = \varrho(t)$ uniformly hold on $[0,1]$, *where $\gamma(t)$ and $\varrho(t)$ are minimal and maximal solutions of* IPBVP(5), *respectively.*

Yang and Shen in [23], [11], [7], by introducing the concept of lower and upper solutions of IPBVP(5), proved that the method of lower and upper solutions coupled with a monotone iterative technique also works.

In a recent paper [19], Nieto and Rodriguez-Lopez studied the problem

$$\begin{cases} x'(t) = f(t,x(t),[\psi_k x_k](t)) \text{ for a.e. } t \in J, \\ x(t_k^+) - x(t_k) = I_k([\psi_k x_k](t_k)), \ k = 1,2,\dots,p, \\ x(0) = x(T), \end{cases}$$

where the functional dependence is not necessarily a Lipschitz function (this paper may be the first paper concerning the IPBVP(5) with non-Lipschitz functions). The new maximum principle obtained improves and extends previous results; the uniqueness of solution between a lower and an upper solution for a particular nonlinear problem was presented in this paper. The conditions for the existence of extremal solutions in an interval delimited by a lower and an upper solution were also established.

Recently, Chen, Tisdell, and Yuan [1] obtained some new results concerning the existence of solutions to the impulsive first-order, nonlinear ordinary differential equation with periodic boundary conditions. The ideas in [1] involve differential inequalities and Schaefer's fixed-point theorem.

In the recent paper [12], the author considered the following BVP

$$\begin{cases} x'(t) + a(t)x(t) = f(t, x(t), x(\alpha_1(t)), \ldots, x(\alpha_n(t))) \text{ for a.e. } t \in J, \\ x(t_k^+) - x(t_k) = I_k(x(t_k)),\ k = 1, 2, \ldots, p, \\ x(0) = x(T), \end{cases}$$

by using Schaefer's fixed-point theorem, but the assumptions imposed on I_k are either $I_k(x)(2x + I_k(x)) \leqslant 0$ for all $x \in \mathbb{R}$ or $I_k(x)(2x + I_k(x)) \geqslant 0$ for all $x \in \mathbb{R}$; and $\alpha(T) = \int_0^T a(s) \neq 0$ is supposed.

To the best of our knowledge, there was no paper concerned with the existence of solutions of periodic boundary value problems for first order impulsive functional differential equations under the assumptions that f or I_k are superlinear.

In this paper, we are concerned with the periodic boundary value problems for nonlinear impulsive functional differential equations

$$\begin{cases} x'(t) = f(t, x(t), x(\alpha_1(t)), \ldots, x(\alpha_n(t))) \text{ for a.e. } t \in [0, T], \\ \Delta x(t_k) = I_k(x(t_k)),\ k = 1, \ldots, m, \\ x(0) = x(T), \end{cases} \tag{6}$$

where $T > 0$, $0 < t_1 < \ldots < t_m < T$ are constants, $\alpha_k \in C^1([0, T], [0, T])$ for all $k = 1, \ldots, n$, and its inverse function denoted by β_k, f is an impulsive Caratheodory function, while I_k are continuous functions.

The purpose of this paper is to establish further existence results to solutions of IPBVP(6) by using Mawhin's continuation theorem. We do not use Green' functions, nor Schaefer's fixed-point theorem, nor fixed point theorems in cones in Banach spaces, nor upper and lower solution methods, nor monotone iterative techniques, and these are the places where the novelty of this paper lies.

The remainder of this paper is divided as follows: In Section 2, we present preliminary notations and results, the main results in this paper will be given in Section 3, and in Section 4, we give some examples to illustrate the main theorems; these examples cannot be solved by known results, cf. the remarks in Section 4.

2. Preliminary results

Let $u\colon J=[0,T]\to\mathbb{R}$, and $0=t_0<t_1<\ldots<t_m<t_{m+1}=T$; for $k=0,\ldots,m$, define the function $u_k\colon (t_k,t_{k+1}]\to\mathbb{R}$ by $u_k(t)=u(t)$. We will use the following Banach spaces

$$X=\left\{\begin{array}{l} u\colon J\to\mathbb{R},\ u_k\in C^0(t_k,t_{k+1}],\ k=0,\ldots,m,\ \text{there exist the limits}\\ \lim\limits_{t\to t_k^+}u(t),\ \lim\limits_{t\to 0^+}u(t)=u(0)\end{array}\right\}$$

and

$$Y=X\times\mathbb{R}^m$$

with the norms

$$\|x\|=\sup_{t\in[0,T]}|x(t)|$$

for $x\in X$ and

$$\|y\|=\max\bigl\{\|u\|,\max_{1\leqslant k\leqslant m}\{|x_k|\}\bigr\}$$

for $y=\{u,x_1,\ldots,x_m\}\in Y$.

A function F is an impulsive Caratheodory function if

* $F(\bullet,u_0,u_1,\ldots,u_n)\in X$ for each $u=(u_0,\ldots,u_n)\in\mathbb{R}^{n+1}$;
* $F(t,\bullet,\ldots,\bullet)$ is continuous for a.e. $t\in J$;
* for each $r>0$ there is $h_r\in L^1(J)$ so that

$$|F(t,u_0,u_1,\ldots,u_n)|\leqslant h_r(t)\ \text{for a.e. } t\in J\setminus\{t_1,\ldots,t_m\}$$

for every u satisfying $\|(u_0,u_1,\ldots,u_n)\|>r$.

By a solution of IPBVP(6) we mean a function $u\in X$ satisfying all the equations in (6).

Now, we define the linear operator $L\colon D(L)\subseteq X\to Y$ and the nonlinear operator $N\colon X\to Y$ by

$$Lx(t)=\begin{pmatrix}x'(t)\\ \Delta x(t_1)\\ \vdots\\ \Delta x(t_m)\end{pmatrix}\quad\text{for } x\in D(L),$$

where $D(L)=\{u\in X,\ u_k'\in C^0(t_k,t_{k+1}],\ k=0,1,\ldots,m,\ \lim\limits_{t\to 0}u(t)=u(0),\ x(0)=x(T)\}$ and

$$Nx(t)=\begin{pmatrix}f(t,x(t),x(\alpha_1(t)),\ldots,x(\alpha_n(t)))\\ I_1(x(t_1))\\ \vdots\\ I_m(x(t_m))\end{pmatrix}\quad\text{for } x\in X.$$

Since f and I_k are continuous, it is easy to prove the following:

(i) $\operatorname{Ker} L = \{x(t) = c,\ t \in [0,T],\ c \in \mathbb{R}\}$.

