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# RATE INDEPENDENT KURZWEIL PROCESSES

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*Abstract.* The Kurzweil integral technique is applied to a class of rate independent processes with convex energy and discontinuous inputs. We prove existence, uniqueness, and continuous data dependence of solutions in $BV$ spaces. It is shown that in the context of elastoplasticity, the Kurzweil solutions coincide with natural limits of viscous regularizations when the viscosity coefficient tends to zero. The discontinuities produce an additional positive dissipation term, which is not homogeneous of degree one.

*Keywords*: Kurzweil integral, rate independence

*MSC 2010*: 49J40, 49K40, 74C15

## Introduction

As an extension of [6], we propose here the Kurzweil integral approach to rate independent processes in a reflexive Banach space $X$ that may formally be described by the inclusion

$$0 \in \partial_\xi E(t, \xi(t)) + \partial M_{K(t)}(\dot{\xi}(t)), \tag{0.1}$$

where $E$ is an energy functional and $M_{K(t)}$ is a dissipation potential represented by the Minkowski functional of a moving convex closed set $K(t)$. Recall that the Minkowski functional $M_{\tilde{K}}\colon X \to [0,\infty]$ of a convex closed set $\tilde{K} \subset X$ containing 0 is defined as

$$M_{\tilde{K}}(x) = \inf\Big\{ s > 0\colon \frac{1}{s} x \in \tilde{K} \Big\}. \tag{0.2}$$

Inclusion (0.1) can be considered as a constitutive law of nonlinear elastoplasticity with or without hardening/softening. The energetic method for solving such problems has been developed in [10] under the hypothesis that the dependence $t \mapsto E(t, \xi)$

for fixed $\xi$ is absolutely continuous and $K$ is fixed. An extension to moving state dependent sets $K$ has been done in [9] as an energetic reformulation of the quasi-variational inequality considered in [2]. The results of [6] are stated in terms of the Young integral in the case that $K$ is independent of $t$, and $E$ is quadratic in $\xi$ and regulated (cf. Definition 1.7) in $t$. Since the Young integral is a special case of the Kurzweil integral (see [7]) and the Kurzweil calculus is simpler, we decided for the latter and show that the Kurzweil integral setting (2.8)–(2.10) explained below allows to remove some restrictions on $E$ and $K$ and solve a more general problem in the space of left continuous functions of bounded variation. It is true, however, that our technique does not cover the whole range of problems treated in [10], in particular, further constraints on the state space or nonstrictly convex energies. Note that for nonstrictly convex energies, the rate independent evolution problem is generically ill-posed, see Example 4.3 below.

The solution is constructed first for piecewise constant inputs as a minimization problem for the conjugate energy functional; the general case then follows from the convergence properties of the Kurzweil integral. If we reformulate the problem in the energetic setting of [9], [10], it turns out that the dissipation is no longer homogeneous of degree one as in the continuous case, but additional dissipation terms related to the discontinuities occur. For a quadratic energy $E$, this dissipation is quadratic and can be obtained as the limit of the viscous dissipation as the viscosity parameter tends to zero. We propose an example (Example 4.2) showing that this additional dissipation cannot be neglected.

The following text is divided into four sections. In Section 1, we give a brief overview of the Kurzweil theory of integration as presented in [13]. The main results are stated in Section 2. Section 3 is devoted to the proof of existence and uniqueness in the general case. In Section 4, we prove the viscous approximation result for quadratic energies.

## 1. The Kurzweil integral

In this section we recall the definition and some basic properties of the Kurzweil integral introduced in [8] as a framework for solving ODEs with singular right-hand sides. We cite most of the results without proof, and an interested reader can find more information also in [5], [7], [14], [15].

The basic concept in the Kurzweil integration theory is that of a *$\delta$-fine partition.* Consider a nondegenerate closed interval $[a,b] \subset \mathbb{R}$, and denote by $\mathcal{D}_{a,b}$ the set of all divisions of the form

$$d = \{t_0, \ldots, t_m\}, \quad a = t_0 < t_1 < \ldots < t_m = b. \tag{1.1}$$

With a division $d = \{t_0, \dots, t_m\} \in \mathcal{D}_{a,b}$ we associate *partitions* $D$ defined as

$$D = \{(\tau_j, [t_{j-1}, t_j])\colon\ j = 1, \dots, m\}; \quad \tau_j \in [t_{j-1}, t_j]\ \ \forall\, j = 1, \dots, m. \tag{1.2}$$

We define the set

$$\Gamma(a, b) := \{\delta\colon\ [a, b] \to \mathbb{R}\colon\ \delta(t) > 0 \ \text{ for every } \ t \in [a, b]\}. \tag{1.3}$$

An element $\delta \in \Gamma(a, b)$ is called a *gauge*. For $t \in [a, b]$ and $\delta \in \Gamma(a, b)$ we denote

$$I_\delta(t) := (t - \delta(t), t + \delta(t)). \tag{1.4}$$

**Definition 1.1** ([13]). Let $\delta \in \Gamma(a, b)$ be a given gauge. A partition $D$ of the form (1.2) is said to be *$\delta$-fine* if for every $j = 1, \dots, m$ we have

$$\tau_j \in [t_{j-1}, t_j] \subset I_\delta(\tau_j),$$

and the following implications hold:

$$\tau_j = t_{j-1} \Rightarrow j = 1, \quad \tau_j = t_j \Rightarrow j = m.$$

The set of all $\delta$-fine partitions is denoted by $\mathcal{F}_\delta(a, b)$.

It is easy to see that $\mathcal{F}_\delta(a, b)$ is nonempty for every $\delta \in \Gamma(a, b)$; this follows e.g. from [5, Lemma 1.2].

Consider a reflexive Banach space $X$ endowed with a norm $|x|$ for $x \in X$. The duality between $X$ and its dual $X^*$ will be denoted by $\langle\cdot,\cdot\rangle$, and $|\cdot|_*$ will be the dual norm in $X^*$. For given functions $f\colon\ [a, b] \to X^*$, $g\colon\ [a, b] \to X$ and a partition $D$ of the form (1.2), we define the *Kurzweil integral sum* $K_D(f, g)$ by the formula

$$K_D(f, g) = \sum_{j=1}^{m} \big\langle f(\tau_j), g(t_j) - g(t_{j-1})\big\rangle. \tag{1.5}$$

**Definition 1.2.** Let $f\colon\ [a, b] \to X^*$ and $g\colon\ [a, b] \to X$ be given. We say that $J \in \mathbb{R}$ is the *Kurzweil integral* over $[a, b]$ of $f$ with respect to $g$ and denote

$$J = \int_a^b \big\langle f(t), \mathrm{d}g(t)\big\rangle, \tag{1.6}$$

if for every $\varepsilon > 0$ there exists $\delta \in \Gamma(a, b)$ such that for every $D \in \mathcal{F}_\delta(a, b)$ we have

$$|J - K_D(f, g)| \leqslant \varepsilon. \tag{1.7}$$

Using the fact that the implication

$$\delta \leqslant \min\{\delta_1, \delta_2\} \Longrightarrow \mathcal{F}_\delta(a,b) \subset \mathcal{F}_{\delta_1}(a,b) \cap \mathcal{F}_{\delta_2}(a,b) \tag{1.8}$$

holds for every $\delta, \delta_1, \delta_2 \in \Gamma(a,b)$, we easily check that the value of $J$ in Definition 1.2 is uniquely determined.

We list below in Propositions 1.3, 1.4 some standard properties common to most integral concepts.

**Proposition 1.3.** *Let $f, f_1, f_2\colon [a,b] \to X^*$, $g, g_1, g_2\colon [a,b] \to X$ be any functions. Then the following implications hold.*

(i) *If $\int_a^b \langle f_1(t), \mathrm{d}g(t)\rangle$, $\int_a^b \langle f_2(t), \mathrm{d}g(t)\rangle$ exist, then $\int_a^b \langle f_1(t)+f_2(t), \mathrm{d}g(t)\rangle$ exists and*

$$\int_a^b \langle f_1(t) + f_2(t), \mathrm{d}g(t)\rangle = \int_a^b \langle f_1(t), \mathrm{d}g(t)\rangle + \int_a^b \langle f_2(t), \mathrm{d}g(t)\rangle. \tag{1.9}$$

(ii) *If $\int_a^b \langle f(t), \mathrm{d}g_1(t)\rangle$, $\int_a^b \langle f(t), \mathrm{d}g_2(t)\rangle$ exist, then $\int_a^b \langle f(t), \mathrm{d}(g_1+g_2)(t)\rangle$ exists and*

$$\int_a^b \langle f(t), \mathrm{d}(g_1 + g_2)(t)\rangle = \int_a^b \langle f(t), \mathrm{d}g_1(t)\rangle + \int_a^b \langle f(t), \mathrm{d}g_2(t)\rangle. \tag{1.10}$$

(iii) *If $\int_a^b \langle f(t), \mathrm{d}g(t)\rangle$ exists, then $\int_a^b \langle \lambda f(t), \mathrm{d}g(t)\rangle$, $\int_a^b \langle f(t), \mathrm{d}\lambda g(t)\rangle$ exist for every constant $\lambda \in \mathbb{R}$ and*

$$\int_a^b \langle \lambda f(t), \mathrm{d}g(t)\rangle = \int_a^b \langle f(t), \mathrm{d}\lambda g(t)\rangle = \lambda \int_a^b \langle f(t), \mathrm{d}g(t)\rangle. \tag{1.11}$$

**Proposition 1.4.** *Let $f\colon [a,b] \to X^*$, $g\colon [a,b] \to X$ be given functions and let $s \in (a,b)$ be given.*

(i) *Assume that $\int_a^b \langle f(t), \mathrm{d}g(t)\rangle$ exists. Then $\int_a^s \langle f(t), \mathrm{d}g(t)\rangle$, $\int_s^b \langle f(t), \mathrm{d}g(t)\rangle$ exist.*

(ii) *Assume that $\int_a^s \langle f(t), \mathrm{d}g(t)\rangle$, $\int_s^b \langle f(t), \mathrm{d}g(t)\rangle$ exist. Then $\int_a^b \langle f(t), \mathrm{d}g(t)\rangle$ exists and*

$$\int_a^b \langle f(t), \mathrm{d}g(t)\rangle = \int_a^s \langle f(t), \mathrm{d}g(t)\rangle + \int_s^b \langle f(t), \mathrm{d}g(t)\rangle. \tag{1.12}$$

In order to preserve the consistency of (1.12) also in the limit cases $s = a$ and $s = b$, we set

$$\int_s^s \langle f(t), \mathrm{d}g(t)\rangle = 0 \quad \forall\, s \in [a,b], \ \ \forall\, f\colon [a,b] \to X^*, \ g\colon [a,b] \to X. \tag{1.13}$$

Let us recall some typical formulas. We denote by $\chi_\Omega$ the characteristic function of a set $\Omega \subset [0,T]$.

