Anton Dekrét; Ján Bakša

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ON ASYMPTOTIC MOTIONS OF ROBOT-MANIPULATOR IN HOMOGENEOUS SPACE

Anton Dekrét, Ján Bakša, Zvolen

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Abstract. In this paper the notion of robot-manipulators in the Euclidean space is generalized to the case in a general homogeneous space with the Lie group G of motions. Some kinematic subspaces of the Lie algebra $\mathcal{G}$ (the subspaces of velocity operators, of Coriolis acceleration operators, asymptotic subspaces) are introduced and by them asymptotic and geodesic motions are described.

Keywords: local differential geometry, robotics, Lie algebra, asymptotic motion

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1. Introduction

Roughly speaking the notion of an n-parametric robot-manipulator (briefly an n-robot) in the Euclidean space E_3 is a system of n links bound in n joints such that the motion of the ith link in the ith joint with respect to the preceding link depends on the parameter u_i. The first joint is fixed to the base. The nth link is called the effector of the robot. A motion of the effector with respect to the base is determined by a curve $H(t)$ on the Lie group $E(3)$ of all Euclidean motions in E_3. Let $\mathcal{E}(3)$ be the Lie algebra of the group $E(3)$. To the ith joint a unique element $X_i \in \mathcal{E}(3)$ is assigned such that the motion of the ith link induced by the ith joint is of the form $\exp u_i(t)X_i$. Then the motion of the effector mediated by the work of all joints is of the form $H(t) = \exp u^1(t)X_1 \ldots \exp u^n(t)X_n$. For details of this concept we refer our readers to [6]. In this paper we try to generalize this approach in the following way. Instead of the Euclidean space E_3 we will consider a homogeneous space M with the Lie group G which acts transitively on M from the left side, i.e. for any $L_1, L_2 \in M$ there is an element $g \in G$ such that $g(L_1) := g \circ L_1 = L_2$.

In Chapter 2 we recall the basic notions of differential geometry such as tangent vectors (velocities), 2-tangent vectors (accelerations) on differentiable manifolds, Lie groups and their Lie algebras. In Definition 1 we explain our approach to asymptotic curves on manifolds with connection which is the basic notion of our investigation.

The third chapter contains our conception of robotics in a homogeneous space with its Lie group G. The notion of n-parametric robots is introduced in Definition 7. We deduce the basic form (6) of the covariant acceleration of the robot effector by means of which we introduce some subspaces in the Lie algebra $\mathcal{G}$ connected with kinematics of robots. This subspaces are used in our investigations of asymptotic motions of robots. In Remarks 6, 8 we mention the relation of our investigations to the asymptotic motions of robots in the Euclidean space E_3.

2. Notions on differentiable manifolds, Lie groups and their Lie algebras

In this chapter we roughly recall some notions of differential geometry we will use, see for example [5], [6] for details. Let M, $\dim M = n$ be a smooth manifold with local coordinates $(x^1, \dots, x^n)$. A curve (a motion) on M is an injective map $\gamma\colon (a,b) \to M$, $\gamma(t) = (x^1 = \gamma^1(t), \dots, x^n = \gamma^n(t))$ where $(a,b) \subset \mathbb{R}$ is an open interval, $0 \in (a,b)$, $\gamma^1(t), \dots, \gamma^n(t)$ are smooth functions. Let $\dot\gamma(0) \equiv j_0^1\gamma := \big(\gamma^i(0), \dot\gamma^i(0) = (\mathrm{d}\gamma^i/\mathrm{d}t)(0)\big)$ or $\ddot\gamma(0) \equiv j_0^2\gamma := \big(\gamma^i(0), \dot\gamma^i(0), \ddot\gamma^i(0) = (\mathrm{d}^2\gamma^i/\mathrm{d}t)(0)\big)$ denote respectively a tangent vector (a velocity) or a 2-tangent vector (an acceleration) determined by the curve γ at $x_0 = \gamma(t_0 = 0) \in M$. The set T_xM of all tangent vectors at $x \in M$ has a vector space structure of differentiation operators on the space $\mathcal{F}M$ of all smooth functions on M at x, i.e., if $X = j_0^1\gamma$ then $Xf = \sum\limits_{i=1}^{n} (\partial f/\partial x^i)\dot\gamma^i(0)$. The spaces $TM := \bigsqcup\limits_{x\in M} T_xM$, $T^2M := \bigsqcup\limits_{x\in M} T_x^2M$ have the fibre manifold structures over M (T_x^2M denotes the fibre of all 2-tangent vectors at $x \in M$). A chart (x^i) on M induces the chart $(x^i, x_1^i \equiv \dot x^i)$ on TM, the chart $(x^i, x_1^i, x_{01}^i := \dot x^i, x_{11}^i := \dot x_1^i)$ on $T(TM) = TTM$ and the chart $(x^i, x_1^i \equiv \dot x^i, x_2^i \equiv \ddot x^i)$ on $T^2M \subset TTM$. A vertical vector on TM is a vector $j_0^1\gamma(t)$, where $\gamma(t) = (x_0^i, x_1^i(t))$ is a curve in $T_{x_0}M$ and thus $j_0^1\gamma = (x_0^i, x_1^i(0), 0, \dot x_1^i(0))$ can be identified with $(x_0^i, \dot x_1^i(0)) \in T_{x_0}M$ as $T_{x_0}M$ is a vector space.