(ii) $\operatorname{Im} L = \left\{(y(t), a_1, \ldots, a_m) \in Y,\ \int_0^T y(s)\,\mathrm{d}s + \sum_{k=1}^{m} a_k = 0\right\}$.

(iii) L is a Fredholm operator of index zero.

(iv) There exist projections $P\colon X \to X$ and $Q\colon Y \to Y$ such that $\operatorname{Ker} L = \operatorname{Im} P$, $\operatorname{Ker} Q = \operatorname{Im} L$. Furthermore, let $\Omega \subset X$ be an open bounded subset with $\overline{\Omega} \cap D(L) \neq \emptyset$, then N is L-compact on $\overline{\Omega}$.

(v) $x \in D(L)$ is a solution of IPBVP(6) if and only if x is a solution of the operator equation $Lx = Nx$ in $D(L)$.

We omit the details of the proofs. The projections $P\colon X \to X$ and $Q\colon Y \to Y$, the isomorphism $\wedge\colon \operatorname{Ker} L \to Y/\operatorname{Im} L$ and the generalized inverse $K_p\colon \operatorname{Im} L \to D(L) \cap \operatorname{Im} P$ are defined as follows:

$$
\begin{aligned}
Px(t) &= x(0) \quad \text{for } x \in X,\\
Q(y(t), a_1, \ldots, a_m) &= \left(\frac{1}{T}\int_0^T y(s)\,\mathrm{d}s + \frac{1}{T}\sum_{k=1}^{m} a_k, 0, \ldots, 0\right),\\
\wedge(c) &= (c, 0, \ldots, 0), \quad c \in \mathbb{R},\\
K_p(y(t), a_1, \ldots, a_m) &= \int_0^t y(s)\,\mathrm{d}s + \sum_{0<t_k<t} a_k.
\end{aligned}
$$

The following abstract existence lemma is used in this paper, whose proof can be found in [5].

Lemma 2.1 ([5]). *Let L be a Fredholm operator of index zero and let N be L-compact on Ω. Assume that the following conditions are satisfied:*

(i) *$Lx \neq \lambda Nx$ for every $(x, \lambda) \in [(D(L) \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0,1)$;*

(ii) *$Nx \notin \operatorname{Im} L$ for every $x \in \operatorname{Ker} L \cap \partial\Omega$;*

(iii) *$\deg(\wedge QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) \neq 0$, where $\wedge\colon \operatorname{Ker} L \to Y/\operatorname{Im} L$ is an isomorphism.*

Then the equation $Lx = Nx$ has at least one solution in $D(L) \cap \overline{\Omega}$.

3. Main results

We make the following assumptions which should be used in the main results.

(A_1) $I_k(x)(2x+I_k(x)) \leqslant 0$ for all $x \in \mathbb{R}$ and $k = 1, \ldots, m$.

(A_2) $xI_k(x) \geqslant 0$ for all $x \in \mathbb{R}$ and $k = 1, \ldots, m$.

(C_1) There exist impulsive Caratheodory functions $h\colon [0,T] \times \mathbb{R}^{n+1} \to \mathbb{R}$, $r \in X$, and $g_i\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ such that

(i) $f(t, x_0, \ldots, x_n) = h(t, x_0, \ldots, x_n) + \sum\limits_{i=0}^{n} g_i(t, x_i) + r(t)$ holds for all $(t, x_0, \ldots, x_n) \in [0,T] \times \mathbb{R}^{n+1}$.

(ii) There exist constants $q \geqslant 0$ and $\beta > 0$ such that

$$h(t, x_0, \ldots, x_n)x_0 \leqslant -\beta |x_0|^{q+1}$$

holds for all $(t, x_0, \ldots, x_n) \in [0,T] \times \mathbb{R}^{n+1}$.

(iii) $\lim\limits_{|x|\to\infty} \sup\limits_{t\in[0,T]} |g_i(t,x)|/|x|^q = r_i \in [0, \infty)$ for $i = 0, \ldots, n$.

(C_2) There exist impulsive Caratheodory functions $h\colon [0,T] \times \mathbb{R}^{n+1} \to \mathbb{R}$, $r \in X$, and $g_i\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ so that

(i) $f(t, x_0, \ldots, x_n) = h(t, x_0, \ldots, x_n) + \sum\limits_{i=0}^{n} g_i(t, x_i) + r(t)$ holds for all $(t, x_0, \ldots, x_n) \in [0,T] \times \mathbb{R}^{n+1}$.

(ii) There exist constants $q \geqslant 0$ and $\beta > 0$ such that

$$h(t, x_0, \ldots, x_n)x_0 \geqslant \beta |x_0|^{q+1}$$

holds for all $(t, x_0, \ldots, x_n) \in [0,T] \times \mathbb{R}^{n+1}$.

(iii) $\lim\limits_{|x|\to\infty} \sup\limits_{t\in[0,T]} |g_i(t,x)|/|x|^q = r_i \in [0, \infty)$ for $i = 0, \ldots, n$.

(E) There exists a constant $M_0 > 0$ such that

$$c\biggl(\frac{1}{T}\int_0^T f(t, c, c, \ldots, c)\,\mathrm{d}t + \frac{1}{T}\sum_{k=1}^{m} I_k(c)\biggr) > 0$$

for all $|c| > M_0$ or

$$c\biggl(\frac{1}{T}\int_0^T f(t, c, c, \ldots, c)\,\mathrm{d}t + \frac{1}{T}\sum_{k=1}^{m} I_k(c)\biggr) < 0$$

for all $|c| > M_0$.

Theorem 3.1. *Suppose that* (E), (A_2) *and* (C_2) *hold. Then* IPBVP(6) *has at least one solution if*

$$r_0 + \sum_{k=1}^{n} r_k \|\beta_k'\|_\infty^{q/(1+q)} < \beta, \tag{7}$$

where β_k is the inverse function of α_k, $k = 1, \ldots, n$.

