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SHARP UPPER GLOBAL A POSTERIORI ERROR ESTIMATES FOR NONLINEAR ELLIPTIC VARIATIONAL PROBLEMS*

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Dedicated to Professor Ivan Hlaváček on the occasion of his 75th birthday

Abstract. The paper is devoted to the problem of verification of accuracy of approximate solutions obtained in computer simulations. This problem is strongly related to a posteriori error estimates, giving computable bounds for computational errors and detecting zones in the solution domain where such errors are too large and certain mesh refinements should be performed. A mathematical model embracing nonlinear elliptic variational problems is considered in this work. Based on functional type estimates developed on an abstract level, we present a general technology for constructing computable sharp upper bounds for the global error for various particular classes of elliptic problems. Here the global error is understood as a suitable energy type difference between the true and computed solutions. The estimates obtained are completely independent of the numerical technique used to obtain approximate solutions, and are sharp in the sense that they can be, in principle, made as close to the true error as resources of the used computer allow. The latter can be achieved by suitably tuning the auxiliary parameter functions, involved in the proposed upper error bounds, in the course of the calculations.

Keywords: a posteriori error estimation, error control in energy norm, error estimates of functional type, elliptic equation of second order, elliptic equation of fourth order, second order elasticity system, mixed boundary conditions, gradient averaging

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1. Introduction

Many physical phenomena can be described by means of mathematical models presenting linear and nonlinear boundary value problems of elliptic type [10], [13],

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[30], [38], [47]. Various numerical techniques (such as the finite difference method, the finite element method, the finite volume method, etc.) are well developed for finding approximate solutions for such problems, see, e.g., [10], [30] and references therein.

In order to be practically meaningful, computer simulations always require an accuracy verification of computed approximations. Such a verification is the main purpose of a posteriori error estimation methods. Several approaches have been suggested for deriving various a posteriori estimates for elliptic problems, involving errors measured in global (energy) norms (see [1], [4], [5], [17], [20], [41], [48], [49], [50], [53]) or various local quantities [6], [12], [19], [27], [42], [47], see also references in the above mentioned works.

However, most of the estimates proposed there strongly use the fact that the computed solutions are true finite element (FE) approximations which, in fact, rarely happens in real computations, e.g., due to quadrature rules, forcibly stopped iterative processes, various round-off errors, or even bugs in computer codes.

A different approach, based on functional analysis background, was first developed in [43], see monograph [40] and also [44], [45], and the references therein. Hereby the estimation is developed independently of the numerical method used to obtain the approximation. One can thus obtain sharp estimates for linear problems and for certain nonlinear problems; however, for nonlinear problems in general, these estimates may fail to ensure the best upper bound [40, p. 236].

In this paper, based on a Banach space framework, we present another general functional type technology for obtaining *sharp computable guaranteed error bounds* needed for reliable control of the overall accuracy of computed approximations. Such bounds are again valid for *any conforming approximation* independently of the numerical method used to obtain them. The bounds obtained can be made *arbitrarily close* to the true error by tuning the auxiliary parameters involved. In real calculations this closeness only depends on resources of the concrete computer. Based on the general theory, our error estimates are given for various classes of elliptic problems. We also discuss some issues of the practical realization of the proposed error estimation procedures.

We note that the estimates proposed in this work can be considered a certain generalization of results presented for the first time in [43] and in a unified manner in [40], mentioned above. However, our way of constructing the error estimates is somewhat different and leads to sharp bounds for nonlinear problems. In particular, it does not require tools of the duality theory and is based purely on direct calculations. Estimates based on a similar direct approach are also obtained for linear convection-reaction-diffusion problems in [29], [31] and for Maxwell equations in [18].

Another advantage of the proposed approach is that it only uses a few (at most five) global constants, which do not depend on the computational process and must be computed only once in advance (or in parallel). These constants come from embedding theorems and the nonlinear coefficients of the equation. Many other existing estimation techniques (e.g., the residual-type ones) normally involve much more unknown constants (usually related to patches of computational meshes used). Such constants are very hard to compute (or even sufficiently accurately estimate from above) and their evaluation normally leads to a very big overestimation of the error even in simple cases (cf. [9]). Moreover, those constants have to be always recomputed when we perform adaptive computations and change the computational mesh. To the contrary, our global constants do remain the same under any change of meshes during the whole computational process. Moreover, it suffices to estimate our constants only roughly from above (which can actually rely on a known explicit value or can be cheaply calculated), since the terms which they multiply are normally decreasing towards zero during the tuning process of the error estimate.

Chapters 2–3 are devoted to the general estimation theory in a Banach space. We consider operator equations of the form

$$F(u) + l = 0 \tag{1.1}$$

in a Banach space V with a given nonlinear operator $F\colon V \to V^*$ and a given bounded linear functional $l \in V^*$. We will assume certain monotonicity properties of F that both ensure well-posedness for (1.1) and allow a suitable measuring of the error. We give a proper background in Chapter 2, and present our sharp upper error bounds in Chapter 3. Applications to various classes of elliptic problems including second order problems with Dirichlet and mixed boundary conditions, systems, and fourth order equations are developed in Chapter 4.

2. Theoretical background

2.1. Some elementary definitions and properties

For the reader's convenience we define some basic notions and notation that we will use, together with some well-known properties. See, e.g., [13], [52] for related topics. We will use the following notation. Let V be a given Banach space with a norm $\|\cdot\|_V$. Then its dual space V^* consists of all bounded linear functionals $l\colon V \to \mathbb{R}$ on V. If $l \in V^*$ and $u \in V$, then the value of l at u is denoted by $\langle l, u\rangle$, where $\langle\cdot,\cdot\rangle$ is the duality pairing.

Definition 2.1. The operator $F\colon V \to V^*$ is called a *monotone operator* if

$$\langle F(u) - F(v), u - v\rangle \geqslant 0 \quad (u, v \in V),$$

a *strictly monotone operator* if

$$\langle F(u) - F(v), u - v \rangle > 0 \quad (u, v \in V,\ u \neq v),$$

and a *uniformly monotone operator* if there exists a constant $m > 0$ such that

$$\langle F(u) - F(v), u - v \rangle \geqslant m \|u - v\|_V^2 \quad (u, v \in V).$$

Definition 2.2. The nonlinear operator $F\colon V \to V^*$ has a *bihemicontinuous Gâteaux derivative* if

(i) F is Gâteaux differentiable;

(ii) F' is bihemicontinuous, i.e., for any $u, k, w, h \in V$ the mapping $(s, t) \mapsto F'(u + sk + tw)h$ is continuous from $\mathbb{R}^2$ to V^*.

In addition, this Gâteaux derivative is called *symmetric* if

(iii) for any $u \in V$ the operator $F'(u)$ is *symmetric*, i.e.,

$$\langle F'(u)h, v \rangle = \langle F'(u)v, h \rangle \quad (u, h, v \in V).$$

Accordingly, a nonlinear functional $J\colon V \to \mathbb{R}$ has a *bihemicontinuous symmetric second Gâteaux derivative* if J is twice Gâteaux differentiable and J'' satisfies (ii)–(iii) above.

Definition 2.3. The operator $F\colon V \to V^*$ is called a *potential operator* if there exists a Gâteaux differentiable functional $\phi\colon V \to \mathbb{R}$ such that $\phi'(u) = F(u)$ $(u \in V)$. (Such a ϕ is called a potential of F.)

The following characterization holds, see, e.g., [52]:

Proposition 2.1. *Let an operator $F\colon V \to V^*$ have a bihemicontinuous Gâteaux derivative. Then F is a potential operator if and only if $F'(u)$ is symmetric for any $u \in V$.*

Then we have a related basic well-posedness result:

Theorem 2.1. *Let an operator $F\colon V \to V^*$ have a bihemicontinuous symmetric Gâteaux derivative, and let there exist a constant $m > 0$ such that*

$$\langle F'(u)v, v \rangle \geqslant m \|v\|_V^2 \quad (u, v \in V).$$

Then for any $l \in V^*$ the operator equation (1.1) *has a unique solution $u^* \in V$.*

For a proof see, e.g., [13], [52]. We note that the solution u^* is the unique minimizer of the functional $J(u) := \phi(u) + \langle l, u \rangle$, where ϕ comes from Definition 2.3.

2.2. Error functionals for monotone operators

Let us assume that the operator equation (1.1) has a unique solution $u^* \in V$. (Sufficient conditions will be given later when our main result is presented in Section 3.)

In this paper we consider an approximate solution $u \in V$ of equation (1.1), i.e. $u \approx u^*$ where u^* is the exact solution. Our goal is to estimate the error arising from this approximation. For this purpose, we will use the following (energy type) error functional for equation (1.1):

$$E(u) := \langle F(u) + l, u - u^* \rangle \quad (u \in V) \tag{2.1}$$

or in other form

$$E(u) = \langle F(u) - F(u^*), u - u^* \rangle \quad (u \in V). \tag{2.2}$$

The following facts obviously hold. If F is monotone then $E(u) \geqslant 0 = E(u^*)$ $(u \in V)$. If F is also strictly monotone then $E(u) = 0$ if and only if $u = u^*$.

If F is also uniformly monotone then

$$E(u) \geqslant m\|u - u^*\|_V^2 \quad (u \in V). \tag{2.3}$$

We note that if F is a potential operator and ϕ is a potential of F, then another possible error functional has the form $\hat{E}(u) := J(u) - J(u^*)$, where $J(u) := \phi(u) + \langle l, u \rangle$. If the above monotonicity properties are assumed, then the corresponding statements on E also hold for $\hat{E}$, which can be verified via the corresponding convexity of ϕ. However, in this paper it will be more convenient to use the error functional E. We also note that E and $\hat{E}$ are just the same (up to a constant multiplier 2) for linear problems.

2.3. Error estimates in normed spaces via convex functionals

A thorough study of error estimation for nonlinear variational problems is given in the book [40], which our paper also builds on. Here we briefly outline a setting from this book and some results.

In addition to the Banach space V, let us introduce another Banach space Y, assumed to be reflexive. We also introduce a linear operator $\Lambda\colon V \to Y$, for which there exist constants $c_2 \geqslant c_1 > 0$ such that

$$c_1\|u\|_V \leqslant \|\Lambda u\|_Y \leqslant c_2\|u\|_V \quad (u \in V). \tag{2.4}$$

Then Λ has an adjoint operator $\Lambda^*\colon Y^* \to V^*$ satisfying

$$\langle y^*, \Lambda u \rangle = \langle \Lambda^* y^*, u \rangle \quad (y^* \in Y^*,\ u \in V). \tag{2.5}$$

In [40, Chap. 7], functionals $J\colon V \to \mathbb{R}$ are considered in the form

$$J(u) := G(\Lambda u) + f(u) \quad (u \in V),$$

where $G\colon Y \to \mathbb{R}$ and $f\colon V \to \mathbb{R}$ are given uniformly convex functionals, and error estimates are given with expressions based on duality theory.

More explicit estimates are found for the special case

$$J(u) := G(\Lambda u) + \langle l, u\rangle \quad (u \in V), \tag{2.6}$$

in which $l \in V^*$, further, Y is assumed to be a Hilbert space and G has the form

$$G(y) = \frac{1}{2}\langle Ay, y\rangle + \Phi(y) \quad (y \in Y), \tag{2.7}$$

where $A\colon Y \to Y$ is an invertible bounded self-adjoint linear operator and $\Phi\colon Y \to \mathbb{R}$ is a convex continuous functional. Denoting by u^* the minimizer of J, it is proved in [40, Sec. 7.7] that

$$\left|\!\left|\!\left| \frac{1}{2}(\Lambda(u^* - u)) \right|\!\right|\!\right|^2 \leqslant (1+\beta) D_G(\Lambda u, y^*) + \Big(1 + \frac{1}{\beta}\Big) |\Lambda^* y^* + l|^2, \tag{2.8}$$

where

$$|\!|\!| y |\!|\!| := \langle Ay, y\rangle^{1/2}$$

is the A-norm, $\beta > 0$ is an arbitrary constant,

$$|\Lambda^* y^* + l| := \sup_{\substack{w \in V\\ w \neq 0}} \frac{\langle \Lambda^* y^* + l, w\rangle}{|\!|\!| \Lambda w |\!|\!|}, \tag{2.9}$$

and

$$D_G(\Lambda u, y^*) = G(\Lambda u) + \sup_{y \in Y}(\langle y^*, y\rangle - G(y)) - \langle y^*, \Lambda u\rangle.$$

Remark 2.1. It is pointed out in [40, Sec. 7.7] that the estimate (2.8) is *not sharp*, and finding the best upper bound can only be expected from a further analysis of the particular problem considered.