Let $F\colon M \to N$ be a smooth map of manifolds. Then $TF\colon TM \to TN$ or $T^2F\colon T^2M \to T^2N$ denotes the 1-tangent or 2-tangent prolongation of F, i.e., if $X = j_0^1\gamma(t) \in T_xM$ or $Z = j_0^2\gamma(t) \in T_x^2M$ then $TF(X) = j_0^1F(\gamma(t)) \in T_{F(x)}N$ or $T^2F(Z) = j_0^2F(\gamma(t)) \in T^2_{F(x)}N$, respectively.

A smooth vector field on M is a smooth map $s\colon M \to TM$, $s(x) \in T_xM$. It determines a differentiation operator on the set $\mathcal{F}M$: if $f \in \mathcal{F}M$ is a real function on M, $f = f(x^i)$, $s(x) = (x^i, x_1^i(x))$ then $Xf = \sum\limits_{i=1}^{n} (\partial f(x)/\partial x^i) x_1^i(x)$. The space of all vector fields has the Lie algebra structure when the Lie bracket is defined by $[X,Y]f = X(Yf) - Y(Xf)$.

The asymptotic motions we will deal with are connected with special curves on manifolds with a connection. A linear connection (shortly connection) can be introduced by the so-called parallel transport along curves on a manifold M. We say that a connection Γ is given on M if for any curve γ parallel transport along γ is defined that is a system of such linear maps $\mathcal{P}_\gamma$ independent of parametrization that $\mathcal{P}_\gamma(t_1,t_2)\colon T_{\gamma(t_1)}M \to T_{\gamma(t_2)}M$, $\mathcal{P}_\gamma(t_1,t_1) = \mathrm{Id}|_{T_{\gamma(t_1)}M}$, $\mathcal{P}_\gamma(t_1,t_2) = \mathcal{P}_\gamma(t,t_2) \circ \mathcal{P}_\gamma(t_1,t)$, $t \in \langle t_1, t_2 \rangle$. By a parallel transport we can define the following covariant differentiation of a vector field Y along a curve $\gamma(t)$: $\nabla_{\dot\gamma(t_0)} Y(\gamma(t)) = j^1_{t_0}(\mathcal{P}_\gamma(t,t_0)(Y(\gamma(t))))$.

Definition 1. Let $N \subset M$ be a submanifold on M with connection Γ. We say that a curve γ on N is Γ-asymptotic (Γ-asymptotic at t_0) if $\nabla_{\dot\gamma(t)}\dot\gamma \in T_{\gamma(t)}N$ for any t ($\nabla_{\dot\gamma(t_0)}\dot\gamma \in T_{\gamma(t_0)}N$). A curve γ on M is Γ-geodesic if $\nabla_{\dot\gamma}\dot\gamma = 0$.

A Lie group G has two structures: it is both a smooth manifold and a group where the binary operation $G \times G \to G$, $(g_1, g_2) \mapsto g_1 \circ g_2$ is smooth. Every $g \in G$ introduces the left $g^l\colon x \mapsto g \circ x$ and right $g^r\colon x \mapsto x \circ g$ group translations on G. Their tangential prolongations will be written briefly as $\xi \circ g := Tg^r(\xi)$, $g \circ \xi := Tg^l(\xi)$.

Let $e \in G$ denote a unit in G. Let $X \in T_eG$. Then the rules $X^R\colon g \mapsto X \circ g$ and $X^L\colon g \mapsto g \circ X$ introduce on G the right and left invariant vector fields. Then the tangent space $T_eG \colon\equiv \mathcal{G}$ is a Lie algebra of G where $[X_1, X_2] = [X_1^L, X_2^L](e) = [X_1^R, X_2^R](e)$. By the left (right) invariant vector fields the exponential map $\exp\colon \mathcal{G} \to G$ is defined as follows. If $\gamma(t)$ is the integral curve of a field X^L (X^R) through e, i.e. $\gamma(0) = e$, $j^1_{t_0}\gamma = X^L(\gamma(t_0))$, then $\exp(X) = \gamma(t = 1)$.

We recall two maps connected with the Lie algebra $\mathcal{G}$:

a) Let $GL(\mathcal{G})$ denote the group of all regular linear maps on $\mathcal{G}$. Then the map $Ad\colon G \to Gl(\mathcal{G})$, $g \mapsto Ad_g$, is defined by $Ad_g(X) = g \circ X \circ g^{-1}$, $X \in \mathcal{G}$.

b) Using the tangential prolongation $TAd\colon TG \to TGl(\mathcal{G})$ we obtain the adjoint map $T_eAd \equiv ad\colon T_eG \equiv \mathcal{G} \to L(\mathcal{G})$ where $L(\mathcal{G})$ is the space of linear maps on $\mathcal{G}$. We have $ad_X(Y) = [X, Y]$.