Proof. To apply Lemma 2.1, we define an open bounded subset Ω of X centered at the origin such that (i), (ii) and (iii) of Lemma 2.1 hold. It is based upon three steps to obtain Ω. The proof of this theorem is divided into four steps.

Step 1. Set $\Omega_1 = \{x \in D(L)\colon Lx = Nx,\ \lambda \in (0,1)\}$. We prove that Ω_1 is bounded. Suppose $x \in \Omega_1$. Then

$$\begin{cases} x'(t) = \lambda f(t, x(t), x(\alpha_1(t)), \ldots, x(\alpha_n(t))),\ t \in [0,T],\ t \neq t_k,\ k = 1, \ldots, m, \\ \Delta x(t_k) = \lambda I_k(x(t_k)),\ k = 1, \ldots, m, \\ x(0) = x(T). \end{cases} \tag{8}$$

We do the following two substeps.

Substep 1.1. Prove that there is a constant $M > 0$ so that $\int_0^T |x(s)|^{q+1}\,\mathrm{d}s \leqslant M$ for each $x \in \Omega_1$.

Multiplying both sides of the equation (8) by $x(t)$ and integrating it from 0 to T, we get

$$\begin{aligned} &\frac{1}{2}(x(T))^2 - \frac{1}{2}(x(0))^2 - \frac{1}{2}\sum_{k=1}^{m}[(x(t_k^+))^2 - (x(t_k^-))^2] \\ &\quad = \lambda \int_0^T f(s, x(s), x(\alpha_1(s)), \ldots, x(\alpha_n(s)))x(s)\,\mathrm{d}s \\ &\quad = \lambda\Big(\int_0^T h(s, x(s), x(\alpha_1(s)), \ldots, x(\alpha_n(s)))x(s)\,\mathrm{d}s + \int_0^T g_0(s, x(s))x(s)\,\mathrm{d}s \\ &\qquad + \sum_{i=1}^{n}\int_0^T g_i(s, x(\alpha_i(s))x(s)\,\mathrm{d}s + \int_0^T r(s)x(s)\,\mathrm{d}s\Big). \end{aligned}$$

It follows from (A_2) that

$$\begin{aligned} (x(t_k^+))^2 - (x(t_k^-))^2 &= (x(t_k^+) - x(t_k^-))(x(t_k^+) + x(t_k^-)) \\ &= \Delta x(t_k^-)(2x(t_k^-) + \Delta x(t_k^-)) \\ &= \lambda I_k(x(t_k^-))(2x(t_k^-) + \lambda I_k(x(t_k^-))) \\ &\geqslant 2\lambda x(t_k^-)I_k(x(t_k^-)) \geqslant 0. \end{aligned}$$

We get

$$\int_0^T h(s,x(s),x(\alpha_1(s)),\dots,x(\alpha_n(s)))x(s)\,\mathrm{d}s+\int_0^T g_0(s,x(s))x(s)\,\mathrm{d}s$$
$$+\sum_{i=1}^n\int_0^T g_i(s,x(\alpha_i(s))x(s)\,\mathrm{d}s+\int_0^T r(s)x(s)\,\mathrm{d}s\leqslant 0.$$

It follows from (C_2) that

$$\begin{aligned}\beta\int_0^T &|x(s)|^{q+1}\,\mathrm{d}s\\ &\leqslant -\int_0^T g_0(s,x(s))x(s)\,\mathrm{d}s-\sum_{i=1}^n\int_0^1 g_i(s,x(\alpha_i(s))x(s)\,\mathrm{d}s-\int_0^T r(s)x(s)\,\mathrm{d}s\\ &\leqslant \int_0^T |g_0(s,x(s))||x(s)|\,\mathrm{d}s+\sum_{i=1}^n\int_0^T |g_i(s,x(\alpha_i(s))||x(s)|\,\mathrm{d}s+\int_0^T |r(s)||x(s)|\,\mathrm{d}s.\end{aligned}$$

Choose $\varepsilon>0$ such that

$$(r_0+\varepsilon)+\sum_{k=1}^n(r_k+\varepsilon)\|\beta_k'\|_\infty^{q/(q+1)}<\beta. \tag{9}$$

For such $\varepsilon>0$, there is $\delta>0$ so that for every $i=0,1,\dots,n$,

$$|g_i(t,x)|<(r_i+\varepsilon)|x|^q\quad\text{uniformly for } t\in[0,T] \text{ and } |x|>\delta. \tag{10}$$

Let, for $i=1,\dots,n$, $\Delta_{1,i}=\{t\colon t\in[0,T],\,|x(\alpha_i(t))|\leqslant\delta\}$, $\Delta_{2,i}=\{t\colon t\in[0,T],\,|x(\alpha_i(t))|>\delta\}$, $g_{\delta,i}=\max\limits_{t\in[0,T],|x|\leqslant\delta}|g_i(t,x)|$, and $\Delta_1=\{t\in[0,T],\,|x(t)|\leqslant\delta\}$, $\Delta_2=\{t\in[0,T],\,|x(t)|>\delta\}$. Then we get