An important special case of the above is formed by linear equations, i.e. when $\Phi \equiv 0$ and G is a quadratic functional to be minimized. Then, by [40, Chap. 6], an estimate similar to (2.8) holds:

$$\frac{1}{2} |\!|\!| \Lambda(u^* - u) |\!|\!|^2 \leqslant (1+\beta) D(\Lambda u, y^*) + \Big(1 + \frac{1}{\beta}\Big)\frac{1}{2} |\Lambda^* y^* + l|^2, \tag{2.10}$$

where

$$D(\Lambda u, y^*) = \frac{1}{2}\langle A(\Lambda u) - y^*, \Lambda u - A^{-1}y^*\rangle.$$

Accordingly,

$$\|\!|\Lambda(u^* - u)|\!\|^2 \leqslant (1+\beta)\langle A(\Lambda u) - y^*, \Lambda u - A^{-1}y^*\rangle + \Big(1 + \frac{1}{\beta}\Big)|\Lambda^* y^* + l|^2. \tag{2.11}$$

Note that here the choice $y^* = A\Lambda u^*$ is known to be optimal, since it provides equality if $\beta \to \infty$, using that here $\Lambda^* y^* + l = 0$. We also note that in this special case, for $F(u) := J'(u)$, (2.2) coincides with $\|\!|\Lambda(u^* - u)|\!\|^2$.

Returning to the nonlinear case (2.6), the minimizer u^* of J is the solution of the equation

$$\langle J'(u), v\rangle = \langle G'(\Lambda u), \Lambda v\rangle + \langle l, v\rangle = 0 \quad (v \in V) \tag{2.12}$$

or $N(u) = 0$ where $N = J'$. It is important to note here that for a twice Gâteaux differentiable Φ, formula (2.7) implies

$$\langle G''(y)p, p\rangle = \langle Ap, p\rangle + \langle \Phi''(y)p, p\rangle \geqslant \langle Ap, p\rangle = \|\!|p|\!\|^2 \quad (y, p \in Y). \tag{2.13}$$

The twice Gâteaux differentiability of J (i.e. the Gâteaux differentiability of N) is not a strong requirement and is satisfied in many practical cases; further, the corresponding condition (2.13) (up to a constant multiplier) is rather close to uniform convexity. Therefore, the assumption

$$\langle G''(y)p, p\rangle \geqslant m\|p\|_Y^2 \quad (y, p \in Y)$$

(with $m > 0$) and the corresponding estimate (2.8) are a proper starting point for our search for sharp estimates. In fact, our goal is to obtain such an estimate as an extension of the more explicit formula (2.11).

2.4. Integral mean operators

Let Y be a Banach space and $A\colon\ Y \to Y^*$ an operator having a bihemicontinuous symmetric Gâteaux derivative.

Definition 2.4. For any vectors $y, z \in Y$ we define $A'_{[y,z]} \in B(Y, Y^*)$, that is, a bounded linear operator $A'_{[y,z]}\colon\ Y \to Y^*$, by the formula

$$A'_{[y,z]} := \int_0^1 A'(y + t(z-y))\,\mathrm{d}t. \tag{2.14}$$

This is an integral of a family of operators, understood via the corresponding bilinear forms:

$$\langle A'_{[y,z]}p, q\rangle = \int_0^1 \langle A'(y + t(z-y))p, q\rangle \, \mathrm{d}t \quad (p, q \in Y). \tag{2.15}$$

The unique existence of $A'_{[y,z]}$ (i.e., that this definition is correct) is ensured by the fact that

$$\int_0^1 \langle A'(y + t(z-y))p, q\rangle \, \mathrm{d}t \leqslant \big(\max_{t\in[0,1]} \|A'(y + t(z-y))\|\big) \|p\|_Y \|q\|_Y \tag{2.16}$$

(where the maximum exists by the continuity of the mapping $t \mapsto A'(y+t(z-y))$ and of the operator norm), which means that the right-hand side of (2.15) is a bounded bilinear form in p and q. Then we obtain by the definition of Y^* that this bilinear form can be represented as the bilinear form of a bounded linear operator from Y to Y^*.

The following properties are direct consequences of the above definition:

Proposition 2.2. *For any $y, z \in Y$*

(i) *the operator $A'_{[y,z]}$ is symmetric , i.e.,*

$$\langle A'_{[y,z]}p, q\rangle = \langle A'_{[y,z]}q, p\rangle \quad (p, q \in Y); \tag{2.17}$$

(ii) $A'_{[y,z]} = A'_{[z,y]}$;

(iii) $A(z) - A(y) = A'_{[y,z]}(z - y)$.

3. A sharp global error estimate in normed spaces

In what follows, our goal is to find upper bounds for $E(u)$. We follow the setting of [28], [40]. Namely, we let $u \in V$ be arbitrary and look for a bound involving some other vector parameters.

3.1. Basic properties

We study the operator equation (1.1) in the Banach space V in the framework of subsection 2.3, in particular, for simplicity, property (2.4) can be replaced by the simplified version (3.2) (which is achieved just by redefining the norm of V by the equivalent norm $\|\Lambda u\|_Y$).

Following subsection 2.3, let $J\colon V \to \mathbb{R}$ be a functional of the form

$$J(u) := G(\Lambda u) + \langle l, u\rangle \quad (u \in V) \tag{3.1}$$

under the following conditions:

A s s u m p t i o n s 3.1.

(i) Y is another Banach space and $\Lambda\colon V \to Y$ is a linear operator for which

$$\|\Lambda u\|_Y = \|u\|_V \quad (u \in V), \tag{3.2}$$

(ii) $G\colon Y \to \mathbb{R}$ is a functional having a bihemicontinuous symmetric second Gâteaux derivative (according to Definition 2.2),

(iii) there exists a constant $m > 0$ such that

$$\langle G''(y)p, p\rangle \geqslant m\|p\|_Y^2 \quad (y, p \in Y), \tag{3.3}$$

(iv) the operator $F\colon V \to V^*$ has the form

$$\langle F(u), v\rangle = \langle G'(\Lambda u), \Lambda v\rangle \quad (u, v \in V). \tag{3.4}$$

Proposition 3.1. *Under Assumptions* 3.1, *for any* $l \in V^*$ *the operator equation* (1.1) *has a unique solution* $u^* \in V$.

P r o o f. The assumptions yield that F has a bihemicontinuous symmetric Gâteaux derivative that satisfies

$$\langle F'(u)v, v\rangle = \langle G''(\Lambda u)\Lambda v, \Lambda v\rangle \geqslant m\|\Lambda v\|_Y^2 = m\|v\|_V^2 \quad (u, v \in V). \tag{3.5}$$

Then Theorem 2.1 implies well-posedness for (1.1). □

We note that the solution u^* of (1.1) is the unique minimizer of J. However, from now on, our calculations will involve the operator G' in (3.4) rather than the functional G. Hence, we study below the solution of equation (1.1) directly, instead of using the corresponding minimization problem.

3.2. Operator formulation and a preliminary estimate

We will replace the minimization problem for (3.1) by the corresponding operator equation, which is a more detailed form of (1.1) for this case. For this purpose, we introduce the operator

$$A := G'. \tag{3.6}$$

Then Assumptions 3.1 are equivalent to

Assumptions 3.2.

(i) Y is another Banach space and $\Lambda\colon V \to Y$ is a linear operator for which

$$\|\Lambda u\|_Y = \|u\|_V \quad (u \in V); \tag{3.7}$$

(ii) the operator $A\colon Y \to Y^*$ has a bihemicontinuous symmetric Gâteaux derivative (according to Definition 2.2);

(iii) there exists a constant $m > 0$ such that

$$\langle A'(y)p, p\rangle \geqslant m\|p\|_Y^2 \quad (y, p \in Y); \tag{3.8}$$

(iv) the operator $F\colon V \to V^*$ has the form

$$\langle F(u), v\rangle = \langle A(\Lambda u), \Lambda v\rangle \quad (u, v \in V). \tag{3.9}$$

Assumptions (ii)–(iii) imply in particular that A is bijective, i.e. $A^{-1}\colon Y^* \to Y$ exists. By (3.9), equation (1.1) can be written as

$$\langle A(\Lambda u), \Lambda v\rangle + \langle l, v\rangle = 0 \quad (v \in V) \tag{3.10}$$

which has a unique solution $u^* \in V$ for any $l \in V^*$ by Proposition 3.1.

We will need some further related properties. First, Proposition 2.2 (i) and (3.8) imply:

Proposition 3.2. *Under Assumptions 3.2, for any $y, z \in Y$ the mapping $p, q \mapsto \langle A'_{[y,z]}p, q\rangle$ is an inner product on Y.*

Proposition 3.3. *Under Assumptions 3.2, the following formulas hold:*

(i) $E(u) = \langle A'_{[\Lambda u^*, \Lambda u]}\Lambda(u - u^*), \Lambda(u - u^*)\rangle \quad (u \in V)$,

(ii) $E(u) \geqslant m\|u - u^*\|_V^2 = m\|\Lambda(u - u^*)\|_Y^2 \quad (u \in V)$,

(iii) $\|A(z) - A(y)\|_{Y^*} \geqslant m\|z - y\|_Y \quad (y, z \in Y)$.

Proof. (i) Using (3.9) and Proposition 2.2 (iii) for $z = \Lambda u$ and $y = \Lambda u^*$,

$$\begin{aligned} E(u) &= \langle F(u) - F(u^*), u - u^*\rangle = \langle A(\Lambda u) - A(\Lambda u^*), \Lambda(u - u^*)\rangle \\ &= \langle A'_{[\Lambda u^*, \Lambda u]}\Lambda(u - u^*), \Lambda(u - u^*)\rangle. \end{aligned} \tag{3.11}$$

(ii) Estimate (3.5) implies that F is uniformly monotone, hence (2.3) and (3.7) yield the required statement.

(iii) Estimate (3.8) implies

$$\langle A(z) - A(y), z - y\rangle \geqslant m\|z - y\|_Y^2 \quad (y, z \in Y), \tag{3.12}$$

whence we obtain the required statement by using the Cauchy-Schwarz inequality.

□

For the V^*-norm of a linear functional $l \in V^*$, we introduce the notation of [40]:

$$|l| := \|l\|_{V^*} \quad (l \in V^*). \tag{3.13}$$

Here (3.7) yields

$$|l| = \sup_{w \in V} \frac{\langle l, w \rangle}{\|w\|_V} = \sup_{w \in V} \frac{\langle l, w \rangle}{\|\Lambda w\|_Y}, \tag{3.14}$$

i.e. we have an analogue of (2.9).

Now we let $y^* \in Y^*$ be an arbitrary vector. We give a preliminary estimate, which is a starting point for our study.

Lemma 3.1. *Let Assumptions* 3.2 *hold and let $u^* \in V$ be the solution of* (1.1). *Let $u \in V$ and $y^* \in Y^*$ be arbitrary, let $z^* := A^{-1}(y^*)$. Then*

$$E(u) \leqslant |\Lambda^* y^* + l| m^{-1/2} E(u)^{1/2} + \langle A'_{[z^*, \Lambda u]}(\Lambda u - z^*), \Lambda(u - u^*) \rangle. \tag{3.15}$$

P r o o f. We have

$$E(u) = \langle F(u) + l, u - u^* \rangle = \langle \Lambda^* y^* + l, u - u^* \rangle + \langle F(u) - \Lambda^* y^*, u - u^* \rangle. \tag{3.16}$$

For the first term, we use (3.13) and Proposition 3.3 (ii) to obtain

$$|\langle \Lambda^* y^* + l, u - u^* \rangle| \leqslant |\Lambda^* y^* + l| \, \|u - u^*\|_V \leqslant |\Lambda^* y^* + l| m^{-1/2} E(u)^{1/2}. \tag{3.17}$$

The second term equals

$$\begin{aligned} &\langle F(u) - \Lambda^* y^*, u - u^* \rangle \\ &\qquad = \langle A(\Lambda u) - y^*, \Lambda(u - u^*) \rangle = \langle A(\Lambda u) - A(z^*), \Lambda(u - u^*) \rangle \\ &\qquad = \langle A'_{[z^*, \Lambda u]}(\Lambda u - z^*), \Lambda(u - u^*) \rangle, \end{aligned} \tag{3.18}$$

where (2.5), (3.9), and Proposition 2.2 (iii) have been used. □

The right-hand side of (3.15) becomes computable if the factor $\Lambda(u - u^*)$ is eliminated.

3.3. The sharp error estimate

A sharp estimation requires a further assumption on the Lipschitz continuity of the derivative of the nonlinear operator. This will be our main result. We amend Assumptions 3.2 by additional conditions:

A s s u m p t i o n s 3.3.