Let Γ be a connection on G. Let $X_1, \ldots, X_r$ be a base in $\mathcal{G}$, $\dim G = r$, let $\tilde X_i$ be the right invariant vector fields on G given by X_i, $i = 1, \ldots, r$. Let $\nabla_{\tilde X_i}\tilde X_j = \Gamma^k_{ij}\tilde X_k$ be the equations of Γ in the base $\tilde X_1, \ldots, \tilde X_r$ with the Christoffel functions Γ^k_{ij}. Let $\gamma(t)$ be a curve on G and let $B = \sum\limits_{i=1}^{r} B^i \tilde X_i$ be a vector field on G, $b(t) = \sum\limits_{i=1}^{r} b^i(t)\tilde X_i(\gamma(t))$, where $b^i(t) = B^i(\gamma(t))$ is the restriction of B to γ. Denote by $\tau(t) = \dot\gamma(\gamma(t)) = \sum\limits_{i=1}^{r} a^i(t)\tilde X_i(\gamma(t))$ the vector field of the tangent vectors of γ. We get $\nabla_{\dot\gamma} B = \nabla_\tau B = \sum\limits_{j=1}^{r} \tau(B^j)\tilde X_j(\gamma) + \sum\limits_{i,j=1}^{r} a^i b^j (\nabla_{\tilde X_i} \tilde X_j)|_\gamma$ and then

$$\nabla_{\dot\gamma} B = \sum_{j=1}^{r} \dot b^j \tilde X_j(\gamma) + \sum_{i,j=1}^{r} a^i b^j \Gamma^k_{ij} \tilde X_k|_\gamma, \quad \dot b^j = \tau[B^j(\gamma(t))], \tag{1}$$

$$\nabla_{\dot\gamma}\dot\gamma = \sum_{j=1}^{r} \dot a^j \tilde X_j(\gamma) + \sum_{i<j}^{r} a^i a^j (\Gamma^k_{ij} + \Gamma^k_{ji}) \tilde X_k|_\gamma + \sum_{i=1}^{r} (a^i)^2 \Gamma^k_{ii} \tilde X_k|_\gamma, \tag{2}$$

where we use the Einstein sum convention with respect to index k: $A^k D_k := \sum\limits_{k=1}^{r} A^k D_k$.

The tangent prolongations of the right group translation on G determine the parallel transport $\mathcal{P}^R$ along curves on G: if $\gamma(t)$ is a curve on G, $X \in T_{\gamma(t_1)}G$ then $\mathcal{P}^R_\gamma(t_1,t_2)(X) = X \circ \gamma^{-1}(t_1) \circ \gamma(t_2) \in T_{\gamma(t_2)}G$. It means that the $\mathcal{P}^R$-transport from $T_{g_1}G$ into $T_{g_2}G$ ($g_1 = \gamma(t_1)$, $g_2 = \gamma(t_2)$) does not depend on curves from g_1 into g_2. Denote by ${}^R\Gamma$ the connection given by $\mathcal{P}^R$. It is invariant. As vector fields $\tilde X_i$ are right invariant, hence $\mathcal{P}^R_\gamma(t,t_2)\tilde X_i$ is constant and thus $\nabla_{\tilde X_i}\tilde X_j = 0$, ${}^R\Gamma^k_{ij} = 0$ in the base $\tilde X_1, \dots, \tilde X_r$. Therefore the relation (2) in the case of the connection ${}^R\Gamma$ is of the form

$${}^R\nabla_{\dot\gamma}\dot\gamma = \sum_{j=1}^{r} \dot a^j \tilde X_j(\gamma). \tag{3}$$

R e m a r k 1. It is well known, see [2], that there is a one-to-one correspondence between the right invariant connections on G and the space of $\mathcal{G}$-value bilinear forms ξ on $\mathcal{G}$, $\nabla_{\tilde X_1}\tilde X_2 = \widetilde{\xi(X_1, X_2)}$. For example, the connection ${}^R\Gamma$ is given by $\xi(X_1,X_2) = 0$. Using (2) it is easy to show that the relation (3) is true for any connection Γ on G which is given by the form $\xi(X_1,X_2) = k[X_1,X_2]$, $k \in \mathbb{R}$. Therefore all these connections have common asymptotic curves.

Definition 2. Let $N \subset G$ be a submanifold of G. A curve $\gamma(t)$ on N is called asymptotic (asymptotic at $\gamma(t_0)$) if it is ${}^R\Gamma$-asymptotic (${}^R\Gamma$-asymptotic at $\gamma(t_0)$). It is called geodesic if it is ${}^R\Gamma$-geodesic.

For any curve $\gamma(t)$ on G we have $\dot\gamma(t) = \sum\limits_{i=1}^{r} a^i(t)\tilde X_i(\gamma(t))$. Denote $Y_\gamma(t) := \dot\gamma(t) \circ \gamma^{-1}(t) = \sum\limits_{i=1}^{r} a^i(t) X_i \in \mathcal{G}$. Then $\dot Y_\gamma(t) = \sum\limits_{i=1}^{r} \dot a^i X_i \in \mathcal{G}$.

Using the equation (3) we obtain

Lemma 1. *Let $\gamma(t)$ be a curve on G. Then $({}^R\nabla_{\dot\gamma}\dot\gamma) \circ \gamma^{-1} = \dot Y_\gamma$.*

Corollary. *A curve γ on G is geodesic iff $\dot{Y}_\gamma = 0$. If γ is an integral curve of a right invariant vector field then γ is geodesic.*

Definition 3. An element $\dot{Y}_\gamma$ will be said to be an operator of the covariant acceleration given by a curve γ on G.