$$\begin{aligned}\beta\int_0^T &|x(s)|^{q+1}\,\mathrm{d}s\\ &\leqslant \int_{\Delta_1}|g_0(s,x(s))||x(s)|\,\mathrm{d}s+\int_{\Delta_2}|g_0(s,x(s))||x(s)|\,\mathrm{d}s\\ &\quad+\sum_{i=1}^n\int_{\Delta_{1,i}}|g_i(s,x(\alpha_i(s))||x(s)|\,\mathrm{d}s+\sum_{i=1}^n\int_{\Delta_{2,i}}|g_i(s,x(\alpha_i(s))||x(s)|\,\mathrm{d}s\\ &\quad+\int_0^T|r(s)||x(s)|\,\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&\leqslant (r_0+\varepsilon)\int_0^T |x(s)|^{q+1}\,\mathrm{d}s + \sum_{k=1}^n (r_k+\varepsilon)\int_0^T |x(\alpha_i(s))|^q |x(s)|\,\mathrm{d}s \\
&\quad + \int_0^T |r(s)||x(s)|\,\mathrm{d}s + \sum_{k=0}^n g_{\delta,i}\int_0^T |x(s)|\,\mathrm{d}s \\
&\leqslant (r_0+\varepsilon)\int_0^T |x(s)|^{q+1}\,\mathrm{d}s \\
&\quad + \sum_{k=1}^n (r_k+\varepsilon)\Big(\int_0^T |x(\alpha_i(s))|^{q+1}\,\mathrm{d}s\Big)^{q/(q+1)}\Big(\int_0^T |x(s)|^{q+1}\,\mathrm{d}s\Big)^{1/(q+1)} \\
&\quad + \Big(\int_0^T |r(s)|^{(q+1)/q}\,\mathrm{d}s\Big)^{q/(q+1)}\Big(\int_0^T |x(s)|^{q+1}\,\mathrm{d}s\Big)^{1/(q+1)} + \sum_{i=0}^n \delta_{\delta,i}\int_0^T |x(s)|\,\mathrm{d}s \\
&= (r_0+\varepsilon)\int_0^T |x(s)|^{q+1}\,\mathrm{d}s \\
&\quad + \sum_{k=1}^n (r_k+\varepsilon)\Big(\int_{\alpha_k(0)}^{\alpha_k(T)} |x(u)|^{q+1}|\beta_k'(u)|\,\mathrm{d}u\Big)^{q/(q+1)}\Big(\int_0^T |x(s)|^{q+1}\,\mathrm{d}s\Big)^{1/(q+1)} \\
&\quad + \Big(\int_0^T |r(s)|^{(q+1)/q}\,\mathrm{d}s\Big)^{q/(q+1)}\Big(\int_0^T |x(s)|^{q+1}\,\mathrm{d}s\Big)^{1/(q+1)} \\
&\quad + \sum_{i=0}^n \delta_{\delta,i}T^{q/(q+1)}\Big(\int_0^T |x(s)|^{q+1}\,\mathrm{d}s\Big)^{1/(q+1)} \\
&\leqslant (r_0+\varepsilon)\int_0^T |x(s)|^{q+1}\,\mathrm{d}s \\
&\quad + \sum_{k=1}^n (r_k+\varepsilon)\|\beta_k'\|_\infty^{q/(q+1)}\Big(\int_0^T |x(u)|^{1+q}|\,\mathrm{d}u\Big)^{q/(q+1)}\Big(\int_0^T |x(s)|^{q+1}\,\mathrm{d}s\Big)^{1/(q+1)} \\
&\quad + \Big(\int_0^T |r(s)|^{(q+1)/q}\,\mathrm{d}s\Big)^{q/(q+1)}\Big(\int_0^T |x(s)|^{q+1}\,\mathrm{d}s\Big)^{1/(q+1)} \\
&\quad + \sum_{i=0}^n \delta_{\delta,i}T^{q/(q+1)}\Big(\int_0^T |x(s)|^{q+1}\,\mathrm{d}s\Big)^{1/(q+1)} \\
&= \Big((r_0+\varepsilon) + \sum_{k=1}^n (r_k+\varepsilon)\|\beta_k'\|_\infty^{q/(q+1)}\Big)\int_0^T |x(s)|^{q+1}\,\mathrm{d}s \\
&\quad + \Big(\int_0^T |r(s)|^{(q+1)/q}\,\mathrm{d}s\Big)^{q/(q+1)}\Big(\int_0^T |x(s)|^{q+1}\,\mathrm{d}s\Big)^{1/(q+1)} \\
&\quad + \sum_{i=0}^n \delta_{\delta,i}T^{q/(q+1)}\Big(\int_0^T |x(s)|^{q+1}\,\mathrm{d}s\Big)^{1/(q+1)}.
\end{aligned}
$$

It follows from (9) that there is a constant $M>0$ so that $\int_0^T |x(s)|^{q+1}\,\mathrm{d}s \leqslant M$.

Substep 1.2. Prove that there is a constant $M_1 > 0$ so that $\|x\|_\infty \leqslant M_1$ for each $x \in \Omega_1$.

It follows from Substep 1.1 that there is $\xi \in [0,T]$ so that $|x(\xi)| \leqslant (M/T)^{1/(q+1)}$. Now, we consider two cases.

Case 1. If $t < \xi$, multiplying both sides of equation (8) by $x(t)$ and integrating it from t to ξ, we get, using (A_2), that

$$\begin{aligned}
\frac{1}{2}(x(t))^2 &= \frac{1}{2}(x(\xi))^2 - \frac{1}{2}\sum_{t\leqslant t_k<\xi}[(x(t_k^+))^2 - (x(t_k^-))^2] \\
&\quad - \lambda\int_t^\xi f(s,x(s),x(\alpha_1(s)),\ldots,x(\alpha_n(s)))x(s)\,\mathrm{d}s \\
&\leqslant \frac{1}{2}(M/T)^{2/(q+1)} - \lambda\int_t^\xi f(s,x(s),x(\alpha_1(s)),\ldots,x(\alpha_n(s)))x(s)\,\mathrm{d}s \\
&\leqslant \frac{1}{2}(M/T)^{2/(q+1)} - \lambda\biggl(\int_t^\xi h(s,x(s),x(\alpha_1(s)),\ldots,x(\alpha_n(s)))x(s)\,\mathrm{d}s \\
&\quad + \int_t^\xi g_0(s,x(s))x(s)\,\mathrm{d}s + \sum_{i=1}^n\int_t^\xi g_i(s,x(\alpha_i(s))x(s)\,\mathrm{d}s + \int_t^\xi r(s)x(s)\,\mathrm{d}s\biggr) \\
&\leqslant \frac{1}{2}(M/T)^{2/(q+1)} - \int_t^\xi g_0(s,x(s))x(s)\,\mathrm{d}s \\
&\quad - \sum_{i=1}^n\int_t^\xi g_i(s,x(\alpha_i(s))x(s)\,\mathrm{d}s - \int_t^\xi r(s)x(s)\,\mathrm{d}s \\
&\leqslant \frac{1}{2}(M/T)^{2/(q+1)} + \int_0^T |g_0(s,x(s))||x(s)|\,\mathrm{d}s \\
&\quad + \sum_{i=1}^n\int_0^T |g_i(s,x(\alpha_i(s))||x(s)|\,\mathrm{d}s + \int_0^T |r(s)||x(s)|\,\mathrm{d}s \\
&\leqslant \frac{1}{2}(M/T)^{2/(q+1)} + \biggl[\biggl((r_0+\varepsilon) + \sum_{k=1}^n (r_k+\varepsilon)\|\beta_k'\|_\infty^{q/(1+q)}\biggr)\int_0^T |x(s)|^{q+1}\,\mathrm{d}s \\
&\quad + \biggl(\int_0^T |r(s)|^{(q+1)/q}\,\mathrm{d}s\biggr)^{q/(q+1)}\biggl(\int_0^T |x(s)|^{q+1}\,\mathrm{d}s\biggr)^{1/(q+1)}\biggr] \\
&\quad + (n+1)\delta T^{q/(q+1)}\biggl(\int_0^T |x(s)|^{q+1}\,\mathrm{d}s\biggr)^{1/(q+1)} \\
&\leqslant \frac{1}{2}(M/T)^{2/(q+1)} + \biggl[\biggl((r_0+\varepsilon) + \sum_{k=1}^n (r_k+\varepsilon)\|\beta_k'\|_\infty^{q/(1+q)}\biggr)M \\
&\quad + \biggl(\int_0^T |r(s)|^{(q+1)/q}\,\mathrm{d}s\biggr)^{q/(q+1)} M^{1/(q+1)}\biggr](n+1)\delta T^{q/(q+1)}M^{1/(q+1)} \\
&\quad =: M_2.
\end{aligned}$$