(i) There exists a subspace $W \subset Y$ with a new norm $\|\cdot\|_W$ such that A' is Lipschitz continuous as an operator from Y to $B(W, Y^*)$.

(ii) There exists a constant $M > 0$ such that

$$\langle A'(y)p, p\rangle \leqslant M\|p\|_Y^2 \quad (y, p \in Y). \tag{3.19}$$

Assumption 3.3 (i) means that there exists a constant $L > 0$ such that

$$\|A'(z) - A'(y)\|_{B(W,Y^*)} \leqslant L\|z - y\|_Y \quad (y, z \in Y), \tag{3.20}$$

or in more detailed form,

$$|\langle (A'(z) - A'(y))w, p\rangle| \leqslant L\|z - y\|_Y\|w\|_W\|p\|_Y \quad (y, z, p \in Y,\ w \in W). \tag{3.21}$$

Lemma 3.2. *Let Assumption* (3.3) (i) *hold. Then the operators defined in* (2.14) *satisfy for all* $y, v, z \in Y$

$$\|A'_{[z,v]} - A'_{[y,v]}\|_{B(W,Y^*)} \leqslant \frac{L}{2}\|z - y\|_Y. \tag{3.22}$$

P r o o f. We have

$$\begin{aligned}\|A'_{[z,v]} - A'_{[y,v]}\|_{B(W,Y^*)} &\leqslant \int_0^1 \|A'(z + t(v - z)) - A'(y + t(v - y))\|_{B(W,Y^*)}\,\mathrm{d}t \\ &\leqslant L\int_0^1 (1 - t)\|z - y\|_Y\,\mathrm{d}t = \frac{L}{2}\|z - y\|_Y.\end{aligned}$$

□

In more detailed form (as in (3.21)), inequality (3.22) means that

$$|\langle (A'_{[z,v]} - A'_{[y,v]})w, p\rangle| \leqslant \frac{L}{2}\|z - y\|_Y\|w\|_W\|p\|_Y \quad (y, v, z, p \in Y,\ w \in W). \tag{3.23}$$

R e m a r k 3.1. For our error estimate in a normed space, there is no restriction on the relation of the norms $\|\cdot\|_W$ and $\|\cdot\|_Y$. In practice the norm $\|\cdot\|_W$ will be stronger, i.e., the range of the values $\|w\|_W/\|w\|_Y$ (where $w \in W$) will run from a positive constant to $+\infty$, see Section 4.

Assumption 3.3 (ii) implies that the upper analogue of Proposition 3.3 (iii) holds:

$$\|A(z) - A(y)\|_{Y^*} \leqslant M\|z - y\|_Y \quad (y, z \in Y). \tag{3.24}$$

Further, we will need the following inequality:

Lemma 3.3. *Let Assumptions* 3.2–3.3 *hold and let* $u^* \in V$ *be the solution of* (1.1). *Let* $y^* \in Y^*$ *be arbitrary and* $z^* := A^{-1}(y^*)$. *Then for any* $h \in V$

$$\|z^* - \Lambda u^*\|_Y \leqslant \frac{M}{m}\|z^* - \Lambda h\|_Y + \frac{1}{m}|\Lambda^* y^* + l|. \tag{3.25}$$

Proof. Let $w^* \in V$ satisfy $F(w^*) = \Lambda^* y^*$. By (3.9), w^* is the solution of the equation

$$\langle A(\Lambda w^*), \Lambda v\rangle = \langle \Lambda^* y^*, v\rangle \quad (v \in V). \tag{3.26}$$

We have

$$\|z^* - \Lambda u^*\|_Y \leqslant \|z^* - \Lambda w^*\|_Y + \|\Lambda(w^* - u^*)\|_Y. \tag{3.27}$$

Here (3.26) implies

$$\langle A(\Lambda w^*), \Lambda v\rangle = \langle y^*, \Lambda v\rangle = \langle A(z^*), \Lambda v\rangle \quad (v \in V),$$

that is

$$\langle A(z^*) - A(\Lambda w^*), \Lambda v\rangle = 0 \quad (v \in V). \tag{3.28}$$

Using (3.12), (3.28), and (3.24), we obtain for any $h \in V$ that

$$\begin{aligned} m\|z^* - \Lambda w^*\|_Y^2 &\leqslant \langle A(z^*) - A(\Lambda w^*), z^* - \Lambda w^*\rangle = \langle A(z^*) - A(\Lambda w^*), z^* - \Lambda h\rangle \\ &\leqslant M\|z^* - \Lambda w^*\|_Y\|z^* - \Lambda h\|_Y, \end{aligned}$$

that is,

$$\|z^* - \Lambda w^*\|_Y \leqslant \frac{M}{m}\|z^* - \Lambda h\|_Y. \tag{3.29}$$

Further, using (3.7), (3.12), (3.26), and the fact that u^* solves (3.10) we conclude that

$$\begin{aligned} m\|w^* - u^*\|_V^2 = m\|\Lambda(w^* - u^*)\|_Y^2 &\leqslant \langle A(\Lambda w^*) - A(\Lambda u^*), \Lambda w^* - \Lambda u^*\rangle \\ &= \langle \Lambda^* y^* + l, w^* - u^*\rangle \leqslant |\Lambda^* y^* + l|\,\|w^* - u^*\|_V, \end{aligned}$$

hence

$$\|w^* - u^*\|_V \leqslant \frac{1}{m}|\Lambda^* y^* + l|. \tag{3.30}$$

Then (3.27), (3.29), and (3.30) give the desired estimate. □

Now we can prove our main result.

Theorem 3.1. *Let Assumptions 3.2–3.3 hold and let $u^* \in V$ be the solution of (1.1). Let $u \in V$ be an approximation of u^* such that $\Lambda u \in W$. Then for arbitrary $y^* \in Y^*$ such that $z^* := A^{-1}(y^*) \in W$ and for arbitrary $h \in V$,*

$$\begin{aligned} E(u) \leqslant \mathrm{EST}(u; y^*, h) := \Big(& m^{-1/2}|\Lambda^* y^* + l| + \frac{L}{2} m^{-3/2} D(u; y^*, h) \\ & + \Big(\langle A(\Lambda u) - y^*, \Lambda u - A^{-1}(y^*) \rangle \\ & + \frac{L}{2m} D(u; y^*, h) \|\Lambda u - A^{-1}(y^*)\|_Y \Big)^{1/2} \Big)^2, \end{aligned} \tag{3.31}$$

where

$$D(u; y^*, h) := (M\|A^{-1}(y^*) - \Lambda h\|_Y + |\Lambda^* y^* + l|)\|\Lambda u - A^{-1}(y^*)\|_W. \tag{3.32}$$

Proof. Lemma 3.1 implies

$$E(u) \leqslant |\Lambda^* y^* + l| m^{-1/2} E(u)^{1/2} + \langle A'_{[z^*, \Lambda u]}(\Lambda u - z^*), \Lambda(u - u^*) \rangle, \tag{3.33}$$

and our further goal is to accurately estimate the second term. First, we observe that

$$\begin{aligned} & \langle A'_{[z^*, \Lambda u]}(\Lambda u - z^*), \Lambda(u - u^*) \rangle \\ & \quad = \langle (A'_{[z^*, \Lambda u]} - A'_{[\Lambda u^*, \Lambda u]})(\Lambda u - z^*), \Lambda(u - u^*) \rangle \\ & \qquad + \langle A'_{[\Lambda u^*, \Lambda u]}(\Lambda u - z^*), \Lambda(u - u^*) \rangle. \end{aligned} \tag{3.34}$$

By virtue of (3.23), the first term of (3.34) satisfies

$$\begin{aligned} & \langle (A'_{[z^*, \Lambda u]} - A'_{[\Lambda u^*, \Lambda u]})(\Lambda u - z^*), \Lambda(u - u^*) \rangle \\ & \quad \leqslant \frac{L}{2} \|z^* - \Lambda u^*\|_Y \|\Lambda u - z^*\|_W \|\Lambda(u - u^*)\|_Y, \end{aligned} \tag{3.35}$$

where $\|z^* - \Lambda u^*\|_Y$ fulfils (3.25) and $\|\Lambda(u - u^*)\|_Y \leqslant m^{-1/2} E(u)^{1/2}$ by Proposition 3.3 (ii), hence

$$\begin{aligned} & \langle (A'_{[z^*, \Lambda u]} - A'_{[\Lambda u^*, \Lambda u]})(\Lambda u - z^*), \Lambda(u - u^*) \rangle \\ & \quad \leqslant \frac{L}{2} m^{-3/2} (M\|z^* - \Lambda h\|_Y + |\Lambda^* y^* + l|)\|\Lambda u - z^*\|_W E(u)^{1/2}. \end{aligned} \tag{3.36}$$

The second term of (3.34) can be estimated via the Cauchy-Schwarz inequality:

$$\begin{aligned} & \langle A'_{[\Lambda u^*, \Lambda u]}(\Lambda u - z^*), \Lambda(u - u^*) \rangle \\ & \quad \leqslant \langle A'_{[\Lambda u^*, \Lambda u]}(\Lambda u - z^*), \Lambda u - z^* \rangle^{1/2} \\ & \qquad \times \langle A'_{[\Lambda u^*, \Lambda u]} \Lambda(u - u^*), \Lambda(u - u^*) \rangle^{1/2}. \end{aligned} \tag{3.37}$$

Proposition 3.3 (i) states that the second factor of (3.37) equals $E(u)^{1/2}$. For the first factor,

$$\begin{aligned}(3.38)\qquad &\langle A'_{[\Lambda u^*,\Lambda u]}(\Lambda u - z^*), \Lambda u - z^*\rangle \\ &= \langle A'_{[z^*,\Lambda u]}(\Lambda u - z^*), \Lambda u - z^*\rangle \\ &\quad + \langle (A'_{[\Lambda u^*,\Lambda u]} - A'_{[z^*,\Lambda u]})(\Lambda u - z^*), \Lambda u - z^*\rangle.\end{aligned}$$

Here Proposition 2.2 (iii) yields

$$\begin{aligned}(3.39)\qquad \langle A'_{[z^*,\Lambda u]}(\Lambda u - z^*), \Lambda u - z^*\rangle &= \langle A(\Lambda u) - A(z^*), \Lambda u - z^*\rangle \\ &= \langle A(\Lambda u) - y^*, \Lambda u - A^{-1}(y^*)\rangle\end{aligned}$$

and (3.23) and (3.25) imply

$$\begin{aligned}(3.40)\quad &\langle (A'_{[\Lambda u^*,\Lambda u]} - A'_{[z^*,\Lambda u]})(\Lambda u - z^*), \Lambda u - z^*\rangle \\ &\leqslant \frac{L}{2}\|\Lambda u^* - z^*\|_Y \|\Lambda u - z^*\|_W \|\Lambda u - z^*\|_Y \\ &\leqslant \frac{L}{2m}(M\|z^* - \Lambda h\|_Y + |\Lambda^* y^* + l|)\|\Lambda u - z^*\|_W \|\Lambda u - z^*\|_Y.\end{aligned}$$

Summing up, (3.33), (3.34), (3.36), (3.38), (3.39), and (3.40) yield

$$\begin{aligned}E(u)^{1/2} &\leqslant m^{-1/2}|\Lambda^* y^* + l| + \frac{L}{2}m^{-3/2}(M\|z^* - \Lambda h\|_Y + |\Lambda^* y^* + l|)\|\Lambda u - z^*\|_W \\ &\quad + \Big(\langle A(\Lambda u) - y^*, \Lambda u - A^{-1}(y^*)\rangle \\ &\quad + \frac{L}{2m}(M\|z^* - \Lambda h\|_Y + |\Lambda^* y^* + l|)\|\Lambda u - z^*\|_W \|\Lambda u - z^*\|_Y\Big)^{1/2} \\ &= m^{-1/2}|\Lambda^* y^* + l| + \frac{L}{2}m^{-3/2} D(u; y^*, h) \\ &\quad + \Big(\langle A(\Lambda u) - y^*, \Lambda u - A^{-1}(y^*)\rangle + \frac{L}{2m} D(u; y^*, h)\|\Lambda u - z^*\|_Y\Big)^{1/2}.\end{aligned}$$

□

The repeated application of the elementary inequality $(a + b)^2 \leqslant (1 + \alpha)a^2 + (1 + 1/\alpha)b^2$ (where $\alpha > 0$) yields

Corollary 3.1. *Under the assumptions and notation of Theorem* 3.1, *for any constants* $\beta > 0$, $\gamma > 0$ *we have*

$$
\begin{aligned}
(3.41)\qquad E(u) \leqslant \overline{\mathrm{EST}}(u; y^*, h, \beta, \gamma)
&:= (1+\beta)(1+\gamma)m^{-1}|\Lambda^* y^* + l|^2 \\
&\quad + (1+\beta)\Big(1+\frac{1}{\gamma}\Big)\frac{L^2}{4}m^{-3}\, D(u; y^*, h)^2 \\
&\quad + \Big(1+\frac{1}{\beta}\Big)\Big(\langle A(\Lambda u) - y^*, \Lambda u - A^{-1}(y^*)\rangle \\
&\quad + \frac{L}{2m} D(u; y^*, h)\|\Lambda u - z^*\|_Y\Big)
\end{aligned}
$$

where $D(u; y^*, h)$ *is defined in* (3.32).