Recall that we say that a Lie group G acts smoothly from the left side on a manifold M if there is a smooth map $F\colon G \times M \to M$ such that for every $g_0 \in G$ the map $\tilde{g}_0\colon M \to M$, $\tilde{g}_0(x) = F(g_0, x)$ is a smooth diffeomorphism and the map $g \mapsto \tilde{g}$ is an injective homomorphism $G \to \operatorname{Diff} M$ where $\operatorname{Diff} M$ denotes the group of all diffeomorphisms on M. So we have $F(e, x) = x$, $F(g_1 \circ g_2, x) = F(g_1, \tilde{g}_2(x))$. Let $TF\colon T(G\times M) = TG\times TM \to TM$ be the tangent prolongation of F. Denote by $\mathcal{F}$: $TF|_{TG\times M} \to TM$ the restriction of TF to $TG \oplus O$ where O denotes the zero section $O\colon M \to TM$, $x \mapsto \overline{0} \in T_xM$, which can be identified with M, $\mathcal{F}(j^1_{t_0} g(t), x) = j^1_{t_0}(F(g(t), x)) = j^1_{t_0}(\tilde{g}(t)(x))$. Let $\gamma(t)$ be a curve on the group G. Without loss of generality we can suppose $\gamma(0) = e$.

Definition 4. The motion $t \mapsto L(t) = \tilde{g}(t)(L^0) = F(g(t), L^0)$ will be called the G-motion of a point $L^0 \in M$.

Let us express velocities and accelerations of G-motions. We get

$$\dot{L}(t_0) := j^1_{t_0}(L(t)) = j^1_{t_0} F(g(t), L^0) = j^1_{t_0}\tilde{g}(t)(L^0) = \mathcal{F}(j^1_{t_0} g(t), L^0).$$

As for $h \in G$ we have $\mathcal{F}(j^1_{t_0} g(t), \tilde{h}(L^0)) = \mathcal{F}(j^1_{t_0}(g(t)) \circ h, L^0)$ and $L^0 = \tilde{g}^{-1}(t_0)(L(t_0))$ we get

$$\begin{aligned}\dot{L}(t_0) &= \mathcal{F}(j^1_{t_0} g(t), L^0) = \mathcal{F}(j^1_{t_0} g(t), \tilde{g}^{-1}(t_0)(L(t_0)))\\ &= \mathcal{F}(j^1_{t_0} g(t) \circ g^{-1}(t_0), L(t_0)) = \mathcal{F}(\dot{g}(t_0) \circ g^{-1}(t_0), L(t_0)) = \mathcal{F}(Y(t_0), L(t_0)),\end{aligned}$$

where $Y(t_0) := \dot{g}(t_0) \circ g^{-1}(t_0)$ will be called the velocity operator of G-motions. Analogously

$$\begin{aligned}\ddot{L}(t_0) &= j^2_{t_0}(L(t)) = j^1_{t_0}(\dot{L}(t))\\ &= j^1_{t_0}\mathcal{F}(Y(t), L(t)) = T\mathcal{F}(\dot{Y}(t_0) + \dot{L}(t_0)) := \ddot{L}_{C_0}(t_0) + T\mathcal{F}(\dot{L}(t_0)).\end{aligned}$$

As $\dot{Y}(t_0)$ is a vertical vector in VTG hence $\ddot{L}_{C_0}$ is a vertical vector in VTM which we can identify with a tangent vector in $T_{L(t_0)}M$. As an element $\dot{Y}(t_0)$ is connected with the covariant derivation ${}^R\nabla$ hence $\ddot{L}_{C_0}(t_0)$ will be called a covariant acceleration of G-motions.

Finally, recall the notion of the homogeneous space. Let $G_L = \{g \in G,\ g(L) = L\}$ or $G(L)$ denote the isotropy subgroup or the orbit of a point $L \in M$, respectively.

Then M is called a homogeneous G-space if $G(L) = M$ for any L and G_{L_1}, G_{L_2} are isomorphic for any $L_1, L_2 \in M$ and thus $M = G/G_L$. As examples we can introduce:

1. The group G which acts on itself by the left group translation: $\tilde{g}_1(g) = g_1 \circ g$. Then $G_g = e$, $G(G) = G$.
2. Euclidean space E_3 with the group $E(3)$ of all Euclidean motions.

3. Robots in G-homogeneous spaces

Let M be a G-homogeneous space, $\dim M = n$, $\dim G = r$. A rigid body in M is a subset of M which is homeomorphic with a closed interval in $\mathbb{R}^n$. Two solid bodies s_1, s_2 are called G-equivalent if there is $g \in G$ such that $\tilde{g}(s_1) = s_2$. We suppose that the group G acts on M so that if $s_1 = \tilde{g}_1(s_2)$, $s_1 = \tilde{g}_2(s_2)$ for solid bodies s_1, s_2 then $g_1 = g_2$.

From the roughly technical point of view an n-parametric robot-manipulator (shortly n-robot) in M is a sequence of n solid bodies called links. Each neighboring links, both the $(i-1)$st and the ith, are connected in the common joint such that the ith link and all succeeding links carry out the motion $\exp u^i X_i$, $X_i \in \mathcal{G}$, forced by the work of the ith joint and controlled by the u^i-parameter. The last link is called an effector $\mathcal{E}$ of the robot. From the mathematical point of view this can be formulated as follows.