**Proposition 1.5.** *For every $g\colon [a,b] \to X$, $a \leqslant r \leqslant s \leqslant b$ and $v \in X^*$ we have*

(i) $\int_a^b \bigl\langle v\,\chi_{\{s\}}(t), \mathrm{d}g(t)\bigr\rangle = \bigl\langle v,g\bigr\rangle(s+) - \bigl\langle v,g\bigr\rangle(s-)$,

(ii) $\int_a^b \bigl\langle v\,\chi_{(r,s)}(t), \mathrm{d}g(t)\bigr\rangle = \bigl\langle v,g\bigr\rangle(s-) - \bigl\langle v,g\bigr\rangle(r+)$

*provided the limits on the right-hand sides exist, with the convention $\bigl\langle v,g\bigr\rangle(a-) = \bigl\langle v,g(a)\bigr\rangle$, $\bigl\langle v,g\bigr\rangle(b+) = \bigl\langle v,g(b)\bigr\rangle$.*

**Proposition 1.6.** *For every $f\colon [a,b] \to X^*$, $a \leqslant r \leqslant s \leqslant b$, and $v \in X$ we have*

(i) $\int_a^b \bigl\langle f(t), \mathrm{d}(v\,\chi_{\{s\}})(t)\bigr\rangle = \begin{cases} 0 & \text{for } s \in (a,b), \\ -\bigl\langle f(a), v\bigr\rangle & \text{for } s = a, \\ \bigl\langle f(b), v\bigr\rangle & \text{for } s = b. \end{cases}$

(ii) $\int_a^b \bigl\langle f(t), \mathrm{d}(v\,\chi_{(r,s)}(t)\bigr\rangle = \bigl\langle f(r) - f(s), v\bigr\rangle$.

We now introduce the concept of *regulated functions*, which goes back to [1].

**Definition 1.7.** Let $Y$ be a Banach space with norm $|\cdot|_Y$. We say that a function $f\colon [a,b] \to Y$ is *regulated* if for every $t \in [a,b]$ there exist both one-sided limits $f(t+), f(t-) \in Y$, with the convention $f(a-) = f(a)$, $f(b+) = f(b)$.

We denote by $G(a,b;Y)$ the set of all regulated functions $f\colon [a,b] \to Y$, and by $G_L(a,b;Y)$ and $G_R(a,b;Y)$ the space of left continuous and right continuous regulated functions on $[a,b]$, respectively. The space $BV(a,b;Y)$ of all functions of bounded variation with values in $Y$ is included in $G(a,b;Y)$. As an important example of regulated functions, let us mention *step functions* $w$ of the form

$$w(t) := \sum_{k=0}^{m} \hat{c}_k\,\chi_{\{t_k\}}(t) + \sum_{k=1}^{m} c_k\,\chi_{(t_{k-1},t_k)}(t), \quad t \in [a,b], \tag{1.14}$$

where $d = \{t_0, \dots, t_m\} \in \mathcal{D}_{a,b}$ is a given division, and $\hat{c}_0, \dots, \hat{c}_m, c_1, \dots, c_m$ are given elements from $Y$. We further set $BV_L(a,b;Y) = BV(a,b;Y) \cap G_L(a,b;Y)$ and $BV_R(a,b;Y) = BV(a,b;Y) \cap G_R(a,b;Y)$. On $G(a,b;Y)$ we introduce a norm $\|\cdot\|_{[a,b]}$ by

$$\|f\|_{[a,b]} := \sup\{|f(\tau)|_Y\colon \tau \in [a,b]\}. \tag{1.15}$$

**Lemma 1.8.**

(i) *Every regulated function is bounded.*

(ii) *The space $G(a,b;Y)$ is complete and non-separable with respect to the norm $\|\cdot\|_{[a,b]}$.*

(iii) *Given* $C > 0$, *the set* $V_C = \{g \in BV(a,b;Y)\colon \operatorname{Var}_{[a,b]} g \leqslant C\}$ *is closed in* $G(a,b;Y)$.

(iv) *For every* $f \in G(a,b;Y)$ *and* $\varepsilon > 0$ *there exists a step function* $w$ *of the form* (1.14) *such that* $\|f-w\|_{[a,b]} \leqslant \varepsilon$, $w(t) \in \bigcup_{\tau\in[a,b]} \{f(\tau)\}$ *for every* $t \in [a,b]$, *and* $\operatorname{Var}_{[a,b]} w \leqslant \operatorname{Var}_{[a,b]} f$.

**Theorem 1.9.** *If* $f \in G(a,b;X^*)$ *and* $g \in BV(a,b;X)$ *or* $f \in BV(a,b;X^*)$ *and* $g \in G(a,b;X)$, *then* $\int_a^b \langle f(t), \mathrm{d}g(t)\rangle$ *exists and satisfies the estimate*

$$(1.16)\quad \Big|\int_a^b \langle f(t), \mathrm{d}g(t)\rangle\Big| \leqslant \min\Big\{\|f\|_{[a,b]} \operatorname*{Var}_{[a,b]} g,\ \Big(|f(a)|_* + |f(b)|_* + \operatorname*{Var}_{[a,b]} f\Big)\|g\|_{[a,b]}\Big\}.$$

The following identity explains the motivation for a Kurzweil solution to the process (0.1) defined in (2.8)–(2.10) below.

**Proposition 1.10.** *If* $f \in G(a,b;X^*)$ *and* $g \in W^{1,1}(a,b;X)$, *then*

$$\int_a^b \langle f(t), \mathrm{d}g(t)\rangle = (\mathrm{L})\int_a^b \langle f(t), \dot g(t)\rangle\,\mathrm{d}t,$$

*where* (L) *denotes the Lebesgue integral.*

The next Proposition 1.11 plays a key role in the construction of a solution to (0.1).

**Proposition 1.11.** *Consider* $f, f_n \in G(a,b;X^*)$ *and* $g, g_n \in BV(a,b;X)$ *for* $n \in \mathbb{N}$ *such that*

$$\lim_{n\to\infty} \|f-f_n\|_{[a,b]} = 0, \qquad \lim_{n\to\infty} \|g-g_n\|_{[a,b]} = 0, \qquad \sup_{n\in\mathbb{N}} \operatorname*{Var}_{[a,b]} g_n = C < \infty.$$

*Then*

$$(1.17)\qquad \int_a^b \langle f(t), \mathrm{d}g(t)\rangle = \lim_{n\to\infty} \int_a^b \langle f_n(t), \mathrm{d}g_n(t)\rangle.$$

The integration by parts formula for the Kurzweil integral contains additional jump terms and reads as follows. The proof is the same as for the Young integral in [6, Theorem 3.14].

**Proposition 1.12.** *For every $f \in G(a,b;X^*)$ and $g \in BV(a,b;X)$ we have*

$$\int_a^b \big\langle f(t), \mathrm{d}g(t)\big\rangle + \int_a^b \big\langle g(t), \mathrm{d}f(t)\big\rangle = \big\langle f(b), g(b)\big\rangle - \big\langle f(a), g(a)\big\rangle$$
$$+ \sum_{t\in[a,b]} \Big(\big\langle f(t) - f(t-), g(t) - g(t-)\big\rangle - \big\langle f(t+) - f(t), g(t+) - g(t)\big\rangle\Big).$$

Note that only countably many points $t$ enter the sum, which is finite due to the bounded variation of $g$.

For a continuously differentiable mapping $E_0\colon X \to \mathbb{R}$, the following integration formula holds.

**Corollary 1.13.** *For every $g \in BV(a,b;X)$ we have*

$$\int_a^b \big\langle E_0'(g(t+)), \mathrm{d}g(t)\big\rangle = E_0(g(b)) - E_0(g(a)) + \sum_{t\in[a,b]} \Delta(g(t+), g(t-)), \tag{1.18}$$

*where $E_0'$ is the Fréchet derivative of $E_0$ and*

$$\Delta(\xi,\eta) := \big\langle E_0'(\xi), \xi - \eta\big\rangle - E_0(\xi) + E_0(\eta) \quad \textit{for } \xi,\eta \in X.$$

Indeed, this can be checked directly for every step function $w$ of the form (1.14) using Propositions 1.3–1.6, which yield, after setting $c_0 = \hat{c}_0$, $c_{m+1} = \hat{c}_m$, that

$$\begin{aligned}
&\int_a^b \big\langle E_0'(w(t+)), \mathrm{d}w(t)\big\rangle \\
&\qquad = \big\langle E_0'(c_{m+1}), c_{m+1}\big\rangle - \big\langle E_0'(c_1), c_0\big\rangle + \sum_{k=1}^{m}\big\langle E_0'(c_k) - E_0'(c_{k+1}), c_k\big\rangle \\
&\qquad = \sum_{k=1}^{m+1}\big\langle E_0'(c_k), c_k - c_{k-1}\big\rangle \\
&\qquad = E_0(c_{m+1}) - E_0(c_0) + \sum_{k=1}^{m+1} \Delta(c_k, c_{k-1}),
\end{aligned}$$

which is precisely (1.18). If $g$ is an arbitrary $BV$-function, then it suffices to use the approximation and convergence argument of Lemma 1.8 (iv) and Proposition 1.11.

## 2. Statement of the problem and main results

In addition to $X$, $X^*$, consider further Banach spaces $U$, $V$ endowed with norms $|\cdot|_U$, $|\cdot|_V$, respectively, and their closed subsets $U_0 \subset U$, $V_0 \subset V$ playing the role of parameter sets. By $\mathrm{Lin}(X \to X^*)$ we denote the space of continuous linear mappings from $X$ to $X^*$, endowed with the norm $\|\cdot\|$. For $\gamma > 0$, we denote by $\mathrm{Sym}_\gamma(X \to X^*)$ the set of all $F \in \mathrm{Lin}(X \to X^*)$ such that

$$\big\langle F\xi, \eta\big\rangle = \big\langle F\eta, \xi\big\rangle, \quad \big\langle F\xi, \xi\big\rangle \geqslant \gamma\,|\xi|^2 \quad \forall\,\xi, \eta \in X. \tag{2.1}$$

Indeed, if $\mathrm{Sym}_\gamma(X \to X^*)$ is nonempty, then $X$ can be considered as a Hilbert space endowed with the scalar product $\big\langle \xi, \eta\big\rangle_F = \big\langle F\xi, \eta\big\rangle$ with some fixed $F \in \mathrm{Sym}_\gamma(X \to X^*)$.

We are given a family $K(v) \subset X$ of convex closed sets depending on a parameter $v \in V_0$, and assume that $0 \in K(v)$ for all $v \in V_0$. The *polar set* $K^*(v) \subset X^*$ of $K(v)$ is defined as

$$K^*(v) = \{y \in X^* \colon \big\langle y, \xi\big\rangle \leqslant 1 \;\forall\,\xi \in K(v)\}. \tag{2.2}$$

Since $K(v)$ is convex, closed, and contains 0, we have $(K^*(v))^* = K(v)$. This and other convex analysis concepts and results used here can be found in [12] and [3, Chapter 2].