R e m a r k 3.2. It may be convenient to reformulate Theorem 3.1 for $z^* = A^{-1}(y^*)$ in order to avoid the computation of A^{-1}. Then for arbitrary $z^* \in W$ and for arbitrary $h \in V$,

$$
\begin{aligned}
(3.42)\quad E(u) \leqslant \widetilde{\mathrm{EST}}(u; z^*, h)
&:= \Big(m^{-1/2}|\Lambda^* A(z^*) + l| + \frac{L}{2}m^{-3/2}\tilde{D}(u; z^*, h) \\
&\quad + \Big(\langle A(\Lambda u) - A(z^*), \Lambda u - z^*\rangle + \frac{L}{2m}\tilde{D}(u; z^*, h)\|\Lambda u - z^*\|_Y\Big)^{1/2}\Big)^2,
\end{aligned}
$$

where

$$
(3.43)\qquad \tilde{D}(u; z^*, h) := (M\|z^* - \Lambda h\|_Y + |\Lambda^* A(z^*) + l|)\|\Lambda u - z^*\|_W.
$$

R e m a r k 3.3. If A is a linear operator then A' is constant, hence its Lipschitz constant is $L = 0$. In this case all terms containing h vanish, and we have in (3.41)

$$
\begin{aligned}
\overline{\mathrm{EST}}(u; y^*, h, \beta, \gamma) &= (1+\beta)(1+\gamma)m^{-1}|\Lambda^* y^* + l|^2 \\
&\quad + \Big(1+\frac{1}{\beta}\Big)\langle A(\Lambda u) - y^*, \Lambda u - A^{-1}y^*\rangle,
\end{aligned}
$$

that is,

$$
\begin{aligned}
(3.44)\qquad \widehat{\mathrm{EST}}(u; y^*, h, \beta) &:= \overline{\mathrm{EST}}(u; y^*, h, \beta, 0) \\
&= (1+\beta)m^{-1}|\Lambda^* y^* + l|^2 \\
&\quad + \Big(1+\frac{1}{\beta}\Big)\langle A(\Lambda u) - y^*, \Lambda u - A^{-1}y^*\rangle,
\end{aligned}
$$

which is nothing but (2.11). (The factor m^{-1} is not present in (2.11) since it is included in the norm $|\!|\!| \cdot |\!|\!|$.) This shows that our estimate is a direct extension of (2.11) for nonlinear problems.

Now we can turn to the problem of sharpness.

Proposition 3.4. *Estimates* (3.31) *and* (3.41) *are sharp in the following sense: denoting* $A(W) := \{A(v)\colon\ v \in W\}$, *we have*

$$\min_{\substack{y^* \in A(W),\\ h \in V}} \mathrm{EST}(u; y^*, h) = E(u), \qquad \inf_{\substack{y^* \in A(W),\\ h \in V,\\ \beta,\gamma > 0}} \overline{\mathrm{EST}}(u; y^*, h, \beta, \gamma) = E(u)$$

provided $\Lambda u^* \in W$.

P r o o f. Let us choose

$$y^* := A(\Lambda u^*) \quad \text{and} \quad h := u^*. \tag{3.45}$$

Then $z^* = A^{-1}(y^*) = \Lambda u^* \in W$. Consequently, y^* satisfies the assumption of Theorem 3.1. Here $y^* = A(\Lambda u^*)$ satisfies $\Lambda^* y^* + l = 0$, similarly to the linear case (see after (2.11)). Hence, the first term in both $\mathrm{EST}(u; y^*, h)$ and $\overline{\mathrm{EST}}(u; y^*, h, \beta, \gamma)$ is zero in this case, further, $A^{-1}(y^*) - \Lambda h = \Lambda u^* - \Lambda u^* = 0$, therefore $D(u; A(\Lambda u^*), \Lambda u^*) = 0$ and thus the terms containing $D(u; y^*, h)$ are also zero in this case. That is,

$$\mathrm{EST}(u; A(\Lambda u^*), \Lambda u^*) = \langle A(\Lambda u) - A(\Lambda u^*), \Lambda u - \Lambda u^* \rangle = E(u)$$

where (3.11) has been used. Similarly,

$$\begin{aligned} \overline{\mathrm{EST}}(u; A(\Lambda u^*), \Lambda u^*, \beta, \gamma) &= \Bigl(1 + \frac{1}{\beta}\Bigr) \langle A(\Lambda u) - A(\Lambda u^*), \Lambda u - \Lambda u^* \rangle \\ &= \Bigl(1 + \frac{1}{\beta}\Bigr) E(u), \end{aligned}$$

hence

$$\inf_{\substack{y^* \in Y^*,\\ h \in V,\\ \beta,\gamma > 0}} \overline{\mathrm{EST}}(u; y^*, h, \beta, \gamma) \leqslant \inf_{\beta > 0} \Bigl(1 + \frac{1}{\beta}\Bigr) E(u) = E(u).$$

□

R e m a r k 3.4 (Finding the optimal h in a Hilbert space). In practice, y^* is obtained as an approximation of the optimal unknown value $A(\Lambda u^*)$ (cf. (3.45)). For given y^*, one can determine the optimal h via projection when Y is a Hilbert space. (In this case $\langle \cdot, \cdot \rangle$ means the inner product.) This is achieved as follows. Let $z^* := A^{-1}(y^*)$ and let h_{opt} be the solution of the problem

$$\langle \Lambda h_{\mathrm{opt}}, \Lambda v \rangle = \langle z^*, \Lambda v \rangle \quad (v \in V), \tag{3.46}$$

i.e., $h_{\rm opt}$ is the orthogonal projection of z^* on the range of Λ. Then for all $h \in V$

$$z^* - \Lambda h = (z^* - \Lambda h_{\rm opt} + (\Lambda h_{\rm opt} - \Lambda h),$$

where (3.46) for $v := h_{\rm opt} - h$ shows that the terms on the right are orthogonal. Therefore,

$$\|z^* - \Lambda h_{\rm opt}\|_Y \leqslant \|z^* - \Lambda h\|_Y.$$

That is, $h_{\rm opt}$ provides the smallest value of $\|z^* - \Lambda h\|_Y$ in (3.43).

Remark 3.5 (The Lipschitz condition for scalar nonlinearities). The following class of operators A is an important example of the type discussed above, which occurs in many practical models (see Section 4) and has the Lipschitz property from Assumption 3.3 (i).

Let $\mathcal{E}$ be a Euclidean space with a scalar product $[\cdot,\cdot]$, and let Y be the function space $L^2(\Omega,\mathcal{E})$, i.e.,

$$Y := \{p\colon\ \Omega \to \mathcal{E}\colon\ \text{the function } [p,p] \in L^2(\Omega)\}.$$

Then Y is a Hilbert space with the inner product $\langle p,q\rangle = \int_\Omega [p,q]$, hence Y is a Banach space as well and $Y^* = Y$. Then we define the operator $A\colon Y \to Y$ as $A(p) := a([p,p])p$, or equivalently (in a test function form)

$$\langle A(p),q\rangle = \int_\Omega (a([p,p])[p,q]) \quad (p,q \in Y), \tag{3.47}$$

where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is a scalar C^2 function with the following properties: there exist constants $M \geqslant m > 0$ such that

$$0 < m \leqslant a(t) \leqslant M, \quad 0 < m \leqslant \frac{\mathrm{d}}{\mathrm{d}t}(a(t^2)t) \leqslant M \quad (t \geqslant 0), \tag{3.48}$$

further, there exists a constant $L_1 > 0$ such that

$$\left|\frac{\mathrm{d}^2}{\mathrm{d}t^2}(a(t^2)t)\right| \leqslant L_1 \quad (t \geqslant 0). \tag{3.49}$$

Let

$$L := \max\{L_1, 3L_2\}, \quad \text{where} \quad L_2 := \sup_{t\geqslant 0} \frac{\mathrm{d}}{\mathrm{d}t}(a(t^2)). \tag{3.50}$$

Then (3.8) implies that A has a bihemicontinuous symmetric Gâteaux derivative satisfying

$$m\|p\|_Y^2 \leqslant \langle A'(y)p,p\rangle \leqslant M\|p\|_Y^2 \quad (y,p \in Y) \tag{3.51}$$

(see, e.g., [13]), that is, Assumptions 3.2 (ii)–(iii) and Assumption 3.3 (ii) hold. Further, let

$$W := \{p \in Y\colon\ [p,p] \in L^\infty(\Omega)\}, \quad \|p\|_W := \||p|_{\mathcal{E}}\|_{L^\infty(\Omega)},$$

where $|x|_{\mathcal{E}} := [x,x]^{1/2}$ $(x \in \mathcal{E})$. Then, as proved in [24], A' is Lipschitz continuous as an operator from Y to $B(W, Y^*)$, with the Lipschitz constant L from (3.50). That is, for all $p, q, s \in Y$, $r \in W$ we have

$$|\langle (A'(p) - A'(q))r, s\rangle| \leqslant L\|p-q\|_Y \|r\|_W \|s\|_Y, \tag{3.52}$$

which is (3.21), that is, Assumption 3.3 (i) holds as well.

We underline that (3.49) is a natural condition for functions satisfying (3.48). (The latter does not imply (3.49) but only due to some pathological counterexamples.) In particular, if $(\mathrm{d}^2/\mathrm{d}t^2)(a(t^2)t)$ is monotone for sufficiently large t, then it is elementary to verify that (3.48) implies (3.49).

The above results (3.51) and (3.52) obviously remain valid under natural generalizations of the conditions (3.48)–(3.49). First, one can allow dependence on x: we let $a\colon\ \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ be a scalar-valued function that is measurable and bounded with respect to the variable $x \in \Omega$ and C^2 in the variable $t \in \mathbb{R}$, and satisfies

$$0 < m \leqslant a(x,t) \leqslant M, \quad 0 < m \leqslant \frac{\partial}{\partial t}(a(x,t^2)t) \leqslant M \quad (x \in \Omega,\ t \geqslant 0), \tag{3.53}$$

$$\Bigl|\frac{\partial^2}{\partial t^2}(a(x,t^2)t)\Bigr| \leqslant L \quad (x \in \Omega,\ t \geqslant 0). \tag{3.54}$$

The operator A, where $a([p,p])$ in (3.47) is replaced by $a(x,[p,p])$, then satisfies (3.51) and (3.52). Further, the sum of such operators also inherits this property. For instance, the results hold for

$$\langle A(p), q\rangle = \int_\Omega (a(x,[p,p])[p,q] + b(x,\{p,p\})\{p,q\}) \quad (p,q \in Y), \tag{3.55}$$

where $[\cdot,\cdot]$ and $\{\cdot,\cdot\}$ are two different semi-scalar products on $\mathcal{E}$, such that the sum $[x,y] + \{x,y\}$ for $x,y \in \mathcal{E}$ is already a scalar product on $\mathcal{E}$; further, a and b are functions each satisfying (3.53)–(3.54). Finally, it is enough to require a to be C^2 except for finitely many points.

4. Sharp global error estimates for nonlinear elliptic problems

This section is the main part of the paper, where we apply the previous abstract results to obtain sharp global error estimates for various concrete nonlinear elliptic problems. The problems considered include second order problems with both the Dirichlet and the mixed boundary conditions, fourth order problems and second order systems. The restrictions are that they are in divergence form and consist of the principal part only: however, as will be pointed out, we thus cover many important real-life models.

To avoid extra length, we detail the exposition for second order Dirichlet problems and only sketch the analogous results for the other types of problems considered.

4.1. Second order Dirichlet problems

We consider the problem

$$\begin{cases} -\operatorname{div} f(\nabla u) = g, \\ u_{|\partial\Omega} = 0 \end{cases} \tag{4.1}$$

under the following assumptions:

Assumptions 4.1.

(i) $\Omega \subset \mathbb{R}^d$ is a bounded domain with a piecewise C^2 boundary, locally convex at the corners.