One sees that

$$x^2(t) \leqslant 2M_2 =: M_3 \quad \text{for } t \in [0,\xi].$$

This implies $x^2(0) \leqslant M_3$. Hence, $x^2(T) = x^2(0) \leqslant M_3$. For $t \in [\xi, T]$ we have

$$\begin{aligned}\frac{1}{2}(x(t))^2 = \frac{1}{2}(x(T))^2 - \frac{1}{2} \sum_{\xi \leqslant t_k < t} [(x(t_k^+))^2 - (x(t_k^-))^2] \\ - \lambda \int_t^T f(s, x(s), x(\alpha_1(s)), \ldots, x(\alpha_n(s)))x(s)\,\mathrm{d}s.\end{aligned}$$

Similarly to the above discussion, we get that there is $M_4 > 0$ so that $x^2(t) \leqslant M_4$ for $t \in [\xi, T]$. Altogether this implies that there is $M_1 > 0$ such that $|x(t)| \leqslant M_1$. Thus $\|x\|_\infty \leqslant M_1$.

It follows that Ω_1 is bounded.

Step 2. Let

$$\Omega_2 = \{x \in \operatorname{Ker} L,\ Nx \in \operatorname{Im} L\}.$$

We prove Ω_2 is bounded. Suppose that $x \in \Omega_2$. Then $x(t) = c \in \mathbb{R}$ and

$$\int_0^T f(t, c, c, \ldots, c)\,\mathrm{d}t + \sum_{k=1}^m I_{n-1,k}(c, 0, \ldots, 0) = 0.$$

It follows from (E) that $|c| \leqslant M_0'$.

Step 3. If the first case in (E) holds, let

$$\Omega_3 = \{x \in \operatorname{Ker} L,\ \lambda \wedge x + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\},$$

where $\wedge\colon \operatorname{Ker} L \to \operatorname{Im} Q$ is the linear isomorphism given by $\wedge(c) = c$ for all $c \in \mathbb{R}$, $\lambda \in [0,1]$. Now we show that Ω_3 is bounded. Suppose $x_n(t) = c_n \in \Omega_3$ and $|c_n| \to \infty$ as n tends to infinity. Then

$$\lambda \wedge (c_n) + (1-\lambda)\biggl(\frac{1}{T}\int_0^T f(t, c_n, c_n, \ldots, c_n)\,\mathrm{d}t + \frac{1}{T}\sum_{k=1}^m I_{n-1,k}(c_n, 0, \ldots, 0)\biggr) = 0.$$

Consequently,

$$\lambda c_n^2 = -(1-\lambda)c_n\biggl(\frac{1}{T}\int_0^T f(t, c_n, \ldots, c_n)\,\mathrm{d}t + \frac{1}{T}\sum_{k=1}^m I_{n-1,k}(c_n, 0, \ldots, 0)\biggr).$$

If $\lambda = 1$, then $c_n = 0$. If $\lambda \in [0,1)$ and $|c_n| > M_0$, then $\lambda c_n^2 < 0$, which is a contradiction. Hence, $|c_n| \leqslant M_0$. So Ω_3 is bounded.

If the second case in (E) holds, let

$$\Omega_3 = \{x \in \operatorname{Ker} L,\ \lambda \wedge x - (1-\lambda)QNx = 0,\ \lambda \in [0,1]\}.$$

Similarly as above, we get Ω_3 is bounded.

In the following, we shall show that all the conditions of Lemma 2.1 are satisfied. Let Ω be a non-empty open bounded subset of X centered at zero such that $\Omega \supset \bigcup_{i=1}^{3} \overline{\Omega_i}$ centered at zero. By Lemma 2.1, L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$. By the definition of Ω, we have

(a) $Lx \neq \lambda Nx$ for $x \in (D(L) \setminus \operatorname{Ker} L) \cap \partial\Omega$ and $\lambda \in (0,1)$;

(b) $Nx \notin \operatorname{Im} L$ for $x \in \operatorname{Ker} L \cap \partial\Omega$.

Step 4. We prove (c): $\deg(QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) \neq 0$.