Definition 5. Let $X_1, \dots, X_n$ be elements of the Lie algebra $\mathcal{G}$. An n-parametric robot-manipulator (shortly n-robot) in M is a smooth map $\mathcal{R}\colon U \to G$, $(u^1, \dots, u^n) \mapsto \exp u^1 X_1 \circ \dots \circ \exp u^n X_n$, where $U \subset \mathbb{R}^n$ is an open neighborhood of $0 \in \mathbb{R}^n$ of admissible parameters.

We shorten $(u^1, \dots, u^n) \equiv (u) := \mathcal{R}(u)$ and speak about the position (u) of the effector $\mathcal{E}$. A controlling curve is a curve $\mathcal{H}(t) = (u^1(t), \dots, u^n(t))$ in U. It determines a curve $\gamma(t) = \exp u^1(t) X_1 \circ \dots \circ \exp u^n(t) X_n$ in G and then the motion $\tilde{\gamma}(t)(\mathcal{E})$ of the effector, see Definition 4. In the sense of our considerations in Chapter 2 the elements $\dot{\gamma}(t_0) = j^1_{t_0}\gamma(t)$, $\ddot{\gamma}(t_0) = j^2_{t_0}\gamma(t)$, ${}^R\nabla_{\dot{\gamma}(t_0)}\dot{\gamma}(t)$, $\dot{Y}(t_0) = \ddot{\gamma}(t_0) \circ \gamma^{-1}(t_0)$ will be called the velocity, the acceleration, the covariant acceleration operators of the effector $\mathcal{E}$, respectively. The map $J_u\colon T\mathbb{R}^n \to \mathcal{G}$, $\dot{\mathcal{H}} \mapsto T\mathcal{R}(\dot{\mathcal{H}}) \circ (\mathcal{R}(\mathcal{H}))^{-1}$ will be called the Jacobian of the robot at (u).

A u^i-curve across a position $(u_0) = (u_0^1, \dots, u_0^n)$ on G is the curve $\gamma_i(t, u_0) = \mathcal{R}(\mathcal{H}_i(t, u_0))$, $\mathcal{H}_i(t, u_0) = (u_0^1, \dots, u_0^{i-1}, u_0^i + t, u_0^{i+1}, \dots, u_0^n)$. The motion of the effector $\mathcal{E}$ determined by $\mathcal{H}_i(t, u_0)$ will be called the i-basic motion of $\mathcal{E}$ forced by the ith joint. Recall that $\delta_i(t) = \exp(u_0^i + t) X_i$ is the integral curve of the right-invariant vector field X_i^R through $\exp u_0^i X_i$ and thus $j^1_0\delta_i(t) = \dot{\delta}_i(t) = X_i \circ \exp u_0^i X_i$, which is the image of X_i at the right group translation $\exp u_0^i X_i$. Then $j^i_0\gamma_i(t, u_0) = (\exp u_0^1 X_1 \circ \dots \circ \exp u_0^{i-1} X_{i-1}) X_i (\exp u_0^i X_i \circ \dots \circ \exp u_0^n X_n)$. Denote $\partial_i(u_0) :=$

$j_0^1\gamma_i(t,u_0) = T\mathcal{R}j_0^1\mathcal{H}_i(t,u_0) = T\mathcal{R}(u_0^1,\dots,u_0^n;0,\dots,0,1,0,\dots,0)$. Then for a general curve $\gamma(t) = \mathcal{R}(u^1(t),\dots,u^n(t)) = \exp u^1(t)X_1 \circ \dots \circ \exp u^n X_n$ we have

$$\begin{aligned}\dot\gamma(t_0) = j_0^1\gamma(t) &= T\mathcal{R}(u^1(t_0),\dots,u^n(t_0),\dot u^1(t_0),\dots,\dot u^n(t_0))\\ &= \dot u^1(t_0)\partial_1(u_0) + \dots + \dot u^n(t_0)\partial_n(u_0).\end{aligned}$$

Using the right group translation by the element $\gamma^{-1}(t_0)$ we get

$$Y_i(u_0) \equiv Y_i(t_0) := \partial_i(u_0)\circ\gamma^{-1}(t_0) = Ad_{g_i(t_0)}X_i, \tag{4}$$

where $g_i(t_0) = \exp u^1(t_0)X_1 \circ \dots \circ \exp u^{i-1}(t_0)X_{i-1}$, $(u_0) = \gamma(t_0)$,

$$Y(u_0) = \dot\gamma(t_0)\circ\gamma^{-1}(t_0) = \dot u^1(t_0)Y_1(u_0) + \dots + \dot u^n(t_0)Y_n(u_0), \quad (u_0) = \gamma(t_0). \tag{5}$$

By the relation (4), $Y_i(t) = Ad_{g_i(t)}X_i$. Differentiating it with respect to t we get $\dot Y_i(t_0) = j_{t_0}^1 Y_i(t) = j_{t_0}^1(Ad_{g_i(t)}X_i) = (j_{t_0}^1 Ad_{g_i(t)})(X_i)$, where $j_{t_0}^1 Ad_{g_i(t)} = TAd(j_{t_0}^1 g_i(t)) = T_{g_i(t_0)}Ad(\dot g_i(t_0))$.