(ii) $f \in C^1(\mathbb{R}^d, \mathbb{R}^d)$, the Jacobians $f'(\eta) := \partial f(\eta)/\partial \eta$ are symmetric and there exist constants $M \geqslant m > 0$ such that

$$m|\xi|^2 \leqslant f'(\eta)\xi \cdot \xi \leqslant M|\xi|^2 \quad (\eta, \xi \in \mathbb{R}^d). \tag{4.2}$$

(iii) $f'\colon \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is Lipschitz continuous with a Lipschitz constant L.

(iv) $g \in L^2(\Omega)$.

Let $H_0^1(\Omega)$ denote the usual Sobolev space with the inner product

$$\langle u, v\rangle_{H_0^1} := \int_\Omega \nabla u \cdot \nabla v, \tag{4.3}$$

further, let

$$H(\operatorname{div}) := \{y \in L^2(\Omega)^d\colon \operatorname{div} y \in L^2(\Omega)\}.$$

We will also use the space $L^2(\Omega)^d$ with the usual inner product $\langle y, z\rangle_{L^2(\Omega)^d} := \int_\Omega y \cdot z$.

Assumptions (ii) and (iv) imply that problem (4.1) has a unique weak solution $u^* \in H_0^1(\Omega)$, i.e., such that it satisfies

$$\int_\Omega f(\nabla u^*) \cdot \nabla v - \int_\Omega gv = 0 \quad (v \in H_0^1(\Omega)). \tag{4.4}$$

We consider an approximate solution $u \in H_0^1(\Omega)$ and measure the error by the functional

$$\begin{aligned} E(u) &:= \int_\Omega (f(\nabla u) - f(\nabla u^*)) \cdot (\nabla u - \nabla u^*) \\ &= \int_\Omega f(\nabla u) \cdot (\nabla u - \nabla u^*) - \int_\Omega g(u - u^*). \end{aligned} \tag{4.5}$$

We note that by (2.3),

$$\|u - u^*\|_{H_0^1}^2 \leqslant m^{-1} E(u).$$

4.1.1. The error estimation

Now we formulate and prove our main result on the error estimation for (4.1) for the approximate solution u.

Theorem 4.1. *Let $u \in W^{1,\infty}(\Omega)$. Then for arbitrary $y^* \in H(\operatorname{div}) \cap L^\infty(\Omega)^d$ and arbitrary $h \in H_0^1(\Omega)$,*

$$\begin{aligned} E(u) \leqslant\ & \mathrm{EST}(u; y^*, h) \\ := & \Big(m^{-1/2} C_\Omega \|\operatorname{div} y^* + g\|_{L^2(\Omega)} + \frac{L}{2} m^{-3/2} D(u; y^*, h) \\ & + \Big(\langle f(\nabla u) - y^*, \nabla u - f^{-1}(y^*)\rangle_{L^2(\Omega)^d} \\ & + \frac{L}{2m} D(u; y^*, h) \|\nabla u - f^{-1}(y^*)\|_{L^2(\Omega)^d}\Big)^{1/2}\Big)^2, \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} D(u; y^*, h) := & (M\|f^{-1}(y^*) - \nabla h\|_{L^2(\Omega)^d} + C_\Omega \|\operatorname{div} y^* + g\|_{L^2(\Omega)}) \\ & \times \|\nabla u - f^{-1}(y^*)\|_{L^\infty(\Omega)^d}. \end{aligned} \tag{4.7}$$

Proof. Let $V := H_0^1(\Omega)$ and $Y := L^2(\Omega)^d$. We will use Theorem 3.1, to which end we must verify that Assumptions 3.2–3.3 hold for the corresponding spaces and operators.

First, Assumption 3.2 (i) is valid for the operator $\Lambda := \nabla$, since (4.3) just yields that (3.7) holds. Now let $A\colon\ L^2(\Omega)^d \to L^2(\Omega)^d$ be defined by

$$A(y) := f(y) \quad \text{(or, more precisely, } f \circ y), \tag{4.8}$$

that is, outer composition with f. Such an operator is often called a Nemyckiĭ operator (see, e.g., [52]), and it follows in a standard way [13], [52] from our condition $f \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ and from the assumed symmetry of the Jacobians that A has a bihemicontinuous symmetric Gâteaux derivative according to Definition 2.2, i.e., Assumption 3.2 (ii) holds. The Gâteaux derivative of A satisfies

$$\langle A'(y)p, q\rangle_{L^2(\Omega)^d} = \int_\Omega f'(y)p \cdot q \quad (y, p, q \in L^2(\Omega)^d), \tag{4.9}$$

hence by (4.2) we have

$$m\|p\|^2_{L^2(\Omega)^d} \leqslant \langle A'(y)p, p\rangle_{L^2(\Omega)^d} \leqslant M\|p\|^2_{L^2(\Omega)^d} \quad (y, p \in L^2(\Omega)^d). \tag{4.10}$$

The left-hand side of (4.10) coincides with Assumption 3.2 (iii). Finally, defining the operator $F\colon\ H^1_0(\Omega) \to H^{-1}(\Omega)$ via

$$\langle F(u), v\rangle \equiv \int_\Omega f(\nabla u) \cdot \nabla v \quad (u, v \in H^1_0(\Omega)), \tag{4.11}$$

we obtain the equality (3.9), required for Assumption 3.2 (iv) to hold.

To verify Assumption 3.3 (i), let us define $W := L^\infty(\Omega)^d$ with the standard norm $\|y\|_{L^\infty(\Omega)^d} := \operatorname{ess\,sup}_\Omega |y|$. For the required Lipschitz continuity of A' from $L^2(\Omega)^d$ to $B(L^\infty(\Omega)^d, L^2(\Omega)^d)$ we must prove (3.21) for (4.8). In fact, we have imposed in Assumption 4.1 (iii) the Lipschitz continuity of f' with a constant $L > 0$, i.e.,

$$\|f'(\xi) - f'(\eta)\| \leqslant L|\xi - \eta| \quad (\xi, \eta \in \mathbb{R}^d). \tag{4.12}$$

Therefore,

$$\begin{aligned} |\langle (A'(z) - A'(y))w, p\rangle| &= \left| \int_\Omega (f'(z) - f'(y))w \cdot p \right| \\ &\leqslant L \int_\Omega |z - y||w||p| \\ &\leqslant L\|z - y\|_{L^2(\Omega)^d} \|w\|_{L^\infty(\Omega)^d} \|p\|_{L^2(\Omega)^d} \\ &\qquad (y, z, p \in L^2(\Omega)^d,\ w \in L^\infty(\Omega)^d), \end{aligned} \tag{4.13}$$

which is the desired estimate. Assumption 3.3 (ii) for (4.8) coincides with the right-hand side of (4.10).

It is left to check the remaining assumptions of Theorem 3.1. Defining the linear functional $l\colon\ H_0^1(\Omega)\to\mathbb{R}$ as

$$\langle l,v\rangle \equiv -\int_\Omega gv \quad (v\in H_0^1(\Omega)) \tag{4.14}$$

and using (4.11), the weak formulation (4.4) of our problem becomes

$$\langle F(u^*),v\rangle + \langle l,v\rangle = 0,$$

i.e. u^* is the solution of (1.1) indeed. We have chosen u to satisfy $u\in W^{1,\infty}(\Omega)$, hence $u\in V=H_0^1(\Omega)$ and $\Lambda u=\nabla u\in W=L^\infty(\Omega)^d$. Further, we have assumed $y^*\in W=L^\infty(\Omega)^d$, and the left-hand side of (4.2) implies trivially that f^{-1} carries bounded sets into bounded sets (since it grows at most linearly with a factor $1/m$), therefore $z^*:=A^{-1}(y^*)=f^{-1}(y^*)\in L^\infty(\Omega)^d=W$. Finally, $h\in H_0^1(\Omega)=V$. That is, all the assumptions of Theorem 3.1 hold, therefore (3.31) is valid for our problem.

It remains to show that the general estimate (3.31) for our problem becomes estimate (4.6). Here, using $y^*\in H(\mathrm{div})$, we obtain

$$\langle\Lambda^*y^*,v\rangle=\langle y^*,\Lambda v\rangle=\int_\Omega y^*\cdot\nabla v=-\int_\Omega(\operatorname{div}y^*)v \quad (v\in H_0^1(\Omega)),$$

hence $\Lambda^*y^*=-\operatorname{div}y^*$. Then, by (3.14),

$$\begin{aligned}|\Lambda^*y^*+l| &= \sup_{\|v\|_{H_0^1}=1}|\langle\Lambda^*y^*+l,v\rangle|\\ &= \sup_{\|v\|_{H_0^1}=1}\left|-\int_\Omega(\operatorname{div}y^*+g)v\right|\\ &\leqslant \sup_{\|v\|_{H_0^1}=1}\|\operatorname{div}y^*+g\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}\\ &\leqslant C_\Omega\|\operatorname{div}y^*+g\|_{L^2(\Omega)}\end{aligned}$$

(where $C_\Omega>0$ comes from the Friedrichs inequality), see also [40]. Inserting the latter into (3.31) and (3.32) and replacing V, Y, W, Λ and A by $H_0^1(\Omega)$, $L^2(\Omega)^d$, $L^\infty(\Omega)^d$, ∇ and f, respectively, we obtain (4.6). □

R e m a r k 4.1. Following Remark 3.2, it is convenient to reformulate Theorem 4.1 for $z^*:=f^{-1}(y^*)$ in order to avoid the computation of f^{-1}. Then, letting $u\in W^{1,\infty}(\Omega)$ be any approximate solution, for arbitrary $z^*\in L^\infty(\Omega)^d$ such that $f(z^*)\in$

$H(\mathrm{div})$, and for arbitrary $h \in H_0^1(\Omega)$, we have

$$
\begin{aligned}
E(u) \leqslant \widetilde{\mathrm{EST}}(u; z^*, h) \\
&:= \Big(m^{-1/2} C_\Omega \|\operatorname{div} f(z^*) + g\|_{L^2(\Omega)} + \frac{L}{2} m^{-3/2} \tilde{D}(u; z^*, h) \\
&\quad + \Big(\langle f(\nabla u) - f(z^*), \nabla u - z^* \rangle_{L^2(\Omega)^d} \\
&\quad + \frac{L}{2m} \tilde{D}(u; z^*, h) \|\nabla u - z^*\|_{L^2(\Omega)^d} \Big)^{1/2} \Big)^2,
\end{aligned} \tag{4.15}
$$

where

$$
\begin{aligned}
\tilde{D}(u; z^*, h) := \Big(M \|z^* - \nabla h\|_{L^2(\Omega)^d} + C_\Omega \|\operatorname{div} f(z^*) + g\|_{L^2(\Omega)} \Big) \\
\times \|\nabla u - z^*\|_{L^\infty(\Omega)^d}.
\end{aligned} \tag{4.16}
$$

We note that one can further estimate (4.6) using quadratic terms as in Corollary 3.1, which we leave to the reader. Now we state the sharpness of the estimate:

Proposition 4.1. *Estimate* (4.6) *is sharp, that is,*

$$
\min_{\substack{y^* \in H(\mathrm{div}) \cap L^\infty(\Omega)^d, \\ h \in H_0^1(\Omega)}} \mathrm{EST}(u; y^*, h) = E(u).
$$

Proof. By [33], the weak solution of (4.1) satisfies $u^* \in C^{1,\alpha}(\overline{\Omega})$ with some $0 < \alpha < 1$, hence $\nabla u^* \in L^\infty(\Omega)^d = W$. Therefore, we can apply Proposition 3.4 to obtain the desired statement. □

We note that by (3.45) the optimal values for "free" parameters in the estimate are

$$
y^* := f(\nabla u^*) \quad \text{and} \quad h := u^*. \tag{4.17}
$$

The practical approximations of these will be discussed in subsection 4.1.2.