In fact, let $H(x,\lambda) = \lambda \wedge x \pm (1-\lambda)QNx$. According to the definition of Ω, we know $H(x,\lambda) \neq 0$ for $x \in \partial\Omega \cap \operatorname{Ker} L$, thus by the homotopy property of the degree,

$$\begin{aligned}
\deg(QN|\operatorname{Ker} L, \Omega \cap \operatorname{Ker} L, 0) &= \deg(H(\cdot,0), \Omega \cap \operatorname{Ker} L, 0)\\
&= \deg(H(\cdot,1), \Omega \cap \operatorname{Ker} L, 0)\\
&= \deg(I, \Omega \cap \operatorname{Ker} L, 0) \neq 0, \text{ since } 0 \in \Omega.
\end{aligned}$$

Thus by Lemma 2.1, $Lx = Nx$ has at least one solution in $D(L) \cap \overline{\Omega}$, which is a solution of IPBVP(6). The proof is complete. □

Theorem 3.2. *Suppose that* (E), (A_1), *and* (C_1) *hold. Then* IPBVP(6) *has at least one solution if*

$$r_0 + \sum_{k=1}^{n} r_k \|\beta_k'\|_\infty^{q/(q+1)} < \beta. \tag{11}$$

P r o o f. The proof is similar to that of Theorem 3.1. Consider the system (8). It follows from (A_1) that

$$\begin{aligned}
(x(t_k^+))^2 - (x(t_k^-))^2 &= (x(t_k^+) - x(t_k^-))(x(t_k^+) + x(t_k^-))\\
&= \Delta x(t_k^-)(2x(t_k^-) + \Delta x(t_k^-))\\
&= \lambda I_k(x(t_k^-))(2x(t_k^-) + \lambda I_k(x(t_k^-)))\\
&\leqslant \lambda I_k(x(t_k^-))(2x(t_k^-) + I_k(x(t_k^-))) \leqslant 0.
\end{aligned}$$

Hence, one sees that

$$\begin{aligned}
&\int_0^T h(s, x(s), x(\alpha_1(s)), \ldots, x(\alpha_n(s)))x(s)\,\mathrm{d}s + \int_0^T g_0(s, x(s))x(s)\,\mathrm{d}s\\
&\quad + \sum_{i=1}^{n} \int_0^T g_i(s, x(\alpha_i(s))x(s)\,\mathrm{d}s + \int_0^T r(s)x(s)\,\mathrm{d}s \geqslant 0.
\end{aligned}$$

All steps in the remainder of the proof are similar to those of Theorem 3.1 and are omitted.

Now, we suppose the following:

(A_3) $I_k(x) \leqslant 0$ for all $x \in \mathbb{R}$ and $k = 1, \ldots, m$.

(A_4) $I_k(x) \geqslant 0$ for all $x \in \mathbb{R}$ and $k = 1, \ldots, m$.

(C_3) There exist impulsive Caratheodory functions $h\colon [0,T] \times \mathbb{R}^{n+1} \to \mathbb{R}$, $r \in X$, and $g_i\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ such that

(i) $f(t, x_0, \ldots, x_n) = h(t, x_0, \ldots, x_n) + \sum\limits_{i=0}^{n} g_i(t, x_i) + r(t)$ holds for all $(t, x_0, \ldots, x_n) \in [0,T] \times \mathbb{R}^{n+1}$.

(ii) There exist constants $q \geqslant 0$ and $\beta > 0$ such that

$$h(t, x_0, \ldots, x_n) \leqslant -\beta |x_0|^q$$

holds for all $(t, x_0, \ldots, x_n) \in [0,T] \times \mathbb{R}^{n+1}$.

(iii) $\lim\limits_{|x|\to\infty} \sup\limits_{t\in[0,T]} |g_i(t,x)|/|x|^q = r_i \in [0, \infty)$ for $i = 0, \ldots, n$.

(C_4) There exist impulsive Caratheodory functions $h\colon [0,T] \times \mathbb{R}^{n+1} \to \mathbb{R}$, $r \in X$, and $g_i\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ such that

(i) $f(t, x_0, \ldots, x_n) = h(t, x_0, \ldots, x_n) + \sum\limits_{i=0}^{n} g_i(t, x_i) + r(t)$ holds for all $(t, x_0, \ldots, x_n) \in [0,T] \times \mathbb{R}^{n+1}$.

(ii) There exist constants $q \geqslant 0$ and $\beta > 0$ such that

$$h(t, x_0, \ldots, x_n) \geqslant \beta |x_0|^q$$

holds for all $(t, x_0, \ldots, x_n) \in [0,T] \times \mathbb{R}^{n+1}$.

(iii) $\lim\limits_{|x|\to\infty} \sup\limits_{t\in[0,T]} |g_i(t,x)|/|x|^q = r_i \in [0, \infty)$ for $i = 0, \ldots, n$.

(E_1) There exist functions $R, p_i \in X$ such that

$$|f(t, x_0, \ldots, x_n)| \leqslant \sum_{k=0}^{n} p_i(t) |x_i|^q + R(t).$$

(E_2) There exist constants $\alpha_k \geqslant 0$ such that $|I_k(x)| \leqslant \alpha_k |x|$ for all $x \in \mathbb{R}$ and $k = 1, \ldots, m$, with $\sum\limits_{k=1}^{m} \alpha_k < 1$.

Theorem 3.3. *Assume that* (E), (E_1), (E_2), (C_3), *and* (A_3) *hold. Furthermore, suppose that*

$$r_0 + \sum_{k=1}^{n} r_k \|\beta_k'\|_\infty < \beta. \tag{12}$$

Then IPBVP(6) *has at least one solution.*

Proof. Suppose $\lambda \in (0,1)$ and consider problem (8). Integrating the first equation of (8), we get

$$x(T) - x(0) - \lambda \sum_{k=1}^{m} I_k(x(t_k)) = \lambda \int_0^T f(t, x(s), x(\alpha_1(s)), \ldots, x(\alpha_n(s)))\,\mathrm{d}s.$$

Since $I_k(x) \leqslant 0$ due to (A_3), we get

$$\int_0^T [h(t, x(s), x(\alpha_1(s)),\, \ldots, x(\alpha_n(s))) + g_0(t, x(s)) \\ + \sum_{k=1}^{m} g_i(s, x(\alpha_i(s))) + r(s)]\,\mathrm{d}s \geqslant 0.$$

(C_3) implies that

$$\beta \int_0^T |x(s)|^q\,\mathrm{d}s \leqslant \int_0^T [g_0(t, x(s)) + \sum_{k=1}^{m} g_i(s, x(\alpha_i(s))) + r(s)]\,\mathrm{d}s.$$