To measure the distance between sets in $X^*$, we define the *Hausdorff distance* $d_H(A, B)$ of the sets $A, B \subset X^*$ as

$$d_H(A, B) = \max\Big\{\sup_{a\in A} \mathrm{dist}(a, B), \sup_{b\in B} \mathrm{dist}(b, A)\Big\},$$

where $\mathrm{dist}(a, B) = \inf\{|a - b|_* \colon b \in B\}$ etc. For each $v \in V_0$ we define the *projection* $Q_v(x)$ of an element $x \in X^*$ onto $K^*(v)$ as the set of all $z \in K^*(v)$ such that

$$|x - z|_* = \min\{|x - z'|_* \colon z' \in K^*(v)\}. \tag{2.3}$$

For $v_1, v_2 \in V_0$ we obviously have the implication

$$x \in K^*(v_1),\ z \in Q_{v_2}x \Longrightarrow |x - z|_* \leqslant d_H(K^*(v_1), K^*(v_2)). \tag{2.4}$$

We will assume in the sequel that there exists a constant $C_H > 0$ such that

$$d_H(K^*(v_1), K^*(v_2)) \leqslant C_H\,|v_1 - v_2|_V \quad \forall\,v_1, v_2 \in V_0. \tag{2.5}$$

Assume that $E\colon U_0 \times X \to \mathbb{R}$ is a functional which with each $u \in U_0$ and $\xi \in X$ associates the *stored energy* corresponding to $u$ and $\xi$. The *conjugate energy functional* $E^*\colon U_0 \times X^* \to \mathbb{R}$ is defined by the Legendre transform

$$E^*(u,y) = \sup_{\xi\in X}\{\langle y,\xi\rangle - E(u,\xi)\} \quad \text{for } (u,y) \in U_0 \times X^*. \tag{2.6}$$

We assume the following hypothesis to hold.

**Hypothesis 2.1.** *Let $\partial_\xi E\colon U_0\times X \to X^*$, $\partial_\xi^2 E\colon U_0\times X \to \mathrm{Lin}(X \to X^*)$ denote the first and the second partial Fréchet derivatives of $E$ with respect to $\xi$.*

(i) *There exists a constant $L > 0$ such that for every $u_1, u_2 \in U_0$ and $\xi \in X$ we have*
$$|\partial_\xi E(u_1,\xi) - \partial_\xi E(u_2,\xi)|_* \leqslant L\,|u_1 - u_2|_U.$$
(ii) *There exists a constant $\gamma > 0$ such that $\partial_\xi^2 E(u,\xi) \in \mathrm{Sym}_\gamma(X \to X^*)$ for every $(u,\xi) \in U_0 \times X$.*
(iii) *For every $R > 0$ there exists $C(R) > 0$ such that for all $u_1, u_2 \in U_0$ and $\xi_1, \xi_2 \in X$, $|\xi_i| \leqslant R$ for $i = 1,2$, we have*
$$\|\partial_\xi^2 E(u_1,\xi_1) - \partial_\xi^2 E(u_2,\xi_2)\| \leqslant C(R)\,(|u_1 - u_2|_U + |\xi_1 - \xi_2|).$$

As a consequence of Hypothesis 2.1, we see that both $E(u,\cdot)$ and $E^*(u,\cdot)$ are strictly convex and twice continuously differentiable. As a classical property of the Legendre transform, we have

$$x = \partial_\xi E(u,\xi) \Longleftrightarrow \xi = \partial_x E^*(u,x). \tag{2.7}$$

It is easy to see that the symmetry of $\partial_\xi^2 E$ in (ii) follows from the continuity property (iii). Indeed, for all $s,t \in (0,1)$, $u \in U_0$, and $\xi,\eta,\theta \in X$ we have the identities

$$\begin{aligned}
&E(u,\xi+s\theta+t\eta) - E(u,\xi+s\theta) - E(u,\xi+t\eta) + E(u,\xi)\\
&\quad= \int_0^t\int_0^s \big\langle(\partial_\xi^2 E(u,\xi+\sigma\theta+\tau\eta) - \partial_\xi^2 E(u,\xi))\,\theta,\eta\big\rangle\,\mathrm{d}\sigma\,\mathrm{d}\tau + st\,\big\langle\partial_\xi^2 E(u,\xi)\,\theta,\eta\big\rangle\\
&\quad= \int_0^s\int_0^t \big\langle(\partial_\xi^2 E(u,\xi+\sigma\theta+\tau\eta) - \partial_\xi^2 E(u,\xi))\,\eta,\theta\big\rangle\,\mathrm{d}\tau\,\mathrm{d}\sigma + st\,\big\langle\partial_\xi^2 E(u,\xi)\,\eta,\theta\big\rangle,
\end{aligned}$$

hence, by Hypothesis 2.1 (iii),

$$\big|\big\langle\partial_\xi^2 E(u,\xi)\,\eta,\theta\big\rangle - \big\langle\partial_\xi^2 E(u,\xi)\,\theta,\eta\big\rangle\big| \leqslant C\,(t+s)$$

with a constant $C$ depending only on the norms of $\xi$, $\eta$, $\theta$, and it suffices to let $t$, $s$ tend to 0.

The Kurzweil integral setting of Problem (0.1) is defined as follows.

**Problem 2.2.** *For given input functions $u \in BV_L(0,T;U_0)$, $v \in BV_L(0,T;V_0)$ and an initial condition $x_0 \in K^*(v(0))$, we look for a function $\xi \in BV_L(0,T;X)$ such that*

$$x(t) := \partial_\xi E(u(t),\xi(t)) \in K^*(v(t)) \quad \forall\, t \in [0,T], \tag{2.8}$$

$$\partial_\xi E(u(0),\xi(0)) = x_0, \tag{2.9}$$

$$\int_0^T \langle x(t+) - y(t), \mathrm{d}\xi(t)\rangle \leqslant 0 \tag{2.10}$$

*for every $y \in G(0,T;X^*)$ such that $y(t) \in K^*(v(t+))$ for every $t \in [0,T]$.*

Note that every solution to Problem 2.2 satisfies

$$\int_s^t \langle x(\tau+) - y(\tau), \mathrm{d}\xi(\tau)\rangle \leqslant 0 \tag{2.11}$$

for every test function $y$ as in Theorem 2.3 and for every $0 \leqslant s < t \leqslant T$. Indeed, it suffices to set

$$\tilde{y}(\tau) = \begin{cases} y(\tau) & \text{for } \tau \in [s,t), \\ x(\tau+) & \text{for } \tau \in [0,s) \cup [t,T], \end{cases}$$

and check, by virtue of Propositions 1.4–1.5 and the left continuity of $\xi$, that

$$\begin{aligned} 0 \geqslant \int_0^T \langle x(\tau+) - \tilde{y}(\tau), \mathrm{d}\xi(\tau)\rangle &= \int_0^T \langle \chi_{[s,t)}(\tau)(x(\tau+) - y(\tau)), \mathrm{d}\xi(\tau)\rangle \\ = \int_s^t \langle x(\tau+) - y(\tau), \mathrm{d}\xi(\tau)\rangle &+ \int_0^s \langle \chi_{\{s\}}(\tau)(x(\tau+) - y(\tau)), \mathrm{d}\xi(\tau)\rangle \\ - \int_s^t \langle \chi_{\{t\}}(\tau)(x(\tau+) - y(\tau)), \mathrm{d}\xi(\tau)\rangle & \\ = \int_s^t \langle x(\tau+) - y(\tau), \mathrm{d}\xi(\tau)\rangle. & \end{aligned}$$

Proposition 1.10 enables us to understand the relation between (2.8)–(2.10) and (0.1). In fact, we can formally rewrite (2.8)–(2.10) as

$$\dot{\xi}(t) \in \partial I_{-K^*(v(t))}(-\partial_\xi E(u(t),\xi(t))), \tag{2.12}$$

where $I_{\tilde{K}}$ is the indicator function of an arbitrary set $\tilde{K}$, and this is in turn equivalent to

$$-\partial_\xi E(u(t), \xi(t)) \in \partial M_{-K(v(t))}(\dot{\xi}(t)), \tag{2.13}$$

which is precisely (0.1) with $K(t)$ replaced by $-K(v(t))$.

We prove the following existence and uniqueness result.

**Theorem 2.3.** *Let Hypothesis* 2.1 *and inequality* (2.5) *hold. Then for every* $u \in BV_L(0, T; U_0)$, $v \in BV_L(0, T; V_0)$ *and* $x_0 \in K^*(v(0))$, *Problem* 2.2 *has a unique solution* $\xi \in BV_L(0, T; X)$. *Moreover, for every* $D > 0$ *there exists* $C_D > 0$ *such that for all input functions* $u_i$, $v_i$, $i = 1, 2$, *such that*

$$\|u_i\|_{[0,T]} + \|v_i\|_{[0,T]} + \operatorname{Var}_{[0,T]} u_i + \operatorname{Var}_{[0,T]} v_i \leqslant D, \quad i = 1, 2,$$

*the solutions* $\xi_1$ *and* $\xi_2$ *corresponding to* $u_1$, $v_1$ *and* $u_2$, $v_2$ *and to initial conditions* $x_1^0$, $x_2^0$, *respectively, satisfy the inequality*

$$\|\xi_1 - \xi_2\|^2_{[0,T]} \leqslant C_D\big(|x_1^0 - x_2^0|_*^2 + \|u_1 - u_2\|_{[0,T]} + \|v_1 - v_2\|_{[0,T]}\big). \tag{2.14}$$

Let now $K_0 \subset X$ be a fixed convex closed set containing 0. We define the $K_0$-*variation* of a function $\xi\colon [0, T] \to X$ on an interval $[s, t] \subset [0, T]$ by the formula

$$\operatorname*{Var}_{[s,t]}{}_{K_0}\, \xi = \sup \sum_{i=1}^{p} M_{K_0}(\xi(\sigma_i) - \xi(\sigma_{i-1})),$$

the supremum being taken over all divisions $s = \sigma_0 < \sigma_1 < \ldots < \sigma_p = t$. For a left continuous function $\xi$, an equivalent definition reads

$$\operatorname*{Var}_{[s,t]}{}_{K_0}\, \xi = \sup\bigg\{\int_s^t \big\langle y(\tau), \mathrm{d}\xi(\tau)\big\rangle\colon\ y \in G(s, t; X^*),\ y(\tau) \in K_0^*\ \forall\, \tau \in [s, t]\bigg\}. \tag{2.15}$$

Assuming still that Hypothesis 2.1 holds, consider now the special case of Problem 2.2, where $E$ is of the form $E(u, \xi) = E_0(\xi) - \big\langle u, \xi\big\rangle$ for $u \in U := X^*$ and $\xi \in X$, and $K(v) = -K_0$. According to [10], the energetic solution to (2.13) with an absolutely continuous input $u$ is defined by the *stability condition*

$$(\mathcal{S}) \qquad E(u(t), \xi(t)) \leqslant E(u(t), \eta) + M_{K_0}(\eta - \xi(t)) \quad \text{a.e. } \forall\, \eta \in X,$$

and by the *energy inequality*

$$(\mathcal{E}) \qquad E(u(t), \xi(t)) - E(u(s), \xi(s)) + \operatorname*{Var}_{[s,t]}{}_{K_0}\, \xi \leqslant -(\mathrm{L})\!\int_s^t \big\langle \xi(\tau), \dot{u}(\tau)\big\rangle\, \mathrm{d}\tau \qquad \forall\, 0 \leqslant s < t \leqslant T,$$

where the right-hand side corresponds to the energy supply, $\operatorname{Var}_{K_0} \xi$ is the dissipation, and the symbol (L) denotes again the Lebesgue integral. For differentiable energies, condition $(\mathcal{S})$ is equivalent to the inclusion $\partial_\xi E(u(t), \xi(t)) \in -K_0^*$, which is precisely (2.8).