Remark 4.2. Our result is a direct extension of earlier sharp error estimates obtained for linear problems first in [43] by means of the duality theory (see also [40]), and later in [44] (via the Helmholtz decomposition) and in [28] (via integral identities). Namely, if we have the linear equation

$$
-\operatorname{div}(\mathcal{A} \nabla u) = g
$$

for some symmetric and uniformly positive definite matrix $\mathcal{A}$ in (4.1), which corresponds to the case $f(\eta) := \mathcal{A}\eta$, then we can use Remark 3.3. Now the Lipschitz constant is $L = 0$, i.e. the terms containing L vanish, and (3.44) yields

$$E(u) \leqslant (1+\beta)m^{-1}\|\operatorname{div} y^* + g\|^2_{L^2(\Omega)} + \Big(1+\frac{1}{\beta}\Big)\langle \mathcal{A}\nabla u - y^*, \nabla u - \mathcal{A}^{-1}y^*\rangle_{L^2(\Omega)^d}.$$

4.1.2. Practical considerations

Finite element solution. The most important practical case is when finite element approximation is used to find an approximate solution. In general, let V_h be a given FEM subspace and $u_h \in V_h$ the corresponding FEM approximation of the exact solution u^*. Then our error measure is

$$E(u_h) = \langle F(u_h) - F(u^*), u_h - u^*\rangle. \tag{4.18}$$

Here u_h is a continuous piecewise polynomial, hence the condition $u_h \in W^{1,\infty}(\Omega)$ in Theorem 4.1 is satisfied. If we choose y^* to be any continuous piecewise polynomial function, e.g. a function from another FEM subspace, and an arbitrary $w \in H^1_0(\Omega)$, then $y^* \in H(\operatorname{div}) \cap L^\infty(\Omega)^d$, hence Theorem 4.1 can be applied, which gives

$$\begin{aligned} E(u_h) &\leqslant \operatorname{EST}(u_h; y^*, w) \\ &:= \Big(m^{-1/2}C_\Omega\|\operatorname{div} y^* + g\|_{L^2(\Omega)} + \frac{L}{2}m^{-3/2}\,D(u_h; y^*, w) \\ &\quad + \Big(\langle f(\nabla u_h) - y^*, \nabla u_h - f^{-1}(y^*)\rangle_{L^2(\Omega)^d} \\ &\quad + \frac{L}{2m}D(u_h; y^*, w)\|\nabla u_h - f^{-1}(y^*)\|_{L^2(\Omega)^d}\Big)^{1/2}\Big)^2, \end{aligned} \tag{4.19}$$

where

$$\begin{aligned} D(u_h; y^*, w) &:= (M\|f^{-1}(y^*) - \nabla w\|_{L^2(\Omega)^d} + C_\Omega\|\operatorname{div} y^* + g\|_{L^2(\Omega)}) \\ &\qquad \times \|\nabla u_h - f^{-1}(y^*)\|_{L^\infty(\Omega)^d}. \end{aligned} \tag{4.20}$$

We note that it is useful to replace $f^{-1}(y^*)$ by z^* as in (4.15)–(4.16) to avoid the computation of f^{-1}. The expressions obtained are directly computable integrals.

Determining optimal y^* and w in $\operatorname{EST}(u_h; y^*, w)$. By virtue of (4.17), the optimal value of the parameter y^* should be a sufficiently accurate approximation of $f(\nabla u^*)$. For finite element solutions, a common and "computationally cheap" way to achieve this goal is to use an averaging procedure, i.e., to replace the unknown function ∇u^* (the gradient of the exact solution) by $G_h(\nabla u_h)$, where G_h is some averaging operator. For the case of linear finite elements, $G_h(\nabla u_h)$ is closer to ∇u^* than ∇u_h by an order of magnitude, namely, the original approximation order $\|\nabla u^* - \nabla u_h\|_{L^2} = O(h)$ can be thus improved to $\|\nabla u^* - G_h(\nabla u_h)\|_{L^2} = O(h^2)$ if u^* is sufficiently smooth, see [8] or [21, Part I] for details. Accordingly, we can define

$$y^* := f(G_h(\nabla u_h)), \quad z^* = f^{-1}(y^*) = G_h(\nabla u_h) \tag{4.21}$$

as a first candidate for the parameter y^* (or z^*).

Remark 4.3. The value (4.21) for the parameter y^* (or z^*) can still give a too rough bound in general, in which case we normally execute a minimization process for y^*, see [40, Section 6.10] or [31], [18] for more details.

Next, according to Remark 3.4, the optimal w for this z^* is given as the solution of the following linear auxiliary problem: find $w_{\rm opt} \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla w_{\rm opt} \cdot \nabla v = \int_\Omega z^* \cdot \nabla v \quad (v \in H_0^1(\Omega)), \tag{4.22}$$

that is, the weak solution of the Poisson problem

$$\begin{cases} -\Delta w_{\rm opt} = -\operatorname{div} z^*, \\ w_{{\rm opt}|\partial\Omega} = 0. \end{cases} \tag{4.23}$$

This means that for a given y^*, the optimal estimate for the second parameter w is found by solving a kind of adjoint or auxiliary equation; however, the latter is linear, hence its numerical solution costs much less than that of the original one. For piecewise linear FEM, if (4.23) is solved numerically on the same mesh as used for u_h, then its right-hand side $-\operatorname{div} z^* = -\operatorname{div} G_h(\nabla u_h)$ is constant on each element, hence it requires minimal numerical integration and is therefore a cheap auxiliary problem. On the other hand, using a finer (or just different) mesh for (4.23) than the one used for u_h may considerably increase the accuracy of the estimate, similarly to the adjoint problems for linear equations [19], [27] (see also [47]), with low extra cost due to the linearity of (4.23).

Calculating the required constants. The constants used in estimate (4.6) are C_Ω, m, M, and L. The only one depending on the domain is C_Ω, which can be easily estimated from above (it is sufficient for the estimation purposes) as in [35, p. 8]. Further, the three remaining constants m, M and L come from the given nonlinearity, see Assumptions 4.1 (ii)–(iii), where we note that the crucial point in our sharp estimates is the existence of L, i.e., the condition of Lipschitz continuity of the derivative of f. Based on Remark 3.5, one can see that this Lipschitz condition usually means no restriction in practice, since it is satisfied for most real problems. Namely, problems of the type (4.1) in real models are generally of the following special form, involving a scalar nonlinearity:

$$\begin{cases} -\operatorname{div}(a(|\nabla u|^2)\nabla u) = g, \\ u_{|\partial\Omega} = 0 \end{cases} \tag{4.24}$$

(which corresponds to $f(\eta) = a(|\eta|^2)\eta$ in (4.1)), where $a\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a scalar C^2 function with properties (3.48)–(3.49). Such nonlinearities form the main examples for (4.1), arising, e.g., in elasto-plastic torsion [23], or in electromagneticity, see the presentation for nonlinear Maxwell equations in [30] and for nonlinear magnetostatic field in [11]. One may even have explicit formulae for the function a, such as

$$a(t) = \frac{1}{\mu_0}\left(\alpha + (1-\alpha)\frac{t^8}{t^8+\beta}\right) \quad (t \geqslant 0), \tag{4.25}$$

which characterizes the reluctance of stator sheets in the cross-sections of an electrical motor [30], or

$$a(t) = \left(1 - (c-d)\frac{1}{t^2+c}\right) \quad (t \geqslant 0), \tag{4.26}$$

which describes the magnetostatic field [11]; the constants in these formulas are given positive characteristic physical values. Using Remark 3.5, condition (3.49) implies the Lipschitz continuity for f. It has also been pointed out in Remark 3.5 that condition (3.49) follows from the standard elliptic property (3.48) except for some unrealistic special cases.

Summing up, it follows that the bounds m and M and the Lipschitz constant L, needed to calculate $E(u)$, can be determined from lower or upper bounds, respectively, for the scalar functions in (3.48)–(3.49). These only require an elementary numerical calculation. Moreover, if the parameters y^* and w are close to the optimal choice, then (using Proposition 3.4) all terms containing these constants (as well as C_Ω) in $\mathrm{EST}(u_h; y^*, w)$ are close to zero, hence the global constants need not be estimated from above too accurately.

4.2. Other problems

We sketch the results, analogous to the above, for some other nonlinear elliptic problems. The detailed exposition is found in the preprint version [26].

4.2.1. Second order mixed problems

Let us first consider second order problems with mixed boundary conditions. Here we also allow dependence of the nonlinearity f on x, which was not included in (4.1) for simplicity. That is,

$$\begin{cases} -\operatorname{div} f(x, \nabla u) = g, \\ u_{|\Gamma_D} = 0, \\ f(x, \nabla u) \cdot \nu_{|\Gamma_N} = \gamma \end{cases} \tag{4.27}$$

(where ν denotes the outer normal unit vector). Here Assumptions 4.1 are amended with the following conditions: Γ_D, Γ_N are disjoint open subsets of $\partial\Omega$ such that $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$ and $\Gamma_D \neq \emptyset$, further, $\gamma \in L^2(\Gamma_N)$; finally, in assumption (ii), the conditions on $f'(\eta)$ are replaced in an obvious way with those for $f'(x, \eta) := \partial f(x, \eta)/\partial \eta$.

The treatment of this problem uses the Sobolev space

$$H^1_D(\Omega) := \{u \in H^1(\Omega)\colon\ u_{|\Gamma_D} = 0 \text{ in trace sense}\} \tag{4.28}$$

with the inner product $\langle u, v\rangle_{H^1_D} := \int_\Omega \nabla u \cdot \nabla v$; further, let

$$H(\operatorname{div}, \Gamma_N) := \{y \in L^2(\Omega)^d\colon\ \operatorname{div} y \in L^2(\Omega),\ y \cdot \nu \in L^2(\Gamma_N)\}.$$

We now use the estimates

$$\|v\|_{L^2(\Omega)} \leqslant C'_\Omega \|\nabla v\|_{L^2(\Omega)^d}, \quad \|v\|_{L^2(\Gamma_N)} \leqslant C_{\Gamma_N} \|\nabla v\|_{L^2(\Omega)^d} \quad (v \in H^1_D(\Omega)) \tag{4.29}$$

with some suitable constants C'_Ω, $C_{\Gamma_N} > 0$, for the proofs, see [37].

To formulate the main result, we note that by Assumption 4.2.1 (ii), for all fixed $x \in \Omega$, the function $f(x, \cdot)$ is invertible on $\mathbb{R}^d$ with respect to η. We will denote by f^{-1} the inverse with respect to η, i.e.

$$f(x, \eta) = \xi \Rightarrow f^{-1}(x, \xi) := \eta. \tag{4.30}$$

Then one can prove the main results similarly to the above:

Theorem 4.2. *Let $u \in W^{1,\infty}(\Omega)$. Then for arbitrary $y^* \in H(\operatorname{div}, \Gamma_N) \cap L^\infty(\Omega)^d$ and arbitrary $h \in H^1_D(\Omega)$,*

$$\begin{aligned} E(u) &\leqslant \operatorname{EST}(u; y^*, h) \\ &:= \Big(m^{-1/2} C'_\Omega \|\operatorname{div} y^* + g\|_{L^2(\Omega)} + m^{-1/2} C_{\Gamma_N} \|y^* \cdot \nu - \gamma\|_{L^2(\Gamma_N)} \\ &\quad + \frac{L}{2} m^{-3/2} D(u; y^*, h) \\ &\quad + \Big(\langle f(x, \nabla u) - y^*, \nabla u - f^{-1}(x, y^*)\rangle_{L^2(\Omega)^d} \\ &\quad + \frac{L}{2m} D(u; y^*, h) \|\nabla u - f^{-1}(x, y^*)\|_{L^2(\Omega)^d} \Big)^{1/2} \Big)^2 \end{aligned} \tag{4.31}$$

where

$$(4.32)\qquad D(u;y^*,h) := \big(M\|f^{-1}(x,y^*) - \nabla h\|_{L^2(\Omega)^d} + C'_\Omega\|\operatorname{div} y^* + g\|_{L^2(\Omega)} + C_{\Gamma_N}\|y^*\cdot\nu - \gamma\|_{L^2(\Gamma_N)}\big)\|\nabla u - f^{-1}(x,y^*)\|_{L^\infty(\Omega)^d}.$$

Turning to the sharpness problem, Proposition 3.4 yields

Proposition 4.2. *Estimate* (4.31) *is sharp, that is,*

$$(4.33)\qquad \min_{\substack{y^*\in H(\operatorname{div},\Gamma_N)\cap L^\infty(\Omega)^d,\\ h\in H^1_0(\Omega)}} \operatorname{EST}(u;y^*,h) = E(u)$$

provided the exact solution satisfies $u^* \in W^{1,\infty}(\Omega)$.

R e m a r k 4.4. Analogues of Theorem 4.2 can be proved similarly if (4.27) is replaced by one of the following problems:

(a) Neumann problem. Allowing $\Gamma_D = \emptyset$ in Assumption 4.2.1 (i), we have

$$(4.34)\qquad \begin{cases} -\operatorname{div} f(x,\nabla u) = g,\\ f(x,\nabla u)\cdot \nu_{|\partial\Omega} = \gamma. \end{cases}$$

Then Theorem 4.2 remains true if we substitute the factorized space $V := \dot H^1(\Omega) := \{u\in H^1(\Omega)\colon \int_\Omega u = 0\}$ instead of $H^1_D(\Omega)$ and replace Γ_N by $\partial\Omega$ in the formulas. In particular, the resulting constant $C_{\partial\Omega}$ to satisfy the second inequality in (4.29) for all $v\in \dot H^1(\Omega)$ is the smallest positive eigenvalue of $-\Delta$ with Neumann boundary conditions.