Let us consider the curve $h_i(t) = g_i(t)\circ g_i^{-1}(t_0)$. Recall that $Ad_{h\circ g} = Ad_h Ad_g$. We have two expressions of $j_{t_0}^1 Ad_{h_i(t)}$:

$$j_{t_0}^1(Ad_{h_i(t)}) = \begin{cases} T_{h_i(t_0)}Ad(\dot h_i(t_0)) = ad(\dot h_i(t_0)) \equiv ad_{\dot h_i(t_0)}, & \text{as } h_i(t_0) = e,\\ j_{t_0}^1(Ad_{g_i(t)})Ad_{g_i^{-1}(t_0)}. \end{cases}$$

Then we get $ad_{\dot h_i(t_0)} = j_{t_0}^1(Ad_{g_i(t)})Ad_{g_i^{-1}(t_0)}$, i.e. $j_{t_0}^1(Ad_{g_i(t)}) = ad_{\dot h_i(t_0)}Ad_{g_i(t_0)}$, and thus

$$(j_{t_0}^1 Ad_{g_i(t)})(X_i) = ad_{\dot h_i(t_0)}(Y_i(t_0)) = [\dot h_i(t_0), Y_i(t_0)].$$

As

$$\begin{aligned} j_{t_0}^1 g_i(t) = \dot g_i(t_0) &= TR(u^1(t_0),\dots,u^{i-1}(t_0),0,\dots,0;\dot u^1(t_0),\dots,\dot u^{i-1}(t_0),0,\dots,0)\\ &= \dot u^1(t_0)\partial_1(u_0) + \dots + \dot u^{i-1}(t_0)\partial_{i-1}(u_0)\end{aligned}$$

we get

$$\dot h_i(t_0) = \dot g_i(t_0)\circ g_i^{-1}(t_0) = \dot u^1(t_0)Y_1(t_0) + \dots + \dot u^{i-1}(t_0)Y_{i-1}(t_0),$$

$$\dot Y_i(t_0) = (j_{t_0}^1 Ad_{g_i(t)})(X_i) = [\dot h_i(t_0), Y_i(t_0)] = \sum_{j=1}^{i-1}\dot u^i(t_0)[Y_j(t_0), Y_i(t_0)].$$

Then by (5) we conclude for the covariant acceleration operator $\dot Y(t_0)$:

$$\dot Y(t_0) = \sum_{i=1}^n \ddot u^i(t_0)Y_i(t_0) + \sum_{k<i}[Y_k(t_0), Y_i(t_0)]\dot u^k(t_0)\dot u^i(t_0) := \dot Y_J + \dot Y_C. \tag{6}$$

It means that the covariant acceleration operator consists of two terms: $\dot Y_C$ will be called the C-acceleration operator and $\dot Y_J$ will be referred to as the joint acceleration operator.

R e m a r k 2. The relation (6) is essential for our investigation. Its form is the same as in the case of the robots in the Euclidean space E_3. We had to use another method to develop it, instead of the matrix calculus used in E_3, see [3], [6].

The relation (6) inspires us to introduce the following subspaces connected with kinematics of robots, which are crucial for our approach to asymptotic motions.

Definition 6. Denote

- $VT(u) := \operatorname{span}(Y_1(u),\dots,Y_n(u))$ is the velocity operators subspace at (u), shortly called the V-subspace at (u).
- $AC(u) := \operatorname{span}([Y_1(u),Y_2(u)],\dots,[Y_{n-1}(u),Y_n(u)])$ is called the C-subspace at (u).
- $AS(u) := \operatorname{span}(VT(u)\cap AC(u))$ is called the asymptotic subspace at (u), briefly the AS-subspace.
- $COV(u) := \operatorname{span}(VT(u)+AC(u))$ is called the COV-subspace (the subspace of covariant acceleration operators at (u)).

For $(u)=(0)$ we use the short notation VT, AC, AS, COV. The rank of a robot is $d=\max\limits_{(u)\in U}(\dim VT(u))$.

R e m a r k 3. Evidently the rank of the Jacobian J_u is equal to $\dim VT(u)$. If $d=n=\dim VT$ then there is a neighborhood V of $0\in\mathbb{R}_n$ such that $\mathcal{R}(V)\subset G$ is an immersed submanifold, see for example [5].

Now we turn to the investigation of asymptotic motions of robots. By our Definition 1 an asymptotic curve is introduced as a special curve on a submanifold of a manifold with a connection. In Definition 2 the asymptotic curve on a Lie group G is introduced as an asymptotic curve with respect to the connection ${}^R\Gamma$ determined by the right group translation. By Remark 1 we can use instead ${}^R\Gamma$ any connection given by the bilinear vector form $k[X_1,X_2]$ on the Lie algebra $\mathcal{G}$ of G.

Let us denote $T(u) := \operatorname{span}(\partial_1(u),\dots,\partial_n(u))$. If $\dim VT=n$ then $T(u)$ is the tangent subspace of the immersed submanifold $\mathcal{R}(V)$. As $VT(u)$ is the image of $T(u)$ in a group translation therefore $\dim VT(u)=\dim T(u)$. By Lemma 1 the covariant acceleration operator $\dot{Y}(u)$ from $COV(u)$ is the image of ${}^R\nabla_{\dot\gamma}\dot\gamma$ in the same group translation. Therefore the element ${}^R\nabla_{\dot\gamma}\dot\gamma$ belongs to $T(u)$ iff $\dot{Y}(u)\in VT(u)$.