Let $0 \leqslant s < t \leqslant T$ be arbitrarily chosen. For $\xi \in BV_L(0, T; X)$ and $u \in BV_L(0, T; X^*)$ we have, by Proposition 1.12 and Corollary 1.13, that

$$\int_s^t \big\langle u(\tau+), \mathrm{d}\xi(\tau)\big\rangle + \int_s^t \big\langle \xi(\tau), \mathrm{d}u(\tau)\big\rangle = \big\langle u(t), \xi(t)\big\rangle - \big\langle u(s), \xi(s)\big\rangle \tag{2.16}$$

and

$$\int_s^t \big\langle \partial_\xi E_0(\xi(\tau+)), \mathrm{d}\xi(\tau)\big\rangle = E_0(\xi(t)) - E_0(\xi(s)) + \sum_{\tau \in [s,t]} \Delta(\xi(\tau+), \xi(\tau)), \tag{2.17}$$

where Hypothesis 2.1 (ii) implies the lower bound

$$\Delta(\xi, \eta) = \big\langle \partial_\xi E_0(\xi), \xi - \eta\big\rangle - E_0(\xi) + E_0(\eta) \geqslant \frac{\gamma}{2}|\xi - \eta|^2 \quad \forall\, \xi, \eta \in X.$$

Using (2.15), we can take the supremum in (2.11) over all regulated functions $y$ with values in $-K_0^*$ and obtain

$$\int_s^t \big\langle x(\tau+), \mathrm{d}\xi(\tau)\big\rangle + \operatorname*{Var}_{[s,t]}{}_{K_0} \xi \leqslant 0.$$

Since $x(\tau+)$ also belongs to $-K_0^*$, we have in fact the identity

$$\int_s^t \big\langle x(\tau+), \mathrm{d}\xi(\tau)\big\rangle + \operatorname*{Var}_{[s,t]}{}_{K_0} \xi = 0. \tag{2.18}$$

From identities (2.16)–(2.18) we derive for the process described by (2.8)–(2.10) the energy balance equation in the form

$$\begin{aligned} E(u(t), \xi(t)) - E(u(s), \xi(s)) + \sum_{\tau \in [s,t]} \Delta(\xi(\tau+), \xi(\tau)) + \operatorname*{Var}_{[s,t]}{}_{K_0} \xi \\ = -\int_s^t \big\langle \xi(\tau), \mathrm{d}u(\tau)\big\rangle. \end{aligned} \tag{2.19}$$

Conversely, the energy inequality

$$\begin{aligned} E(u(T), \xi(T)) - E(u(0), \xi(0)) + \sum_{\tau \in [0,T]} \Delta(\xi(\tau+), \xi(\tau)) + \operatorname*{Var}_{[0,T]}{}_{K_0} \xi \\ \leqslant -\int_0^T \big\langle \xi(\tau), \mathrm{d}u(\tau)\big\rangle \end{aligned} \tag{2.20}$$

implies (2.10) by virtue of (2.15)–(2.17).

If we compare (2.19) or (2.20) with condition $(\mathcal{E})$, we see that in addition to the homogeneous dissipation $\mathrm{Var}_{K_0}\,\xi$ of degree 1, there is in the discontinuous case a non-homogeneous jump dissipation $\sum \Delta(\xi(\tau+), \xi(\tau))$. We show below in Example 4.2 that it cannot be omitted.

As an even more special case, we assume now that $X$ is a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and norm $|\xi| = \sqrt{\langle \xi, \xi \rangle}$, $U = X$, and $K_0 \subset X$ is a *bounded* convex closed set containing 0. Then there exists $r > 0$ such that

$$B_r(0) \subset K_0^*, \tag{2.21}$$

where $B_r(0)$ is the ball centered at 0 with radius $r$.

Let us consider the energy functional

$$E(u, \xi) = \frac{1}{2}|\xi|^2 - \langle u, \xi \rangle. \tag{2.22}$$

For a given initial condition $x_0 \in -K_0^*$, Problem 2.2 then has the form

$$x(t) := \xi(t) - u(t) \in -K_0^* \quad \forall\, t \in [0, T], \tag{2.23}$$

$$\xi(0) = u(0) + x_0, \tag{2.24}$$

$$\int_0^T \langle u(t+) - \xi(t+) - y(t), \mathrm{d}\xi(t) \rangle \geqslant 0 \tag{2.25}$$
$$\text{for every } y \in G(0, T; X) \text{ such that } y(t) \in K_0^* \text{ for every } t \in [0, T],$$

which can formally be written similarly to (2.12)–(2.13) as

$$\partial M_{K_0}(\dot{\xi}(t)) + \xi(t) \ni u(t) \Longleftrightarrow \dot{\xi}(t) \in \partial I_{K_0^*}(u(t) - \xi(t)). \tag{2.26}$$

We compare the solution $\xi$ to (2.23)–(2.25) with the solution $\xi_\varepsilon$ to the regularized problem

$$\partial M_{K_0}(\dot{\xi}_\varepsilon(t)) + \varepsilon\, \dot{\xi}_\varepsilon(t) + \xi_\varepsilon(t) \ni u(t) \tag{2.27}$$

with $\varepsilon > 0$ and the same initial condition

$$\xi_\varepsilon(0) = u(0) + x_0. \tag{2.28}$$

In mechanical interpretation, (2.26) is the constitutive relation of a parallel elastoplastic model, where $u$ stands for the dimensionless stress, $\xi$ is the strain, $K_0^*$ is the admissible plastic stress domain, and its boundary $\partial K_0^*$ is the yield surface. Inclusion (2.27) can again be interpreted as a parallel viscoelastoplastic constitutive relation between the dimensionless stress $u$ and strain $\xi$, with a viscosity coefficient $\varepsilon$.

**Theorem 2.4.** *Let $u \in G_L(0,T;X)$ and $x_0 \in -K_0^*$ be given. Then problem (2.23)–(2.25) admits a unique solution $\xi \in BV_L(0,T;X)$, problem (2.27)–(2.28) admits a unique solution $\xi_\varepsilon \in W^{1,\infty}(0,T;X)$ for every $\varepsilon > 0$, and we have*

$$\lim_{\varepsilon\to 0+} |\xi_\varepsilon(t) - \xi(t)| = 0 \quad \forall\, t \in [0,T]. \tag{2.29}$$

*Moreover, for every $0 \leqslant s < t \leqslant T$ we have*

$$\lim_{\varepsilon\to 0+} \left(\varepsilon\,(\mathrm{L})\!\int_s^t |\dot\xi_\varepsilon(\tau)|^2\,\mathrm{d}\tau + \operatorname*{Var}_{[s,t]}{}_{K_0}\,\xi_\varepsilon\right) = \frac{1}{2}\sum_{\tau\in[s,t]} |\xi(\tau+) - \xi(\tau)|^2 + \operatorname*{Var}_{[s,t]}{}_{K_0}\,\xi. \tag{2.30}$$

In Theorem 2.4, we do not have to assume that $u$ has bounded variation. This is due to the regularizing property of the nonempty interior condition (2.21), see [6]. It would be interesting to establish a similar result for the general system (2.8)–(2.10).

We focus here on the case that $u$ is allowed to be discontinuous. It cannot be expected that the convergence $\xi_\varepsilon \to \xi$ is uniform, since all $\xi_\varepsilon$ are continuous while the discontinuities of $u$ give rise to discontinuities of $\xi$.

The right-hand side of (2.30) is the rate independent dissipation as in (2.19), while the left-hand side is the dissipation of the approximating process (2.27). We see that the second order jump dissipation can be interpreted as the remainder of the viscous one when the viscosity coefficient $\varepsilon$ tends to zero.

Theorem 2.3 will be proved in the next section, the proof of Theorem 2.4 is postponed to Section 4.

## 3. Proof of Theorem 2.3

Consider first step functions $u$ and $v$ of the form

$$u(t) = u_0\,\chi_{\{0\}}(t) + \sum_{k=1}^{m} u_k\,\chi_{(t_{k-1},t_k]}(t), \tag{3.1}$$

$$v(t) = v_0\,\chi_{\{0\}}(t) + \sum_{k=1}^{m} v_k\,\chi_{(t_{k-1},t_k]}(t), \tag{3.2}$$

where $u_0,\dots,u_m \in U_0$ and $v_0,\dots,v_m \in V_0$ are given, and $0 = t_0 < t_1 < \ldots < t_m = T$ is a division of the interval $[0,T]$. By virtue of Propositions 1.5–1.6, the function

$$\xi(t) = \xi_0\,\chi_{\{0\}}(t) + \sum_{k=1}^{m} \xi_k\,\chi_{(t_{k-1},t_k]}(t) \tag{3.3}$$

is a solution to Problem 2.2 if and only if

$$x_k = \partial_\xi E(u_k, \xi_k) \in K^*(v_k) \quad \text{for } k = 0, 1, \ldots, m, \tag{3.4}$$

$$\big\langle x_k - y, \xi_k - \xi_{k-1}\big\rangle \leqslant 0 \quad \text{for every } y \in K^*(v_k),\ k = 1, \ldots, m. \tag{3.5}$$

By (2.7) we have

$$x_k = \partial_\xi E(u_k, \xi_k) \Longleftrightarrow \xi_k = \partial_x E^*(u_k, x_k). \tag{3.6}$$

For $k = 0$ this gives the initial value $\xi_0$. For $k \geqslant 1$ we check that $x_k$ satisfies (3.4)–(3.5) if and only if it is the (unique) solution of the minimization problem

$$x_k = \operatorname{argmin}(x \mapsto E^*(u_k, x) - \big\langle x, \xi_{k-1}\big\rangle + I_{K^*(v_k)}(x)). \tag{3.7}$$

Indeed, if (3.7) holds, then $x_k \in K^*(v_k)$, and

$$E^*(u_k, x_k) - \big\langle x_k, \xi_{k-1}\big\rangle \leqslant E^*(u_k, x_k + \alpha(y - x_k)) - \big\langle x_k + \alpha(y - x_k), \xi_{k-1}\big\rangle \tag{3.8}$$

for all $y \in K^*(v_k)$ and $\alpha \in (0, 1]$. This yields, letting $\alpha$ tend to $0+$, that

$$\big\langle \partial_x E^*(u_k, x_k) - \xi_{k-1}, x_k - y\big\rangle \leqslant 0, \tag{3.9}$$

which is precisely (3.4)–(3.5). The inverse implication (3.4)–(3.5) $\Rightarrow$ (3.7) follows from the convexity of $E^*(u_k, \cdot)$. Note that (3.7) can also be equivalently stated as in [10] in the energetic form

$$\xi_k = \operatorname{argmin}(\xi \mapsto E(u_k, \xi) + M_{K(v_k)}(\xi_{k-1} - \xi)). \tag{3.10}$$

We will make repeated use of the following "discrete Gronwall lemma".