Similarly as in the proof of Theorem 3.1, choose $\varepsilon > 0$ such that

$$(r_0 + \varepsilon) + \sum_{k=1}^{n} (r_k + \varepsilon) \|\beta_k'\|_\infty^{q/(q+1)} < \beta. \tag{13}$$

For such $\varepsilon > 0$ there exists $\delta > 0$ such that for every $i = 0, 1, \ldots, n$,

$$|g_i(t,x)| < (r_i + \varepsilon)|x|^q \quad \text{uniformly for } t \in [0,T] \text{ and } |x| > \delta. \tag{14}$$

Let, for $i = 1, \ldots, n$, $\Delta_{1,i} = \{t\colon t \in [0,T],\, |x(\alpha_i(t))| \leqslant \delta\}$, $\Delta_{2,i} = \{t\colon t \in [0,T],\, |x(\alpha_i(t))| > \delta\}$, $g_{\delta,i} = \max\limits_{t\in[0,T],\, |x|\leqslant\delta} |g_i(t,x)|$, and $\Delta_1 = \{t \in [0,T],\, |x(t)| \leqslant \delta\}$, $\Delta_2 = \{t \in [0,T],\, |x(t)| > \delta\}$. Then we get

$$
\begin{aligned}
&\beta\int_0^T |x(s)|^q\,\mathrm{d}s\\
&\quad\leqslant \int_{\Delta_1}|g_0(t,x(s))|\,\mathrm{d}s+\int_{\Delta_2}|g_0(t,x(s))|\,\mathrm{d}s\\
&\qquad+\sum_{k=1}^m\int_{\Delta_{1,i}}|g_i(s,x(\alpha_i(s)))|\,\mathrm{d}s+\sum_{k=1}^m\int_{\Delta_{2,i}}|g_i(s,x(\alpha_i(s)))|\,\mathrm{d}s+\int_0^T|r(s)|\,\mathrm{d}s\\
&\quad\leqslant \sum_{i=0}^n g_{\delta,i}+(r_0+\varepsilon)\int_0^T|x(s)|^q\,\mathrm{d}s+\sum_{i=1}^n(r_i+\varepsilon)\int_0^T|x(\alpha_i(s))|^q\,\mathrm{d}s+T\|r\|\\
&\quad\leqslant \sum_{i=0}^n g_{\delta,i}+(r_0+\varepsilon)\int_0^T|x(s)|^q\,\mathrm{d}s+\sum_{i=1}^n(r_i+\varepsilon)\left|\int_{\alpha_i(0)}^{\alpha_i(T)}|x(u)|^q\,\mathrm{d}\beta_i(u)\right|+T\|r\|\\
&\quad\leqslant \sum_{i=0}^n g_{\delta,i}+(r_0+\varepsilon)\int_0^T|x(s)|^q\,\mathrm{d}s+\sum_{i=1}^n(r_i+\varepsilon)\|\beta_i'\|_\infty\int_0^T|x(u)|^q\,\mathrm{d}u+T\|r\|.
\end{aligned}
$$

It follows from (13) that there is $M>0$ such that $\int_0^T|x(s)|^q\,\mathrm{d}s\leqslant M$. Hence, there exists $\xi\in[0,T]$ such that $|x(\xi)|\leqslant(M/T)^{1/q}$. So ($\mathrm{E}_1$) and ($\mathrm{E}_2$) imply that

$$
\begin{aligned}
|x(t)|&\leqslant|x(\xi)|+\sum_{t\leqslant t_k<\xi\text{ or }\xi\leqslant t_k<t}|I_k(x(t_k))|+\left|\int_\xi^t x'(s)\,\mathrm{d}s\right|\\
&\leqslant(M/T)^{1/q}+\sum_{k=1}^m\alpha_k\|x\|_\infty+\int_0^T|x'(s)|\,\mathrm{d}s\\
&\leqslant(M/T)^{1/q}+\sum_{k=1}^m\alpha_k\|x\|_\infty+\int_0^T|f(s,x(s),x(\alpha_1(s)),\dots,x(\alpha_n(s)))|\,\mathrm{d}s\\
&\leqslant(M/T)^{1/q}+\sum_{k=1}^m\alpha_k\|x\|_\infty+\int_0^T p_0(s)|x(s)|^q\,\mathrm{d}s\\
&\qquad+\sum_{k=1}^n\int_0^T p_i(s)|x_i(\alpha_i(s))|^q\,\mathrm{d}s+\int_0^T|R(s)|\,\mathrm{d}s\\
&\leqslant(M/T)^{1/q}+\sum_{k=1}^m\alpha_k\|x\|_\infty+\|p_0\|_\infty\int_0^T|x(s)|^q\,\mathrm{d}s\\
&\qquad+\sum_{k=1}^n\|p_i\|_\infty\|\beta_i'\|_\infty\int_0^T|x_i(s)|^q\,\mathrm{d}s+\int_0^T|R(s)|\,\mathrm{d}s\\
&\leqslant(M/T)^{1/q}+\sum_{k=1}^m\alpha_k\|x\|_\infty+\|p_0\|_\infty M\\
&\qquad+\sum_{k=1}^n\|p_i\|_\infty\|\beta_i'\|_\infty M+\int_0^T|R(s)|\,\mathrm{d}s.
\end{aligned}
$$

Then we get

$$\|x\|_\infty \leqslant (M/T)^{1/q} + \sum_{k=1}^{m} \alpha_k \|x\|_\infty + \|p_0\|_\infty M$$
$$+ \sum_{k=1}^{n} \|p_i\|_\infty \|\beta_i'\|_\infty M + \int_0^T |R(s)| \, \mathrm{d}s.$$

Since $\sum\limits_{k=1}^{m} \alpha_k < 1$, one sees that there is a constant $M_1 > 0$ such that $\|x\|_\infty \leqslant M_1$. So Ω_1 is bounded.

The remaining steps of the proof are similar to those of the proof of Theorem 3.1 and are omitted. □

Theorem 3.4. *Assume that* (E), (E_1), (E_2), (C_4), *and* (A_4) *hold. Furthermore, suppose* (12) *holds. Then* IPBVP(6) *has at least one solution.*

P r o o f. The proof is similar to those of Theorem 3.3 and Theorem 3.1 and is omitted. □

4. Examples

In this section we give examples which cannot be solved by the results in known papers, to illustrate the main results.