(b) Interface problems. Let $\Gamma_{\rm int}$ be a piecewise smooth surface lying in the interior of Ω, and let us consider the problem

$$(4.35)\qquad \begin{cases} -\operatorname{div} f(x,\nabla u) = g,\\ u_{|\Gamma_D} = 0,\quad f(x,\nabla u)\cdot\nu_{|\Gamma_N} = \gamma_N,\quad f(x,\nabla u)\cdot \nu_{|\Gamma_{\rm int}} = \gamma_{\rm int}, \end{cases}$$

where the assumptions for the mixed problem are modified so that $\gamma_N\in L^2(\Gamma_N)$ and $\gamma_{\rm int}\in L^2(\Gamma_{\rm int})$. The weak form of this problem is the same as for the mixed problem if Γ_N is replaced by $\Gamma := \Gamma_N\cup\Gamma_{\rm int}$, see [25] for a related setting. Defining $\gamma\in L^2(\Gamma)$ so that its restrictions to Γ_N and $\Gamma_{\rm int}$ are γ_N and $\gamma_{\rm int}$, respectively, Theorem 4.2 remains true if we replace Γ_N by Γ in the formulas.

In practice, to determine suitable y^* and w in $\mathrm{EST}(u_h; y^*, w)$, first y^* should be some approximation of $f(x, \nabla u^*)$. For finite element solutions, using averaging as in (4.21), we can first let

$$y^* := f(x, G_h(\nabla u_h)), \quad z^* = f^{-1}(x, y^*) = G_h(\nabla u_h), \tag{4.36}$$

where G_h is some averaging operator and f^{-1} is understood with respect to η as in (4.30). Averaging for mixed boundary conditions is discussed, e.g., in [21, Part II]. More accurate error bounds can be obtained by suitable further bound minimization with respect to y^* (or z^*) as mentioned in Remark 4.3.

Then by Remark 3.4, the optimal w for this z^* to set in $\mathrm{EST}(u_h; y^*, w)$ is given as the solution of a linear auxiliary problem, which is the modification of (4.22) for mixed boundary conditions. This can be solved on a suitably chosen mesh, either the same as that used for u_h or a finer/different mesh, as discussed in Section 4.1.

The constants used can be obtained easily for most of the practical cases, using a scalar form of the nonlinearity as in (4.24). Some examples are the x-dependent nonlinearity in magnetic potential:

$$a(x,t) = \begin{cases} \dfrac{1}{\mu_0}\Big(\alpha + (1-\alpha)\dfrac{t^8}{t^8+\beta}\Big) & \text{if } x \in \Omega_1,\ t \geqslant 0, \\ \alpha \quad \text{if } x \in \Omega \setminus \Omega_1, \end{cases}$$

where $\alpha > 0$ is a constant magnetic reluctance [15], [30], or that describing air density in a subsonic potential flow, see, e.g., [3]:

$$a(t) = \varrho_\infty \Big(1 + \frac{1}{5}(M_\infty^2 - t)\Big)^{5/2} \quad (t \geqslant 0),$$

where M_∞ is the Mach number at infinity. In the corresponding mixed problem, Γ_D is the wind inblow part and Γ_N consists of the other sides of the wind tunnel section. Altogether, the constants can be therefore determined by elementary numerical calculation.

4.2.2. Fourth order problems

In this subsection we study 4th order Dirichlet problems. The concise presentation requires some basic notation: let D^2u denote the Hessian of a function $u\colon \Omega \to \mathbb{R}$. If $u \in H^2(\Omega)$, we define the elementwise matrix product and the corresponding Frobenius norm in the standard way as

$$P\colon Q := \sum_{i,k=1}^{d} P_{ik} Q_{ik}, \quad |P|_F := (P : P)^{1/2} \quad (P, Q \in \mathbb{R}^{d\times d}), \tag{4.37}$$

further, for a matrix-valued function $P\colon\ \Omega \to \mathbb{R}^{d\times d}$ we let

$$\operatorname{div}^2 P := \sum_{i,k=1}^{d} \frac{\partial^2 P_{ik}}{\partial x_i \partial x_k}$$

provided that these derivatives exist.

Now we can formulate the problems considered, defined via a matrix-valued nonlinearity B, in the form

$$\begin{cases} \operatorname{div}^2 B(x, D^2 u) = g, \\ u_{|\partial\Omega} = \dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0, \end{cases} \tag{4.38}$$

on a bounded domain $\Omega \subset \mathbb{R}^d$ with a piecewise C^1 boundary, with $g \in L^2(\Omega)$ as before, under the following assumptions on the nonlinearity B:

(i) The matrix-valued function $B\colon\ \Omega \times \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ is measurable and bounded with respect to the variable $x \in \Omega$ and C^2 in the matrix variable $\Theta \in \mathbb{R}^{d\times d}$. The Jacobian arrays

$$B'(x,\Theta) := \frac{\partial B(x,\Theta)}{\partial \Theta} = \Big\{ \frac{\partial B_{rs}(x,\Theta)}{\partial \Theta_{ik}} \Big\}_{i,k,r,s=1}^{d} \in \mathbb{R}^{(d\times d)^2}$$

are symmetric, i.e. $\partial B_{rs}/\partial \Theta_{ik} = \partial B_{ik}/\partial \Theta_{rs}$ for all i,k,r,s, and there exist constants $M \geqslant m > 0$ such that

$$m|\Phi|_F^2 \leqslant B'(x,\Theta)\Phi\colon\ \Phi \leqslant M|\Phi|_F^2 \quad (x \in \Omega;\ \Theta, \Phi \in \mathbb{R}^{d\times d}). \tag{4.39}$$

(ii) $B'\colon\ \Omega \times \mathbb{R}^{d\times d} \to \mathbb{R}^{(d\times d)^2}$ is Lipschitz continuous in the matrix variable $\Theta \in \mathbb{R}^{d\times d}$, with a Lipschitz constant L.

In the treatment of this problem we follow the previous sections. Now we use the Lebesgue space

$$L^2(\Omega)^{d\times d} := \{P\colon\ \Omega \to \mathbb{R}^{d\times d}\colon\ P_{ik} \in L^2(\Omega) \text{ for all } i,k = 1,\dots,d\} \tag{4.40}$$

with the inner product $\langle P, Q\rangle_{L^2(\Omega)^{d\times d}} := \int_\Omega P : Q$, and the Sobolev space

$$H_0^2(\Omega) := \Big\{ u \in H^2(\Omega)\colon\ u_{|\partial\Omega} = \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0 \text{ in trace sense} \Big\} \tag{4.41}$$

with the inner product $\langle u, v\rangle_{H_0^2} := \langle D^2 u, D^2 v\rangle_{L^2(\Omega)^{d\times d}} = \int_\Omega D^2 u : D^2 v$. Further, let

$$H(\operatorname{div}^2) := \{P \in L^2(\Omega)^{d\times d}\colon\ \operatorname{div}^2 P \in L^2(\Omega)\}.$$

The actual counterpart of the Friedrichs inequality is

$$\|v\|_{L^2(\Omega)} \leqslant \tilde{C}_\Omega \|D^2 v\|_{L^2(\Omega)^{d\times d}} \quad (v \in H_0^2(\Omega)) \tag{4.42}$$

for a suitable constant $\tilde{C}_\Omega > 0$. Analogously to (4.30), we will denote by B^{-1} the inverse with respect to Θ, i.e.

$$B(x,\Theta) = \Phi \Rightarrow B^{-1}(x,\Phi) := \Theta, \tag{4.43}$$

where B^{-1} exists by virtue of the assumptions on B. Then one can prove the main results similarly to the above:

Theorem 4.3. *Let $u \in W^{2,\infty}(\Omega)$. Then for arbitrary $Y^* \in H(\operatorname{div}^2) \cap L^\infty(\Omega)^{d\times d}$ and arbitrary $h \in H_0^2(\Omega)$,*

$$\begin{aligned} E(u) &\leqslant \operatorname{EST}(u;Y^*,h) \\ &:= \Big(m^{-1/2}\tilde{C}_\Omega \|\operatorname{div}^2 Y^* - g\|_{L^2(\Omega)} + \frac{L}{2} m^{-3/2} D(u;Y^*,h) \\ &\quad + \Big(\langle B(x,D^2u) - Y^*, D^2u - B^{-1}(x,Y^*)\rangle_{L^2(\Omega)^{d\times d}} \\ &\quad + \frac{L}{2m} D(u;Y^*,h) \|D^2u - B^{-1}(x,Y^*)\|_{L^2(\Omega)^{d\times d}}\Big)^{1/2}\Big)^2, \end{aligned} \tag{4.44}$$

where

$$\begin{aligned} D(u;Y^*,h) := \big(&M\|B^{-1}(x,Y^*) - D^2h\|_{L^2(\Omega)^{d\times d}} + \tilde{C}_\Omega\|\operatorname{div}^2 Y^* - g\|_{L^2(\Omega)}\big) \\ &\times \|D^2u - B^{-1}(x,Y^*)\|_{L^\infty(\Omega)^{d\times d}}. \end{aligned} \tag{4.45}$$

Remark 4.5. Following [40, Chap. 6.6], the term $\tilde{C}_\Omega\|\operatorname{div}^2 Y^* - g\|_{L^2(\Omega)}$ in (4.44) can be replaced by

$$\hat{C}_\Omega \|\operatorname{div} Y^* - \eta^*\|_{L^2(\Omega)^{d\times d}} + \tilde{C}_\Omega \|\operatorname{div} \eta^* - g\|_{L^2(\Omega)}$$

for some new parameter function $\eta^* \in H(\operatorname{div})$. In this case the requirement $Y^* \in H(\operatorname{div}^2)$ can be weakened to $Y^* \in H(\operatorname{div})$ (understood row-wise).

Note that our result is a direct extension of earlier sharp error estimates obtained for linear fourth order problems [14], [39], [40]. (This is seen using Remark 3.3 in a similar way to Remark 4.2.) In our case, Proposition 3.4 yields

Proposition 4.3. *Estimate* (4.44) *is sharp, that is,*

$$\min_{\substack{Y^* \in H(\mathrm{div}^2) \cap L^\infty(\Omega)^{d\times d}, \\ h \in H_0^2(\Omega)}} \mathrm{EST}(u; Y^*, h) = E(u) \tag{4.46}$$

provided the exact solution satisfies $u^* \in W^{2,\infty}(\Omega)$.

In practice for FEM, in order to have an approximate solution $u_h \in H_0^2(\Omega)$, one uses C^1-elements (i.e. $u_h \in C^1$ and u_h is piecewise polynomial), see, e.g., [10]. In this case we automatically have $u \in W^{2,\infty}(\Omega)$, which was required for Theorem 4.3 to hold. (Another common FEM approach is to use mixed variables to have less smoothness for u_h. In this case one may expect to reformulate the terms containing D^2u in (4.44) via the mixed variables in a similar vein as in Remark 4.5; however, this is out of the scope of the present paper.) Next, following (3.45), Y^* should be an approximation of $B(x, D^2u^*)$. For finite element solutions, using averaging as before, we can first let

$$Y^* := B(x, G_h(D^2u_h)), \quad Z^* = B^{-1}(x, Y^*) = G_h(D^2u_h), \tag{4.47}$$

where G_h is an averaging operator that defines a C^1-approximation of D^2u_h, and B^{-1} is understood with respect to Θ as in (4.43). More accurate error bounds can be obtained by suitable further bound minimization with respect to Y^* (or Z^*) as mentioned in Remark 4.3. Then by Remark 3.4, the optimal w for this Z^* to set in $\mathrm{EST}(u_h; Y^*, w)$ is the solution of the corresponding linear biharmonic auxiliary problem with the right-hand side $\mathrm{div}^2 Z^*$. Note that Z^* need not be in $H(\mathrm{div}^2)$: in general $\mathrm{div}^2 Z^*$ can be understood in the distributional sense, which exactly means that we need to use the weak form, and thus the weaker condition $Y^* \in H(\mathrm{div})$ (or equivalently $Z^* \in H(\mathrm{div})$) can be used. Altogether, one can define w as the numerical solution of the biharmonic auxiliary problem on a suitably chosen mesh, either the same as used for u_h or a finer mesh, as discussed in Section 4.1.