**Lemma 3.1.** *Let $g_k \in X$ and $F_k \in \mathrm{Sym}_\gamma(X \to X^*)$ be given for $k \in \mathbb{N} \cup \{0\}$. Let there exist constants $B, M \geqslant 0$ such that*

$$\sum_{j=1}^{k} \|F_j - F_{j-1}\| \leqslant B, \quad \sum_{j=1}^{k} \big\langle F_j g_j, g_j - g_{j-1}\big\rangle \leqslant M \quad \forall\, k \in \mathbb{N}. \tag{3.11}$$

*Then for every $n \in \mathbb{N}$ we have*

$$|g_n|^2 \leqslant \frac{\mathrm{e}^{B/\gamma}}{\gamma}\big(2M + \big\langle F_0 g_0, g_0\big\rangle\big).$$

P r o o f of Lemma 3.1. For $k \in \mathbb{N}$ set

$$\beta_k = \|F_k - F_{k-1}\|, \quad r_{k-1} = \frac{1}{2}\big\langle F_{k-1}\, g_{k-1}, g_{k-1}\big\rangle, \quad m_k = \big\langle F_k\, g_k, g_k - g_{k-1}\big\rangle. \tag{3.12}$$

We have by hypothesis on $F_{k-1}$, $F_k$ that

$$\big\langle (F_k - F_{k-1})\, g_{k-1}, g_{k-1}\big\rangle \leqslant \frac{\beta_k}{\gamma}\big\langle F_{k-1}\, g_{k-1}, g_{k-1}\big\rangle. \tag{3.13}$$

It is easy to check the elementary identity

$$\begin{aligned}\big\langle F_k\, g_k, g_k - g_{k-1}\big\rangle &- \frac{1}{2}\big\langle F_k\, g_k, g_k\big\rangle + \frac{1}{2}\big\langle F_{k-1}\, g_{k-1}, g_{k-1}\big\rangle\\ &= \frac{1}{2}\big\langle F_k\,(g_k - g_{k-1}), g_k - g_{k-1}\big\rangle + \frac{1}{2}\big\langle (F_{k-1} - F_k)\, g_{k-1}, g_{k-1}\big\rangle,\end{aligned}$$

which yields

$$r_k - r_{k-1} \leqslant m_k + \frac{\beta_k}{\gamma}\, r_{k-1} \quad \forall\, k \in \mathbb{N}. \tag{3.14}$$

Set

$$A_k = \prod_{j=1}^{k}\Big(1 + \frac{\beta_j}{\gamma}\Big),$$

with the convention $A_0 = 1$. Note that $A_k \geqslant 1$ for all $k$. We divide (3.14) by $A_k$ and obtain for all $k \in \mathbb{N}$ the inequality

$$\frac{r_k}{A_k} - \frac{r_{k-1}}{A_{k-1}} \leqslant \frac{m_k}{A_k}.$$

Summing up over $k = 1, \ldots, n$, we obtain $r_n \leqslant A_n\,(M + r_0)$. For every $k \in \mathbb{N}$ we have

$$\log A_k = \sum_{j=1}^{k} \log\Big(1 + \frac{\beta_j}{\gamma}\Big) \leqslant \frac{1}{\gamma}\sum_{j=1}^{k}\beta_j \leqslant \frac{B}{\gamma}.$$

This implies the bound $r_n \leqslant \mathrm{e}^{B/\gamma}(M + r_0)$, which completes the proof. □

We now use the above result to prove the following "Gronwall-Kurzweil" lemma.

**Lemma 3.2.** *Let $g \in BV_L(0,T;X)$ and $F \in BV_L(0,T;\mathrm{Sym}_\gamma(X \to X^*))$ be given such that $g(0) = 0$. Assume that*

$$\int_0^t \big\langle F(\tau+)g(\tau+), \mathrm{d}g(\tau)\big\rangle \leqslant 0 \quad \forall\, t \in [0,T]. \tag{3.15}$$

*Then $g(t) = 0$ for all $t \in [0,T]$.*

The relation between Lemma 3.2 and the classical Gronwall lemma can be seen easily if $F$ and $g$ are absolutely continuous. Then we may rewrite (3.15) using Proposition 1.10 as

$$\frac{1}{2}\big\langle F(t)g(t), g(t)\big\rangle - \frac{1}{2}(\mathrm{L})\int_0^t \big\langle \dot{F}(\tau)g(\tau), g(\tau)\big\rangle\,\mathrm{d}\tau = (\mathrm{L})\int_0^t \big\langle F(\tau)g(\tau), \dot{g}(\tau)\big\rangle\,\mathrm{d}\tau \leqslant 0$$

with $\dot{F}$ in $L^1(0,T;\mathrm{Lin}(X\to X^*))$.

P r o o f of Lemma 3.2. It suffices to prove that $g(T)=0$. Let $\varepsilon>0$ be arbitrarily given. By Lemma 1.8 (iv), we find step functions of the form

$$\overline{g}(t) = g_0\,\chi_{\{0\}}(t) + \sum_{k=1}^m g_k\,\chi_{(t_{k-1},t_k]}(t), \tag{3.16}$$

$$\overline{F}(t) = F_0\,\chi_{\{0\}}(t) + \sum_{k=1}^m F_k\,\chi_{(t_{k-1},t_k]}(t), \tag{3.17}$$

analogous to (3.1)–(3.2) and such that, taking into account Theorem 1.9,

$$\begin{aligned}
&g_0 = 0, \quad g_m = g(T),\\
&\sup_{t\in[0,T]} |g(t)-\overline{g}(t)| < \varepsilon, \quad \sup_{t\in[0,T]} \|F(t)-\overline{F}(t)\| < \varepsilon,\\
&\mathrm{Var}_{[0,T]}\,\overline{g} \leqslant \mathrm{Var}_{[0,T]}\,g, \quad \mathrm{Var}_{[0,T]}\,\overline{F} = \sum_{k=1}^m \|F_k - F_{k-1}\| \leqslant \mathrm{Var}_{[0,T]}\,F,\\
&\int_0^t \big\langle \overline{F}(\tau+)\overline{g}(\tau+), \mathrm{d}\overline{g}(\tau)\big\rangle \leqslant \varepsilon \quad \forall t\in[0,T].
\end{aligned}$$

For all $k=1,\dots,m$ we have

$$\int_0^{t_k} \big\langle \overline{F}(\tau+)\overline{g}(\tau+), \mathrm{d}\overline{g}(\tau)\big\rangle = \sum_{j=1}^k \big\langle F_j g_j, g_j - g_{j-1}\big\rangle \leqslant \varepsilon.$$

By Lemma 3.1 we have $|g_m|^2 \leqslant C\varepsilon$ with a constant $C$ independent of $m$. Since $\varepsilon$ is arbitrary, we obtain the assertion. $\square$

As a next step, we compare the solutions $\xi_k^{(i)}$ of the form (3.3) corresponding to different input sequences $u_0^{(i)}, u_1^{(i)}, u_2^{(i)}, \ldots \in U_0$ and $v_0^{(i)}, v_1^{(i)}, v_2^{(i)}, \ldots \in V_0$, and different initial conditions $x_0^{(i)}$, $i=1,2$. We do not specify the lengths and consider possibly infinite sequences. We will assume that there exist constants $C_U, C_V > 0$ such that

$$\sum_{k=1}^{\infty} |u_k^{(i)} - u_{k-1}^{(i)}|_U \leqslant C_U, \quad \sum_{k=1}^{\infty} |v_k^{(i)} - v_{k-1}^{(i)}|_V \leqslant C_V \quad \text{for } i = 1, 2. \tag{3.18}$$

In the inequalities (3.5) for $\xi_k^{(i)}$ we choose $y \in Q_{v_k^{(i)}}(x_{k-1}^{(i)})$ and obtain, using Hypotheses 2.1 (i)–(ii) and (2.4)–(2.5), that

$$\begin{aligned} \gamma\, |\xi_k^{(i)} - \xi_{k-1}^{(i)}|^2 &\leqslant \big\langle \partial_\xi E(u_k^{(i)}, \xi_k^{(i)}) - \partial_\xi E(u_k^{(i)}, \xi_{k-1}^{(i)}), \xi_k^{(i)} - \xi_{k-1}^{(i)} \big\rangle \\ &\leqslant \big\langle x_k^{(i)} - x_{k-1}^{(i)}, \xi_k^{(i)} - \xi_{k-1}^{(i)} \big\rangle + L\, |u_k^{(i)} - u_{k-1}^{(i)}|_U\, |\xi_k^{(i)} - \xi_{k-1}^{(i)}| \\ &\leqslant \big\langle y - x_{k-1}^{(i)}, \xi_k^{(i)} - \xi_{k-1}^{(i)} \big\rangle + L\, |u_k^{(i)} - u_{k-1}^{(i)}|_U\, |\xi_k^{(i)} - \xi_{k-1}^{(i)}| \\ &\leqslant \big( L\, |u_k^{(i)} - u_{k-1}^{(i)}|_U + C_H\, |v_k^{(i)} - v_{k-1}^{(i)}|_V \big)\, |\xi_k^{(i)} - \xi_{k-1}^{(i)}|, \end{aligned} \tag{3.19}$$

hence,

$$|\xi_k^{(i)} - \xi_{k-1}^{(i)}| \leqslant \frac{L}{\gamma}\, |u_k^{(i)} - u_{k-1}^{(i)}|_U + \frac{C_H}{\gamma}\, |v_k^{(i)} - v_{k-1}^{(i)}|_V. \tag{3.20}$$

In particular,

$$\begin{aligned} \sum_{k=1}^{\infty} |\xi_k^{(i)} - \xi_{k-1}^{(i)}| &\leqslant \frac{L}{\gamma}\, C_U + \frac{C_H}{\gamma}\, C_V, \\ \sup_{k \in \mathbb{N}} |\xi_k^{(i)}| &\leqslant |\xi_0^{(i)}| + \frac{L}{\gamma}\, C_U + \frac{C_H}{\gamma}\, C_V \quad \text{for } i = 1, 2. \end{aligned} \tag{3.21}$$

We now set

$$R = \max\{|\xi_0^{(1)}|, |\xi_0^{(2)}|\} + \frac{L}{\gamma}\, C_U + \frac{C_H}{\gamma}\, C_V. \tag{3.22}$$