Definition 7. Let $d<\dim G$. A motion $u(t)$ of a robot is said to be asymptotic or asymptotic at (u_0) if $\dot{Y}(u(t))\in VT(u(t))$ for all $u(t)$ or $\dot{Y}(u(t_0))\in VT(u(t_0))$, respectively. A position (u) is said to be flat if any motion through (u) is asymptotic at (u). A robot is flat if there is a neighborhood of flat positions (u). We say that a motion of a robot is geodesic if $\dot{Y}=0$.

R e m a r k 4. Let us recall that a subspace $\mathcal{H}\subset\mathcal{G}$ is called a Lie subalgebra of $\mathcal{G}$ if it is closed with respect to Lie bracket, i.e. if $[\mathcal{H},\mathcal{H}]\subset\mathcal{H}$. If $\mathcal{H}$ is a Lie subalgebra of $\mathcal{G}$ then there is a Lie subgroup H of G such that $\mathcal{H}$ is the Lie algebra of H.

R e m a r k 5. The relation (6) immediately gives:

a) A motion $u(t)$ is asymptotic at (u_0) iff its C-acceleration operator $\dot{Y}_C$ is an element of the asymptotic space $As(u_0)$. At this position we also say that the C-acceleration is tangential.

b) In particular, if $\dot{Y}_C = 0$ then this motion is asymptotic. For example: if $\dot{u}_i(t_0) \neq 0$ and $\dot{u}_j(t_0) = 0$ for all $j \neq i$ (in this case we say that only the ith joint works), then this motion is asymptotic at (u_0).

c) A position (u) is flat iff $AC(u) \cap VT(u) = AC(u)$, i.e. iff $AC(u) \subset VT(u)$, i.e. if $VT(u)$ is a Lie subalgebra. A robot is flat if VT is a Lie subalgebra of $\mathcal{G}$.

R e m a r k 6. The first author who studied asymptotic motions of robots in E_3 with group $E(3)$ of Euclidean motions was Karger, see for example [3], [4]. He introduced asymptotic motions by the Levi-Civita connection of the pseudo-Riemannian manifold $(E(3), Kl)$, where Kl is the Klein bilinear form on the group $E(3)$. As this connection is just the Cartan connection on $E(3)$ determined by the bilinear vector form $\frac{1}{2}[X_1, X_2]$ in the Lie algebra $e(3)$ of $E(3)$, according to Remark 1 our approach is consistent with Karger's one in the case when $\mathcal{R}\colon\, U \to E(3)$ is an immersed submanifold in $E(3)$.

Definition 8. Motions when only such $i_1, \ldots, i_k$-joints work that the subspace $\operatorname{span}(Y_{i_1}, \ldots, Y_{i_k})$ is a subalgebra are called t-asymptotic (for example, all motions when VT is a subalgebra or when they are caused by the work of only one joint are t-asymptotic). The others are called nt-asymptotic.

Proposition 1. *Let at (u_0) the asymptotic space be zero, $AC(u_0) \cap VT(u_0) = 0$ and $\dim AC(u_0) = \binom{n}{2}$. Then a motion is asymptotic at (u_0) iff only one joint works at $(u_0) = u(t_0)$.*

P r o o f. By assumption a motion $u(t)$ is asymptotic iff $\dot{Y}_C(t_0) = 0$. Proof follows from the relation (6) and from the linear independence of $[Y_1, Y_2]_{t_0}, \ldots, [Y_{n-1}, Y_n]_{t_0}$. $\square$

R e m a r k 7. Every geodesic motion is asymptotic and $\dot{Y}_C = -\dot{Y}_J$. An asymptotic motion with zero Coriolis acceleration is geodesic iff also $\dot{Y}_J = 0$. If $n = \dim VT$ then $\dot{Y}_J = 0$ iff all joint velocities are constant, $\dot{u}_i = c_i$, $i = 1, \ldots, n$. To obtain concrete information about the asymptotic motions we need to know the properties of the group G, first of all the subgroups of the Lie group G and the subalgebras of the Lie algebra $\mathcal{G}$. Therefore in the next part of this paper we will investigate two cases $n = 2$, $n = 3 < \dim G$ only.

1) $n = 2$. In this case $AC(u) = \operatorname{span}([Y_1, Y_2])$ and $AC(u) \cap VT(u) = 0$ or $AC(u) \cap VT(u) = AC(u)$. This immediately gives

Proposition 2. *In the case $n = 2 < \dim G$ there are only t-asymptotic motions.*

2) $n = 3$. There are four cases:

a) If $AC(u_0) = 0$ then $VT(u_0)$ is a subalgebra and (u_0) is flat.

b) $\dim AC(u_0) = 1$. Then at least one element from $[Y_1, Y_2]$, $[Y_1, Y_3]$, $[Y_2, Y_3]$ is not zero. Let for example $[Y_1, Y_2] \neq 0$. If $AC(u_0) \cap VT(u_0) = AC(u_0)$ then $VT(u_0)$ is a subalgebra and (u_0) is flat. Let $AC(u_0) \cap VT(u_0) = 0$ and $[Y_1, Y_3] = k_1[Y_1, Y_2]$, $[Y_2, Y_3] = k_2[Y_1, Y_2]$. Then a motion $u(t)$ is asymptotic at (u_0) iff $\dot{Y}_C(u_0) = 0$, i.e. iff $\dot{u}^1\dot{u}^2 + k_1\dot{u}^1\dot{u}^3 + k_2\dot{u}^2\dot{u}^3 = 0$. There are nt-asymptotic motions (for example when $\dot{u}^1 = 0$, $k_2 = 0$).