The most important real-life model that uses fourth order equations like (4.38) describes the elasto-plastic bending of a clamped thin plane plate $\Omega \subset \mathbb{R}^2$, see, e.g., [34]. This problem reads

$$\begin{cases} \mathrm{div}^2(\overline{g}(E(D^2u))\tilde{D}^2u) = \alpha, \\ u_{|\partial\Omega} = \dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0, \end{cases} \tag{4.48}$$

where

$$\tilde{D}^2u := \frac{1}{2}(D^2u + \Delta u \cdot I), \quad E(D^2u) := \frac{1}{2}(|D^2u|_F^2 + (\Delta u)^2)$$

and $\overline{g}$ is a scalar material function satisfying (3.48)–(3.49) (with $\overline{g}$ substituted for a). This problem leads to an operator like (3.47), see more details in [13].

4.2.3. Second order elasticity systems

Symmetric second order systems arise in the description of the elastic behaviour of a body. We follow the presentation of [7], [38] to describe the elasticity of a body $\Omega \subset \mathbb{R}^3$ with nonlinear behaviour of the material.

The involved physical quantities are the displacement vector $u\colon \Omega \to \mathbb{R}^3$, the strain tensor $\varepsilon\colon \Omega \to \mathbb{R}^{3\times 3}$ and the stress tensor $\sigma\colon \Omega \to \mathbb{R}^{3\times 3}$. The basic system of equations is

$$-\operatorname{div}\sigma_i = \varphi_i \text{ in } \Omega, \quad \sigma_i \cdot \nu = \tau_i \text{ on } \Gamma_N, \quad u_i = 0 \text{ on } \Gamma_D \quad (i = 1,2,3) \tag{4.49}$$

where $\sigma_i = (\sigma_{i1}, \sigma_{i2}, \sigma_{i3})$ $(i = 1,2,3)$ is the ith row of the matrix σ, the functions $\varphi\colon \Omega \to \mathbb{R}^3$ and $\tau\colon \Gamma_N \to \mathbb{R}^3$ describe the body and boundary force vectors, respectively, and $\partial\Omega = \Gamma_N \cup \Gamma_D$ is a disjoint measurable subdivision and $\Gamma_D \neq \emptyset$.

The problem (4.49) can be formulated as a second order system in terms of the displacement u. First, the strain tensor $\varepsilon = \varepsilon(u)$ is determined by the displacement via the relation $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^t)$ where $\nabla u^t(x)$ denotes the transpose of the matrix $\nabla u(x) \in \mathbb{R}^{3\times 3}$ for $x \in \Omega$. The connection between strain and stress is given by a matrix-valued function T as follows. For any $\Theta \in \mathbb{R}^{3\times 3}$ let $\operatorname{vol}\Theta = \frac{1}{3}\operatorname{tr}\Theta \cdot I$ and $\operatorname{dev}\Theta = \Theta - \operatorname{vol}\Theta$, where $\operatorname{tr}\Theta = \sum_{i=1}^{3}\Theta_{ii}$ is the trace of Θ and I is the identity matrix. Using this notation, we have

$$\sigma(x) = T(x, \varepsilon(u(x))) \tag{4.50}$$

with $T\colon \Omega \times \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ given by

$$T(x,\Theta) = 3k(x, |\operatorname{vol}\Theta|^2)\operatorname{vol}\Theta + 2\mu(x, |\operatorname{dev}\Theta|^2)\operatorname{dev}\Theta \quad (x \in \Omega,\ \Theta \in \mathbb{R}^{3\times 3}), \tag{4.51}$$

where $k(x,s)$ is the bulk modulus of the material and $\mu(x,s)$ is Lamé's coefficient. (Further properties of k and μ are given below in (4.53).) Then, substituting (4.50) into (4.49), we obtain the system

$$\left.\begin{array}{rl} -\operatorname{div}T_i(x,\varepsilon(u)) = \varphi_i & \text{in } \Omega, \\ T_i(x,\varepsilon(u)) \cdot \nu = \tau_i & \text{on } \Gamma_N, \\ u_i = 0 & \text{on } \Gamma_D \end{array}\right\} \quad (i = 1,2,3). \tag{4.52}$$

The functions $k, \mu \colon \Omega \times \mathbb{R}^+ \to \mathbb{R}$ are measurable and bounded with respect to x and C^1 with respect to the variable $t \in \mathbb{R}^+$. Further, they satisfy

$$0 < m \leqslant \mu(x,t) < \frac{3}{2} k(x,t) \leqslant M, \tag{4.53}$$
$$0 < m \leqslant \frac{\partial}{\partial t}(k(x,t^2)t) \leqslant M, \quad 0 < m \leqslant \frac{\partial}{\partial t}(\mu(x,t^2)t) \leqslant M$$

with constants $M \geqslant m_0$ independent of (x,t), as described in [7]. We impose as an additional condition that k and μ are also piecewise C^2 (i.e. C^2 except for finitely many isolated points, which in practice typically separate the domain of linear and nonlinear behaviour), further, that there exists a constant $L > 0$ such that

$$\Big| \frac{\partial^2}{\partial t^2}(k(x,t^2)t) \Big| \leqslant L, \quad \Big| \frac{\partial^2}{\partial t^2}(\mu(x,t^2)t) \Big| \leqslant L \quad (x \in \Omega,\ t \geqslant 0). \tag{4.54}$$

We note that some concrete measurements or explicit expressions on k and μ are given, e.g., in [2], [38], [40], and k is often considered a constant. With the notation of (4.37), (4.39), (4.51), and (4.53) imply the analogue of (4.39):

$$m|\Phi|_F^2 \leqslant T'(x,\Theta)\Phi \colon \Phi \leqslant M|\Phi|_F^2 \quad (x \in \Omega;\ \Theta, \Phi \in \mathbb{R}^{3\times 3}). \tag{4.55}$$

This property implies well-posedness in $H_D^1(\Omega)^3$ in view of the famous Korn's inequality

$$\kappa \int_\Omega |\nabla u|^2 \leqslant \int_\Omega |\varepsilon(u)|^2 \leqslant \int_\Omega |\nabla u|^2 \quad (u \in H_D^1(\Omega)^3) \tag{4.56}$$

(where $\kappa > 0$), see more details, e.g., in [7], [13], [38].

In the treatment of error estimation for the elasticity problem, we follow the previous sections. Now we use the Lebesgue space

$$L^2(\Omega)^{3\times 3}_{\text{symm}} := \{P \colon \Omega \to \mathbb{R}^{3\times 3} \colon P_{ik} = P_{ki} \in L^2(\Omega) \text{ for all } i,k = 1,2,3\} \tag{4.57}$$

with the inner product $\langle P, Q\rangle_{L^2(\Omega)^{3\times 3}} := \int_\Omega P : Q$, using notation (4.37). Further, we endow the space $H_D^1(\Omega)^3$ with the inner product

$$\langle u, v\rangle_\varepsilon := \langle \varepsilon(u), \varepsilon(v)\rangle_{L^2(\Omega)^{3\times 3}} = \int_\Omega \varepsilon(u) : \varepsilon(v), \tag{4.58}$$

which is equivalent to the standard inner product owing to (4.56). Inequalities (4.29) and (4.56) then imply

$$\|v\|_{L^2(\Omega)^3} \leqslant \kappa^{-1/2} C'_\Omega \|v\|_\varepsilon, \quad \|v\|_{L^2(\Gamma_N)^3} \leqslant \kappa^{-1/2} C_{\Gamma_N} \|v\|_\varepsilon \quad (v \in H_D^1(\Omega)^3). \tag{4.59}$$

We define $L^\infty(\Omega)^{3\times 3}_{\rm symm}$ analogously to (4.57), and finally let

$$H(\mathrm{div}, \mathbb{R}^3; \Gamma_N) := \{P \in L^2(\Omega)^{3\times 3}_{\rm symm} \colon \mathrm{div}\, P \in L^2(\Omega)^3,\ P\cdot\nu \in L^2(\Gamma_N)^3\}.$$

We will use notation T^{-1} in the sense of (4.43).

Theorem 4.4. *Let $u \in W^{1,\infty}(\Omega)^3$. Then for arbitrary $Y^* \in H(\mathrm{div}, \mathbb{R}^3; \Gamma_N) \cap L^\infty(\Omega)^{3\times 3}_{\rm symm}$ and arbitrary $h \in H^1_D(\Omega)^3$,*

$$\begin{aligned}(4.60)\qquad E(u) &\leqslant \mathrm{EST}(u; Y^*, h)\\ &:= \Big((\kappa m)^{-1/2} C'_\Omega \|\mathrm{div}\, Y^* + \varphi\|_{L^2(\Omega)^3}\\ &\quad + (\kappa m)^{-1/2} C_{\Gamma_N} \|Y^*\cdot\nu - \tau\|_{L^2(\Gamma_N)^3} + \frac{L}{2} m^{-3/2} D(u; Y^*, h)\\ &\quad + \Big(\langle T(x, \varepsilon(u)) - Y^*, \varepsilon(u) - T^{-1}(x, Y^*)\rangle_{L^2(\Omega)^{3\times 3}}\\ &\quad + \frac{L}{2m} D(u; Y^*, h) \|\varepsilon(u) - T^{-1}(x, Y^*)\|_{L^2(\Omega)^{3\times 3}}\Big)^{1/2}\Big)^2,\end{aligned}$$

where

$$\begin{aligned}(4.61)\quad & D(u; Y^*, h)\\ &:= \big(M\|T^{-1}(x, Y^*) - \varepsilon(h)\|_{L^2(\Omega)^{3\times 3}} + \kappa^{-1/2} C'_\Omega \|\mathrm{div}\, Y^* + \varphi\|_{L^2(\Omega)^3}\\ &\quad + \kappa^{-1/2} C_{\Gamma_N} \|Y^*\cdot\nu - \tau\|_{L^2(\Gamma_N)^3}\big)\|\varepsilon(u) - T^{-1}(x, Y^*)\|_{L^\infty(\Omega)^{3\times 3}}.\end{aligned}$$

Our result is a direct extension of earlier sharp error estimates obtained for linear elasticity problems [40], [36]. (This is seen by using Remark 3.3 in a way similar to Remark 4.2.) Further, quasi-sharp error estimates for nonlinear elasticity problems have been obtained earlier in [46]. Now Proposition 3.4 yields

Proposition 4.4. *Estimate* (4.60) *is sharp, that is,*

$$(4.62)\qquad \min_{\substack{Y^* \in H(\mathrm{div}, \mathbb{R}^3; \Gamma_N) \cap L^\infty(\Omega)^{3\times 3}_{\rm symm},\\ h \in H^1_D(\Omega)^3}} \mathrm{EST}(u; Y^*, h) = E(u)$$

provided that the exact solution satisfies $u^* \in W^{1,\infty}(\Omega)^3$.

In practice, for finite element solutions, all three coordinate functions of the FEM approximation $u_h \in V_h \subset H^1_D(\Omega)^3$ are continuous piecewise polynomials, hence the condition $u_h \in W^{1,\infty}(\Omega)^3$ in Theorem 4.4 is satisfied. If we choose Y^* to be a symmetric matrix function whose entries are also continuous piecewise polynomial functions, e.g., functions from another FEM subspace, and arbitrary $w \in H^1_D(\Omega)^3$, then $Y^* \in H(\mathrm{div}, \mathbb{R}^3; \Gamma_N) \cap L^\infty(\Omega)^{3\times 3}_{\mathrm{symm}}$, hence Theorem 4.4 can be applied. Next, by virtue of (3.45), Y^* should be an approximation of $T(x, \varepsilon(u^*))$. For finite element solutions, using averaging as before, we can first let

$$Y^* := T(x, G_h(\varepsilon(u_h))), \quad Z^* = T^{-1}(x, Y^*) = G_h(\varepsilon(u_h)). \tag{4.63}$$

Here G_h is an averaging operator, based on [21] where averaging is discussed in the context of elasticity problems, and T^{-1} is understood with respect to Θ as in (4.21). More accurate error bounds can be obtained by a suitable further bound minimization with respect to Y^* (or Z^*) as mentioned in Remark 4.3. Then by Remark 3.4, the optimal w for this Z^* to set in $\mathrm{EST}(u_h; Y^*, w)$ is the solution of the following linear auxiliary problem: find $w_{\mathrm{opt}} \in H^1_D(\Omega)^3$ such that

$$\int_\Omega \varepsilon(w_{\mathrm{opt}}) : \varepsilon(v) = \int_\Omega Z^* : \varepsilon(v) \quad (v \in H^1_D(\Omega)^3). \tag{4.64}$$

Hence one can define w as the numerical solution of (4.64) on a suitable mesh (either the same as used for u_h or a finer mesh, as discussed in Section 4.1). Regarding the required constants, estimates for C'_Ω and C_{Γ_N} can be done similarly to [35], [45], several explicit values and estimates for Korn's constant κ are given in [22], and finally, as pointed out at the end of Remark 3.5, the bounds m and M and the Lipschitz constant L can be calculated numerically from (4.53)–(4.54).

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