In the following estimate we proceed similarly, choosing $y \in Q_{v_k^{(1)}}(x_k^{(2)})$ and $y \in Q_{v_k^{(2)}}(x_k^{(1)})$ in inequality (3.5) for $\xi_k^{(1)}$, $\xi_k^{(2)}$, respectively. Summing up the two resulting inequalities, we obtain from (2.4)–(2.5) that

$$\begin{aligned} &\big\langle x_k^{(1)} - x_k^{(2)}, \xi_k^{(1)} - \xi_{k-1}^{(1)} - \xi_k^{(2)} + \xi_{k-1}^{(2)} \big\rangle \\ &\qquad \leqslant C_H\, |v_k^{(1)} - v_k^{(2)}|_V \big( |\xi_k^{(1)} - \xi_{k-1}^{(1)}| + |\xi_k^{(2)} - \xi_{k-1}^{(2)}| \big). \end{aligned} \tag{3.23}$$

Note that the difference $x_k^{(1)} - x_k^{(2)}$ can be written as

$$x_k^{(1)} - x_k^{(2)} = \big( \partial_\xi E(u_k^{(1)}, \xi_k^{(1)}) - \partial_\xi E(u_k^{(1)}, \xi_k^{(2)}) \big) + \big( \partial_\xi E(u_k^{(1)}, \xi_k^{(2)}) - \partial_\xi E(u_k^{(2)}, \xi_k^{(2)}) \big),$$

where

$$\partial_\xi E(u_k^{(1)}, \xi_k^{(1)}) - \partial_\xi E(u_k^{(1)}, \xi_k^{(2)})$$
$$= (\mathrm{L})\!\int_0^1 \langle \partial_\xi^2 E(u_k^{(1)}, \xi_k^{(2)} + s(\xi_k^{(1)} - \xi_k^{(2)})), \xi_k^{(1)} - \xi_k^{(2)} \rangle \, \mathrm{d}s.$$

We define the mapping $F_k \in \mathrm{Lin}(X \to X^*)$ by the formula

$$F_k = (\mathrm{L})\!\int_0^1 \partial_\xi^2 E(u_k^{(1)}, \xi_k^{(2)} + s(\xi_k^{(1)} - \xi_k^{(2)}))\, \mathrm{d}s. \tag{3.24}$$

The above computations and Hypothesis 2.1 (i) yield

$$\begin{aligned}\big\langle F_k\, (\xi_k^{(1)} - \xi_k^{(2)}), \xi_k^{(1)} - \xi_k^{(2)} - \xi_{k-1}^{(1)} + \xi_{k-1}^{(2)} \big\rangle \\ \leqslant \big(L\, |u_k^{(1)} - u_k^{(2)}|_U + C_H\, |v_k^{(1)} - v_k^{(2)}|_V\big)\big(|\xi_k^{(1)} - \xi_{k-1}^{(1)}| + |\xi_k^{(2)} - \xi_{k-1}^{(2)}|\big).\end{aligned} \tag{3.25}$$

To simplify the notation, we introduce the sequences

$$\begin{aligned} w_k &= |u_k^{(1)} - u_k^{(2)}|_U + |v_k^{(1)} - v_k^{(2)}|_V, \\ \alpha_k &= \max_{i=1,2}(|u_k^{(i)} - u_{k-1}^{(i)}|_U + |v_k^{(i)} - v_{k-1}^{(i)}|_V), \\ g_k &= \xi_k^{(1)} - \xi_k^{(2)}. \end{aligned}$$

By virtue of (3.20), inequality (3.25) can be written in the form

$$\big\langle F_k\, g_k, g_k - g_{k-1} \big\rangle \leqslant C_1\, \alpha_k\, w_k \tag{3.26}$$

for every $k \in \mathbb{N}$ with a constant $C_1 > 0$. Hypothesis 2.1 (ii)–(iii), together with (3.18) and (3.21), implies that we may use Lemma 3.1 to conclude that there exists a constant $C_2 > 0$ such that $|g_k|^2 \leqslant C_2(\big\langle F_0 g_0, g_0 \big\rangle + \sup_{j\in\mathbb{N}} w_j)$. In other words, we have the inequality

$$\sup_{k\in\mathbb{N}} |\xi_k^{(1)} - \xi_k^{(2)}|^2 \leqslant C_3\big(|x_0^{(1)} - x_0^{(2)}|_*^2 + \sup_{k\in\mathbb{N}}\big(|u_k^{(1)} - u_k^{(2)}|_U + |v_k^{(1)} - v_k^{(2)}|_V\big)\big) \tag{3.27}$$

with a suitable constant $C_3 > 0$.

We are now ready to complete the proof of Theorem 2.3. Let $u \in BV_L(0,T;U_0)$, $v \in BV_L(0,T;V_0)$, and $x_0 \in K^*(v(0))$ be arbitrarily given. We first prove the uniqueness. Let $\xi_1, \xi_2$ be two solutions with the expected regularity, and set $x_i(t) =$

$\partial_\xi E(u(t), \xi_i(t)) \in K(v(t))$ for $i = 1, 2$ and $t \in [0, T]$. We may set $y(\tau) = x_{1-i}(\tau+)$ in equation (2.11) for $\xi_i$ on the interval $[0, t]$, and obtain

$$
\begin{aligned}
(3.28) \qquad 0 &\geqslant \int_0^t \big\langle x_1(\tau+) - x_2(\tau+), \mathrm{d}(\xi_1 - \xi_2)(\tau) \big\rangle \\
&= \int_0^t \big\langle F(\tau+) \, (\xi_1(\tau+) - \xi_2(\tau+)), \mathrm{d}(\xi_1 - \xi_2)(\tau) \big\rangle
\end{aligned}
$$

with

$$
F(\tau) = (\mathrm{L}) \int_0^1 \partial_\xi^2 E(u(\tau), \xi_2(\tau) + s(\xi_1(\tau) - \xi_2(\tau))) \, \mathrm{d}s,
$$

and it suffices to use Lemma 3.2 and Hypothesis 2.1 (iii) to obtain $\xi_1 = \xi_2$.

To prove the existence, we use Lemma 1.8 (iv) to find sequences of step functions $u^{(n)} \in BV_L(0, T; U_0)$, $v^{(n)} \in BV_L(0, T; V_0)$ such that $u^{(n)}(0) = u(0)$, $v^{(n)}(0) = v(0)$, $\operatorname{Var}_{[0,T]} u^{(n)} \leqslant \operatorname{Var}_{[0,T]} u$, $\operatorname{Var}_{[0,T]} v^{(n)} \leqslant \operatorname{Var}_{[0,T]} v$, and

$$
\lim_{n \to \infty} \sup_{t \in [0,T]} |u^{(n)}(t) - u(t)|_U = 0, \qquad \lim_{n \to \infty} \sup_{t \in [0,T]} |v^{(n)}(t) - v(t)|_V = 0.
$$

We know by (3.20) that the corresponding solutions $\xi^{(n)} \in BV_L(0, T; X)$ have uniformly bounded variation. Let $n, n' \in \mathbb{N}$ be indices chosen arbitrarily. By inserting additional division points $t_j$, if necessary, we may assume that $u^{(n)}$, $v^{(n)}$, $u^{(n')}$, $v^{(n')}$ are of the form (3.1)–(3.2) with the same division $0 = t_0 < t_1 < \ldots < t_m = T$. It follows from (3.27) that

$$
\begin{aligned}
(3.29) \qquad & \sup_{t \in [0,T]} |\xi^{(n)}(t) - \xi^{(n')}(t)|^2 \\
& \qquad \leqslant C_3 \Big( \sup_{t \in [0,T]} \big( |u^{(n)}(t) - u^{(n')}(t)|_U + |v^{(n)}(t) - v^{(n')}(t)|_V \big) \Big)
\end{aligned}
$$

with a constant $C_3$ independent of $n$. Hence, $\{\xi^{(n)}\}$ is a Cauchy sequence with respect to the sup-norm and admits a uniform limit $\xi \in BV_L(0, T; X)$. Using the continuity of $\partial_\xi E$ and Proposition 1.11, we may pass to the limit as $n \to \infty$ in (2.8)–(2.10) for $\xi^{(n)}$, and check that $\xi$ is the desired solution. The Hölder property (2.14) of the solution mapping follows immediately from (3.27).

## 4. Proof of Theorem 2.4

In the Hilbert framework, the projection $Q_{K_0^*}\colon X \to K_0^*$ analogous to (2.3) can be characterized as

$$z = Q_{K_0^*}(x) \Longleftrightarrow \begin{cases} z \in K_0^* \\ \langle x - z, z - \tilde{z}\rangle \geqslant 0 \quad \forall\, \tilde{z} \in K_0^*. \end{cases} \tag{4.1}$$

We denote $P_{K_0^*}(x) = x - Q_{K_0^*}(x)$ and recall that for every $x \in X$ and $\alpha \geqslant 0$, the projection has the property

$$Q_{K_0^*}(\alpha x + (1-\alpha)Q_{K_0^*}(x)) = Q_{K_0^*}(x), \tag{4.2}$$

or equivalently

$$P_{K_0^*}(Q_{K_0^*}(x) + \alpha P_{K_0^*}(x)) = \alpha P_{K_0^*}(x). \tag{4.3}$$

Note also the following easy relation between the Minkowski functional $M_{K_0}$ and the projection $Q_{K_0^*}$:

$$\forall x, y \in X\colon\ x \in \partial M_{K_0}(y) \Longleftrightarrow x = Q_{K_0^*}(x+y). \tag{4.4}$$

We see in particular that $\partial M_{K_0}(y) \subset K_0^*$ for every $y \in X$. Moreover, for every $y \in X$ we have

$$x \in \partial M_{K_0}(y) \implies \langle x, y\rangle = M_{K_0}(y) = \sup_{z \in K_0^*} \langle z, y\rangle. \tag{4.5}$$

Since $M_{K_0}$ is 1-homogeneous, we may rewrite (2.27) as

$$u(t) - \xi_\varepsilon(t) - \varepsilon\,\dot{\xi}_\varepsilon(t) \in \partial M_{K_0}(\varepsilon\,\dot{\xi}_\varepsilon(t)), \tag{4.6}$$

which is, by virtue of (4.4), in turn equivalent to

$$\varepsilon\,\dot{\xi}_\varepsilon(t) = P_{K_0^*}(u(t) - \xi_\varepsilon(t)). \tag{4.7}$$

The existence and uniqueness of a global absolutely continuous solution $\xi_\varepsilon$ to (4.7) follows from the Lipschitz continuity of the mapping $P_{K_0^*}$. Furthermore, by (2.21) and [6, Proposition 2.2 and Theorem 2.4], for every $u \in G_L(0,T;X)$ there exists a unique solution $\xi \in BV_L(0,T;X)$ to (2.23)–(2.25). As in the previous section, the convergence analysis starts with left continuous step functions of the form

$$u(t) = u_0\,\chi_{\{0\}}(t) + \sum_{k=1}^{m} u_k\,\chi_{(t_{k-1},t_k]}(t), \tag{4.8}$$

where $u_0, u_1, \ldots, u_m$ are given elements of $X$, and $0 = t_0 < t_1 < \ldots < t_m = T$ is a division of the interval $[0,T]$. If $u$ is as in (4.8), then, by [6, Proposition 4.3], the unique solution $\xi$ of (2.23)–(2.25) has also the form of (4.8), more specifically

$$\xi(t) = \xi_0\, \chi_{\{0\}}(t) + \sum_{k=1}^{m} \xi_k\, \chi_{(t_{k-1},t_k]}(t), \tag{4.9}$$

where

$$\xi_0 = u_0 + x_0, \quad \xi_k = \xi_{k-1} + P_{K_0^*}(u_k - \xi_{k-1}) \quad \text{for } k = 1, \ldots, m. \tag{4.10}$$

This is in fact nothing but the classical Moreau formula (see [11]) for time-discrete approximations of a sweeping process. Here, however, it provides the *exact solution* for piecewise constant inputs.