c) $\dim AC(u_0) = 2$. We can suppose $[Y_2, Y_3] = c_2[Y_1, Y_2] + c_3[Y_1, Y_3]$. If $AC(u_0) \cap VT(u_0) = 0$ then a motion is asymptotic at (u_0) iff $\dot{u}^2(\dot{u}^1 + c_2\dot{u}^3) = 0$, $\dot{u}^3(\dot{u}^1 + c_3\dot{u}^2) = 0$ at t_0. If $\dot{u}^2\dot{u}^3 = 0$ then there are only t-asymptotic motions. If $\dot{u}^2\dot{u}^3 \neq 0$ then there are nt-asymptotic motions only in two cases: $c_2 = 0$, $c_3 = 0$ or $c_2c_3 \neq 0$. Let $\dim AC(u) \cap VT(u) = 1$. Then $AC(u) \cap VT(u) = \operatorname{span}(\widehat{Y})$, $\hat{Y} = k_2[Y_1, Y_2] + k_2[Y_1, Y_3]$. Then a motion is asymptotic iff $\dot{Y}_C = \lambda\widehat{Y}$, i.e. iff $\dot{u}^2(\dot{u}^1 + c_2\dot{u}^3) = \lambda k_2$ and $\dot{u}^3(\dot{u}^1 + c_3\dot{u}^2) = \lambda k_3$. There are nt-asymptotic motions in this case. If $\dim AC(u) \cap VT(u) = 2$, i.e. $AC(u) \subset VT(u)$, then $VT(u)$ is a subalgebra and (u) is flat.

d) If $\dim AC(u) = 3$ then $[Y_1, Y_2]$, $[Y_1, Y_3]$, $[Y_2, Y_3]$ is a basis in $AC(u)$. If $AC(u) \cap VT(u) = 0$ then by Proposition 1 a motion is asymptotic iff just one joint works. Let $AC(u) \cap VT(u) \neq 0$, $AC(u) \cap VT(u) \neq AC(u)$. Let $\widehat{Y} \in AC(u) \cap VT(u)$, $\widehat{Y} \neq 0$, $\widehat{Y} = k_{12}[Y_1, Y_2] + k_{13}[Y_1, Y_3] + k_{23}[Y_2, Y_3]$. Then the motion $\dot{Y}_C = \lambda\widehat{Y}$ is asymptotic iff the system of differential equations

$$\dot{u}^1\dot{u}^2 = \lambda k_{12}, \quad \dot{u}^1\dot{u}^3 = \lambda k_{13}, \quad \dot{u}^2\dot{u}^3 = \lambda k_{23} \tag{7}$$

has a solution. For $\lambda = 0$ only u^i-motions are asymptotic. For $\lambda \neq 0$ the solution of the system (7) depends on the coefficients k_{12}, k_{13}, k_{23} and thus on the geometry of robots. There are nt-asymptotic motions. We conclude:

Proposition 3. *In the case $n = 3 < \dim G$ there are nt-asymptotic motions and they are determined by a system of non-linear differential equations for unknown controlling functions $u^i(t)$ containing $\dot{u}^i(t)$ only in the "$\dot{u}^i\dot{u}^j$-product" forms. Their solutions depend on the geometrical properties of the robots only.*

Remark 8. In our paper [1] we described all asymptotic motions of 3-parametric robots in the Euclidean space E_3. For example:

a) Motions of 3-robots when the axis of the rotational joint is orthogonal to the axes of two prismatic joints are t-asymptotic as in this case VT is a subalgebra.

b) Motions when the axis of the rotational joint is parallel to the axis of a prismatic joint are t-asymptotic.

c) Motions of the RTT or TTR robots when all joint axes are complanar, all joints work and the rate of prismatic joint velocities is equal to the rate of the corresponding coordinates of the unit vector of the rotational joint angular velocity in the basis formed by unit vectors of translating velocities of the prismatic joints are nt-asymptotic.

4. Conclusion

In this paper referring to robots in the Euclidean space E_3 we model the n-robot in an arbitrary homogeneous space M with the left action of a Lie group G. Motions of such robots are generated by curves on G. Therefore the properties of the group G influence the properties of the robot motions. On every Lie group G there is a connection ${}^R\Gamma$ the parallel transport of which is determined by the right group translations. So on every n-robot there are at least n asymptotic motions through a position (u) induced by asymptotic curves on G with respect to the connection ${}^R\Gamma$. This result generalizes the ones of Karger [3] and Selig [6] who use the Levi-Civita connection induced by a regular and symmetric bilinear form in the case of an n-robot in the Euclidean space E_3. Our results have geometric character and demonstrate an interesting application of the Lie groups and the Lie algebras in robotics.

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Authors' address: *A. Dekrét, J. Bakša*, Department of Mathematics and Descriptive Geometry, Faculty of Wooden Science, T. G. Masaryka 24, 960 53 Zvolen, Slovak Republic, e-mail: `dekret@fpv.umb.sk`, `baksa@vsld.tuzvo.sk`.