E x a m p l e 4.1. Consider the following IPBVP

$$\begin{cases} x'(t) = \sum\limits_{k=0}^{2q+1} a_k x^k(t) + r(t) \text{ for a.e. } t \in [0,T], \\ \Delta x(t_k) = b_k [x(t_k)]^3, \ k = 1, \dots, m, \\ x(0) = x(T), \end{cases} \tag{15}$$

where $q > 1$ is a positive integer, $T > 0$, $b_k \geqslant 0$ for all $k = 1, \dots, m$, $a_{2q+1} > 0$, and $a_k \in \mathbb{R}$ for all $k = 0, 1, \dots, 2q+1$, $r \in X$. Corresponding to Theorem 3.1, we get

$$I_k(x) = b_k x^3,$$
$$f(t, x_0) = \sum_{k=0}^{2q+1} a_k x_0^k + r(t),$$
$$h(t, x_0) = a_{2q+1} x_0^{2q+1},$$
$$g_0(t, x_0) = \sum_{k=0}^{2q} a_k x_0^k + r(t).$$

On the other hand, one sees that

$$
\begin{aligned}
c\Big(\int_0^T f(t,c,c,\dots,c)\,\mathrm{d}t+\sum_{k=1}^m I_k(c)\Big)
&= c\Big[\int_0^T\Big(\sum_{k=0}^{2q+1} a_k c^k + r(t)\Big)\,\mathrm{d}t+\sum_{k=1}^m b_k c^3\Big]\\
&= c\Big[T\sum_{k=0}^{2q+1} a_k c^k + \int_0^T r(t)\,\mathrm{d}t+\sum_{k=1}^m b_k c^3\Big].
\end{aligned}
$$

Since $q>1$ and $a_{2q+1}>0$, we get that there exists a constant $M>0$ such that

$$
c\Big(\int_0^T f(t,c,c,\dots,c)\,\mathrm{d}t+\sum_{k=1}^m I_k(c)\Big)>0
$$

for each $|c|>M$. Hence, (E), (A_2), (C_2) hold. It follows from Theorem 3.1 that IPBVP(15) has at least one solution.

Remark 4.1. Since the upper and lower solutions and monotone iterative techniques are not used in IPBVP(15), the results in [1], [4], [7], [10]–[12], [16]–[17], [23] cannot solve IPBVP(15). The theorems in [20] cannot solve IPBVP(15), since the I_k in IPBVP(15) are superlinear.

Example 4.2. Consider the following IPBVP

$$
\begin{cases}
x'(t)=\sum\limits_{k=0}^{2q} a_k x^k(t)+\sum\limits_{k=1}^{2q} c_k x^{2m}\Big(\dfrac{1}{k}t\Big)+r(t) \text{ for a.e. } t\in[0,T],\\
\Delta x(t_k)=b_k|x(t_k)|,\ \ k=1,\dots,m,\\
x(0)=x(T),
\end{cases}
\tag{16}
$$

where $q>2$ is a positive integer, $T>0$, $b_k\leqslant 0$ for all $k=1,\dots,m$, $a_{2q}<0$, and $a_k,c_k\in\mathbb{R}$ for all $k=0,1,\dots,2q$, $r\in X$. Corresponding to Theorem 3.3, we get

$$
\begin{aligned}
I_k(x)&=b_k|x|,\\
f(t,x_0,\dots,x_{2q})&=\sum_{k=0}^{2q} a_k x_0^k+\sum_{k=1}^{2q} c_k x_k^{2q}+r(t),\\
h(t,x_0)&=a_{2q}x_0^{2q},\\
g_0(t,x_0)&=\sum_{k=0}^{2q-1} a_k x_0^k,\\
g_i(t,x_i)&=c_i x_i^{2q},\ \ i=1,\dots,2q,\\
\alpha_i(t)&=\frac{1}{i}t,\ \ i=1,\dots,2q.
\end{aligned}
$$

On the other hand, one sees that

$$
\begin{aligned}
&c\biggl(\int_0^T f(t,c,c,\dots,c)\,\mathrm{d}t+\sum_{k=1}^{m} I_k(c)\biggr)\\
&\quad= c\biggl[\int_0^T\biggl(\sum_{k=0}^{2q} a_k c^k+\sum_{k=1}^{2q} c_k c^{2q}+r(t)\biggr)\,\mathrm{d}t+\sum_{k=1}^{m} b_k|c|\biggr]\\
&\quad= c\biggl[T\sum_{k=0}^{2q} a_k c^k+T\sum_{k=1}^{2q} c_k c^{2q}+\int_0^T r(t)\,\mathrm{d}t+\sum_{k=1}^{m} b_k|c|\biggr].
\end{aligned}
$$

It is easy to see that $q>2$ and $a_{2q}+\sum\limits_{k=1}^{2q} c_k<0$ imply that there exists a constant $M>0$ such that

$$c\biggl(\int_0^T f(t,c,c,\dots,c)\,\mathrm{d}t+\sum_{k=1}^{m} I_k(c)\biggr)<0$$

for each $|c|>M$; $q>2$ and $a_{2q}+\sum\limits_{k=1}^{2q} c_k>0$ imply that there exists a constant $M>0$ such that

$$c\biggl(\int_0^T f(t,c,c,\dots,c)\,\mathrm{d}t+\sum_{k=1}^{m} I_k(c)\biggr)>0$$

for each $|c|>M$.

It is easy to see that (E), (E_1), (E_2), (C_3), (A_3) hold. It follows from Theorem 3.3 that IPBVP(16) has at least one solution if

$$\sum_{k=1}^{2q} k|c_k|<-a_{2q},\quad a_{2q}+\sum_{k=1}^{2q} c_k>0$$

or

$$\sum_{k=1}^{2q} k|c_k|<-a_{2q},\quad a_{2q}+\sum_{k=1}^{2q} c_k<0.$$

R e m a r k 4.2. IPBVP(16) cannot be solved by the theorems in [19], [12], [1].

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