We first prove the following result.

**Lemma 4.1.** *Let $u$ be as in* (4.8), *let $\xi$ be given by* (4.9), *and let $\xi_\varepsilon \in W^{1,\infty}(0,T;X)$ be the solution to* (2.27)–(2.28) *for $\varepsilon > 0$. Then*

$$\lim_{\varepsilon \to 0+} |\xi_\varepsilon(t) - \xi(t)| = 0 \quad \forall\, t \in [0,T]. \tag{4.11}$$

P r o o f. Let us denote

$$\xi_k^\varepsilon = \xi_\varepsilon(t_k) \quad \text{for } k = 0, 1, \ldots, m. \tag{4.12}$$

For $t \in (t_{k-1}, t_k]$, Eq. (4.7) has the form

$$\varepsilon\, \dot{\xi}_\varepsilon(t) = P_{K_0^*}(u_k - \xi_\varepsilon(t)), \quad \xi_\varepsilon(t_{k-1}) = \xi_{k-1}^\varepsilon. \tag{4.13}$$

We claim that the solution of (4.13) can be represented in closed form as

$$\xi_\varepsilon(t) = \xi_{k-1}^\varepsilon + \big(1 - \mathrm{e}^{-(t-t_{k-1})/\varepsilon}\big) P_{K_0^*}(u_k - \xi_{k-1}^\varepsilon) \quad \text{for } t \in [t_{k-1}, t_k]. \tag{4.14}$$

Indeed, assuming (4.14), we have by (4.3) that

$$\begin{aligned} P_{K_0^*}(u_k - \xi_\varepsilon(t)) &= P_{K_0^*}\big(Q_{K_0^*}(u_k - \xi_{k-1}^\varepsilon) + \mathrm{e}^{-(t-t_{k-1})/\varepsilon} P_{K_0^*}(u_k - \xi_{k-1}^\varepsilon)\big) \\ &= \mathrm{e}^{-(t-t_{k-1})/\varepsilon} P_{K_0^*}(u_k - \xi_{k-1}^\varepsilon), \end{aligned}$$

hence (4.13) holds.

It suffices to prove the convergence (4.11) only for $t = T$ (the process is causal!). In other words, we have to check that

$$\lim_{\varepsilon\to 0+} |\xi_m^\varepsilon - \xi_m| = 0. \tag{4.15}$$

To this end, we set for $k = 1, \dots, m$

$$z_k = u_k - \xi_k, \quad z_k^\varepsilon = u_k - \xi_k^\varepsilon, \quad e_k^\varepsilon = \mathrm{e}^{-(t_k - t_{k-1})/\varepsilon}, \quad z_0^\varepsilon = -x_0 = u_0 - \xi_0. \tag{4.16}$$

We then have for all $k = 1, \dots, m$ that

$$\begin{aligned} z_k &= Q_{K_0^*}(z_{k-1} + u_k - u_{k-1}), \\ z_k^\varepsilon &= Q_{K_0^*}(z_{k-1}^\varepsilon + u_k - u_{k-1}) + e_k^\varepsilon P_{K_0^*}(z_{k-1}^\varepsilon + u_k - u_{k-1}). \end{aligned} \tag{4.17}$$

This yields in particular that

$$z_k^\varepsilon - z_{k-1}^\varepsilon + (1 - e_k^\varepsilon) P_{K_0^*}(z_{k-1}^\varepsilon + u_k - u_{k-1}) = u_k - u_{k-1}. \tag{4.18}$$

On the other hand, from (4.17) and (4.3) it follows that

$$P_{K_0^*}(z_k^\varepsilon) = e_k^\varepsilon P_{K_0^*}(z_{k-1}^\varepsilon + u_k - u_{k-1}), \tag{4.19}$$

hence

$$z_k^\varepsilon - z_{k-1}^\varepsilon + \frac{1 - e_k^\varepsilon}{e_k^\varepsilon} P_{K_0^*}(z_k^\varepsilon) = u_k - u_{k-1}. \tag{4.20}$$

We have $\langle P_{K_0^*}(z), z\rangle \geqslant 0$ for every $z \in X$. Testing Eq. (4.20) by $z_k^\varepsilon$, we thus obtain

$$|z_k^\varepsilon| \leqslant |z_{k-1}^\varepsilon + u_k - u_{k-1}| \leqslant |z_{k-1}^\varepsilon| + |u_k - u_{k-1}| \tag{4.21}$$

and, in particular,

$$|z_k^\varepsilon| \leqslant |x_0| + \operatorname*{Var}_{[0,T]} u \tag{4.22}$$

for every $k = 1, \dots, m$. Both $Q_{K_0^*}$ and $P_{K_0^*}$ are nonexpansive mappings, $P_{K_0^*}(0) = 0$. Using (4.17) and (4.22), we thus have

$$|z_k^\varepsilon - z_k| \leqslant |z_{k-1}^\varepsilon - z_{k-1}| + (|x_0| + 2 \operatorname*{Var}_{[0,T]} u)\, e_k^\varepsilon. \tag{4.23}$$

Summing up over $k$ we obtain the final estimate

$$|z_m^\varepsilon - z_m| \leqslant (|x_0| + 2 \operatorname*{Var}_{[0,T]} u) \sum_{k=1}^m e_k^\varepsilon, \tag{4.24}$$

and (4.15) follows. □

To prove Theorem 2.4, we fix a sequence $\{u^{(n)}\}$ of left continuous step functions of the form (4.8) and such that

$$\lim_{n\to\infty} \sup_{t\in[0,T]} |u^{(n)}(t) - u(t)| = 0, \tag{4.25}$$

and denote by $\xi^{(n)}$, $\xi_\varepsilon^{(n)}$ the respective solutions to (2.23)–(2.25) and (2.27)–(2.28), with $u$ replaced by $u^{(n)}$. By [6], there exists a constant $C > 0$ independent of $n$ such that

$$\operatorname*{Var}_{[0,T]} \xi^{(n)} \leqslant C, \qquad \lim_{n\to\infty} \sup_{t\in[0,T]} |\xi^{(n)}(t) - \xi(t)| = 0. \tag{4.26}$$

To estimate the total variation of $\xi_\varepsilon^{(n)}$, we use an argument similar to that in [6] that goes back to Section 19.2 of the pioneering Krasnosel'skii and Pokrovskii monograph [4]. As mentined on p. 261 of the Russian edition, this part of the book was written by Alexander Vladimirov.

We fix a division $0 = s_0 < s_1 < \ldots < s_l = T$ such that

$$s_{j-1} < \tau < t \leqslant s_j \Longrightarrow |u(t) - u(\tau)| < \frac{r}{2}, \tag{4.27}$$

where $r$ is as in (2.21). Let now $u^*$ be an arbitrary left continuous regulated function such that

$$\sup_{t\in[0,T]} |u^*(t) - u(t)| \leqslant \frac{r}{6}, \tag{4.28}$$

and let $\xi_\varepsilon^*$ be the solution to (2.27)–(2.28) corresponding to $u^*$. We have

$$\left\langle \dot\xi_\varepsilon^*(t), u^*(t) - \xi_\varepsilon^*(t) - z \right\rangle \geqslant \varepsilon\, |\dot\xi_\varepsilon^*(t)|^2 \quad \text{a.e.} \tag{4.29}$$

for every $z \in K_0^*$. In every interval $(s_{j-1}, s_j]$ we may choose in particular

$$z(t) = \frac{r}{2}\, p(t) + u^*(t) - u^*(s_{j-1}+), \tag{4.30}$$

where

$$p(t) := \begin{cases} \dfrac{\dot\xi_\varepsilon^*(t)}{|\dot\xi_\varepsilon^*(t)|} & \text{if } \dot\xi_\varepsilon^*(t) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

For all $t \in (s_{j-1}, s_j]$ we then have $|z(t)| \leqslant r$, hence $z(t) \in K_0^*$, and from (4.29) we obtain

$$\varepsilon\, |\dot\xi_\varepsilon^*(t)|^2 + \frac{r}{2}\, |\dot\xi_\varepsilon^*(t)| + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} |u^*(s_{j-1}+) - \xi_\varepsilon^*(t)|^2 \leqslant 0 \tag{4.31}$$

a.e. in $(s_{j-1}, s_j]$. Hence,

$$r \operatorname*{Var}_{[s_{j-1},s_j]} \xi_\varepsilon^* + |u^*(s_{j-1}+) - \xi_\varepsilon^*(s_j)|^2 \leqslant |u^*(s_{j-1}+) - \xi_\varepsilon^*(s_{j-1})|^2 \tag{4.32}$$

for all $j = 1, \ldots, l$. Set $x_j^* = u^*(s_j+) - \xi_\varepsilon^*(s_j)$. Then (4.28) and (4.32) yield

$$|x_j^*| \leqslant |u^*(s_{j-1}+) - u^*(s_j+)| + |x_{j-1}^*| \leqslant |u(s_{j-1}+) - u(s_j+)| + |x_{j-1}^*| + \frac{r}{2}, \tag{4.33}$$

hence $|x_j^*| \leqslant C_u$ for all $j = 0, 1, \ldots, l$, where $C_u > 0$ is a constant depending only on $u$ and the fixed division $s_0, s_1, \ldots, s_l$. Using (4.32) once more, we obtain

$$r \operatorname*{Var}_{[0,T]} \xi_\varepsilon^* \leqslant l\, C_u^2. \tag{4.34}$$

We now choose $n_0 \in \mathbb{N}$ sufficiently large such that (4.28) holds with $u^* = u^{(n)}$ for all $n \geqslant n_0$. For all such $n$ we have by virtue of (4.34) that

$$\operatorname*{Var}_{[0,T]} \xi_\varepsilon^{(n)} \leqslant C, \tag{4.35}$$

with a constant $C > 0$ independent of $n$ and $\varepsilon$. Furthermore, the mapping $y \mapsto \partial M_{K_0}(y) + \varepsilon y$ is monotone, hence the equivalent formulation (2.27) of (4.7) yields

$$\left\langle \dot{\xi}_\varepsilon^{(n)}(t) - \dot{\xi}_\varepsilon(t), u^{(n)}(t) - u(t) - \xi_\varepsilon^{(n)}(t) + \xi_\varepsilon(t) \right\rangle \ \geqslant\ 0 \quad \text{a.e.}, \tag{4.36}$$

that is,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\xi_\varepsilon^{(n)}(t) - \xi_\varepsilon(t)|^2 \leqslant (|\dot{\xi}_\varepsilon^{(n)}(t)| + |\dot{\xi}_\varepsilon(t)|)|u^{(n)}(t) - u(t)| \quad \text{a.e.} \tag{4.37}$$

From (4.34) we conclude that

$$\sup_{t\in[0,T]} |\xi_\varepsilon^{(n)}(t) - \xi_\varepsilon(t)|^2 \leqslant \frac{4lC_u^2}{r} \sup_{t\in[0,T]} |u^{(n)}(t) - u(t)|. \tag{4.38}$$

To obtain the convergence (2.29), we have to check that for every $\delta > 0$ and every $t \in [0,T]$ there exists $\varepsilon_0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have

$$|\xi_\varepsilon(t) - \xi(t)| < \delta. \tag{4.39}$$

This follows immediately from Lemma 4.1 and from the uniform convergences $\xi_\varepsilon^{(n)} \to \xi_\varepsilon$ and $\xi^{(n)} \to \xi$.

It remains to prove the convergence in (2.30). Following (2.19), we can rewrite (2.25) in energetic form for every $0 \leqslant s < t \leqslant T$ as

$$\begin{aligned}(4.40)\qquad & \frac{1}{2}|\xi(t)|^2 - \big\langle u(t), \xi(t)\big\rangle - \frac{1}{2}|\xi(s)|^2 + \big\langle u(s), \xi(s)\big\rangle \\ &\qquad + \frac{1}{2}\sum_{\tau\in[s,t]} |\xi(\tau+) - \xi(\tau)|^2 + \operatorname*{Var}_{[s,t]}{}_{K_0}\, \xi = -\int_s^t \big\langle \xi(\tau), \mathrm{d}u(\tau)\big\rangle.\end{aligned}$$

In the simple case (4.8), the energy balance reads

$$\begin{aligned}(4.41)\qquad & \frac{1}{2}|\xi_k|^2 - \big\langle u_k, \xi_k\big\rangle - \frac{1}{2}|\xi_{k-1}|^2 + \big\langle u_{k-1}, \xi_{k-1}\big\rangle \\ &\qquad + \frac{1}{2}|\xi_k - \xi_{k-1}|^2 + M_{K_0}(\xi_k - \xi_{k-1}) = -\big\langle \xi_{k-1}, u_k - u_{k-1}\big\rangle\end{aligned}$$

for every $k = 1, \ldots, m$.

We now derive the energy balance for Eq. (4.7). By definition of $P_{K_0^*}$, we have

$$(4.42)\qquad \big\langle \dot{\xi}_\varepsilon(t), u(t) - \xi_\varepsilon(t) - \varepsilon\, \dot{\xi}_\varepsilon(t) - z\big\rangle \geqslant 0 \quad \text{a.e.}$$

for every $z \in K_0^*$, which in view of (4.5) yields

$$(4.43)\qquad \big\langle \dot{\xi}_\varepsilon(t), u(t) - \xi_\varepsilon(t)\big\rangle = \varepsilon\, |\dot{\xi}_\varepsilon(t)|^2 + M_{K_0}(\dot{\xi}_\varepsilon(t)) \quad \text{a.e.,}$$

hence for every $0 \leqslant s < t \leqslant T$ we have

$$\begin{aligned}(4.44)\qquad & \frac{1}{2}|\xi_\varepsilon(t)|^2 - \big\langle u(t), \xi_\varepsilon(t)\big\rangle - \frac{1}{2}|\xi_\varepsilon(s)|^2 + \big\langle u(s), \xi_\varepsilon(s)\big\rangle \\ &\qquad + \varepsilon\, (\mathrm{L})\!\int_s^t |\dot{\xi}_\varepsilon(\tau)|^2\, \mathrm{d}\tau + \operatorname*{Var}_{[s,t]}{}_{K_0}\, \xi_\varepsilon = -\int_s^t \big\langle \xi_\varepsilon(\tau), \mathrm{d}u(\tau)\big\rangle.\end{aligned}$$

We see that (2.30) follows from (2.29), (4.40), and (4.44), provided we check for every $0 \leqslant s < t \leqslant T$ that

$$(4.45)\qquad \forall\, \delta > 0 \;\; \exists\, \varepsilon_0 > 0 \;\forall\, \varepsilon < \varepsilon_0\colon \; \int_s^t \big\langle (\xi_\varepsilon(\tau) - \xi(\tau)), \mathrm{d}u(\tau)\big\rangle < \delta.$$

By (4.35) and Lemma 1.8 (iii), the functions $\xi_\varepsilon$ and $\xi$, as well as their variations $\operatorname{Var}_{[s,t]} \xi_\varepsilon$ and $\operatorname{Var}_{[s,t]} \xi$, are uniformly bounded by a constant $C$ independent of $\varepsilon$. Using Lemma 1.8 (iv), we find a step function $w$ such that $\|u - w\|_{[s,t]} < \delta/(12C)$. By Theorem 1.9, we have

$$\int_s^t \big\langle (\xi_\varepsilon(\tau) - \xi(\tau)), \mathrm{d}(u - w)(\tau)\big\rangle \leqslant 6C\, \|u - w\|_{[s,t]} \leqslant \frac{\delta}{2}.$$

Since $w$ is a step function, we may refer to Proposition 1.6 and conclude from the pointwise convergence $\xi_\varepsilon(\tau) \to \xi(\tau)$ that

$$\lim_{\varepsilon \to 0+} \int_s^t \big\langle (\xi_\varepsilon(\tau) - \xi(\tau)), \mathrm{d}w(\tau) \big\rangle = 0,$$

whence (4.45) follows. Theorem 2.4 is proved. □

The following example shows that uniqueness of the solution $\xi$ is lost if the jump dissipation term is omitted in (2.20).

E x a m p l e 4.2. Consider the simple case $X = \mathbb{R}$, $K_0 = K_0^* = [-1, 1]$, and

$$u(t) = \begin{cases} u_0 & \text{for } t \in [0, t_0], \\ 0 & \text{for } t \in (t_0, T] \end{cases}$$

with a given $t_0 \in (0, T)$ and $u_0 \geqslant 3$. We look for a left continuous solution $\xi$ to the problem

$$\begin{cases} u(t) - \xi(t) \in K_0^* \quad \forall\, t \in [0, T], \\[2mm] \dfrac{1}{2}|\xi(t)|^2 - \big\langle u(t), \xi(t) \big\rangle - \dfrac{1}{2}|\xi(s)|^2 + \big\langle u(s), \xi(s) \big\rangle + \operatorname*{Var}_{\substack{K_0\\ [s,t]}} \xi \\ \qquad\qquad \leqslant - \displaystyle\int_s^t \big\langle \xi(\tau),\, \mathrm{d}u(\tau) \big\rangle \quad \forall\, 0 \leqslant s < t \leqslant T \end{cases} \tag{4.46}$$

with the initial condition $\xi(0) = u_0$. Every solution $\xi$ is necessarily constant in every interval where $u$ is constant. Indeed, this follows from the inequality

$$\begin{aligned} \frac{1}{2}|\xi(t)|^2 - \big\langle u, \xi(t) \big\rangle - \frac{1}{2}|\xi(s)|^2 + \big\langle u, \xi(s) \big\rangle &= -\frac{1}{2}\big\langle \xi(t) - \xi(s), 2u - \xi(t) - \xi(s) \big\rangle \\ &\geqslant -M_{K_0}(\xi(t) - \xi(s)). \end{aligned}$$

Hence, $\xi$ must have the form

$$\xi(t) = \begin{cases} u_0 & \text{for } t \in [0, t_0] \\ \xi_1 & \text{for } t \in (t_0, T] \end{cases}$$

with $\xi_1 \in [-1, 1]$. By a counterpart of (4.41) without the quadratic dissipation term, we see that $\xi$ is a solution of (4.46) if and only if

$$\frac{1}{2}\xi_1^2 - \frac{1}{2}u_0^2 + u_0^2 + |\xi_1 - u_0| \leqslant u_0^2. \tag{4.47}$$

This inequality is satisfied for all $\xi_1 \in [-1,1]$. Hence, we have a continuum of distinct solutions.

The situation is even worse if we replace (4.46) by

$$
\begin{cases} u(t) - \xi(t) \in K_0^* \quad \forall\, t \in [0,T], \\ \dfrac{1}{2}|\xi(t)|^2 - \langle u(t), \xi(t)\rangle - \dfrac{1}{2}|\xi(s)|^2 + \langle u(s), \xi(s)\rangle + \operatorname*{Var}_{[s,t]}{}_{K_0} \xi \\ \qquad \leqslant - \displaystyle\int_s^t \langle \xi(\tau+), \mathrm{d}u(\tau)\rangle \quad \forall\, 0 \leqslant s < t \leqslant T. \end{cases} \tag{4.48}
$$

Inequality (4.47) is now replaced by

$$
\frac{1}{2}\xi_1^2 - \frac{1}{2}u_0^2 + u_0^2 + |\xi_1 - u_0| \leqslant u_0\, \xi_1, \tag{4.49}
$$

which is never satisfied for $\xi_1 \in [-1,1]$. Hence, there exists no solution to Problem (4.48) for $\xi(0) = u_0$.

The following easy example shows that well-posedness also fails if the energy is nonstrictly convex.

Example 4.3. Consider again $X = \mathbb{R}$, $K_0 = K_0^* = [-1,1]$, and

$$
E(u,\xi) = \frac{1}{2}\big((\xi+1)^-\big)^2 + \frac{1}{2}\big((\xi-1)^+\big)^2 - u\xi,
$$

where $(\cdot)^+$ and $(\cdot)^-$ denote the positive and negative parts, respectively. Let $u(t) \equiv 1$ and $\xi(0) = -1$. We easily check that *every* nondecreasing function $\xi\colon [0,T] \to [-1,1]$ is a solution to (2.20).

A more complicated problem arises if $E$ is strictly convex, but Hypothesis 2.1 (ii) does not hold for any $\gamma > 0$, as in the case $E(u,\xi) = E_0(\xi) - \langle u, \xi\rangle$ with $E_0(\xi) = \frac{1}{4}|\xi|^4$. Following [10, Theorem 6.5], an easy uniqueness proof for continuous solutions to Problem 2.2 can be given provided the so-called *stable set*

$$
S(t) = \{\xi \in X\colon\ \partial_\xi E_0(\xi) \in u(t) + K(v(t))\}
$$

is convex for every $t \in [0,T]$. This condition, however, is very restrictive.

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