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SPACELIKE SUBMANIFOLDS IN INDEFINITE SPACE FORM $M_p^{n+p}(c)$

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Abstract. In this paper, we get an intrinsic inequality for spacelike submanifolds in indefinite space form $M_p^{n+p}(c)$, $(c > 0)$. We also get some rigidity theorems for such spacelike submanifolds.

1. Introduction

Let $M_p^{n+p}(c)$ be $n+p$-dimensional connected semi-Riemannian manifold of constant curvature c whose index is p. It is called indefinite space form of index p. Let M be an n-dimensional Riemannian manifold immersed in $M_p^{n+p}(c)$. The semi-Riemannian metric of $M_p^{n+p}(c)$ induces the Riemannian metric of M, M is called a spacelike submanifold. Spacelike submanifolds in indefinite space form $M_p^{n+p}(c)$ have been of increasing interesting in the recent years. There are many results about these submanifolds, for instance, Dong [3], Wu [6, 7], Liu[4]. In [5], the authors got an intrinsic inequality for spacelike hypersurfaces in de Sitter space form M_1^{n+1} whose index is 1. In this note, we generalize the intrinsic inequality for spacelike hypersurface of de Sitter space to spacelike submanifolds of indefinite space form $M_p^{n+p}(c)$ with index $p \geq 1$. From this inequality, we also get some rigidity theorems for such spacelike submanifolds.

2. Preliminaries

We choose a local field of semi-Riemannian orthonormal frames $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{n+p}\}$ in $M_p^{n+p}(c)$ such that, restricted to M^n, $e_1, \dots, e_n$ are tangent to M^n. Let $\omega_1, \dots, \omega_n$ be its dual frame field such that the semi-Riemannian metric of $M_p^{n+p}(c)$ is given by $ds^2 = \sum_{A=1}^{n+p} \epsilon_A (\omega_A)^2$, where $\epsilon_i = 1$, $i = 1, \dots, n$ and $\epsilon_\alpha = -1$,

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$\alpha = n+1, \ldots, n+p$. Then the structure equations of $M_p^{n+p}(c)$ are given by

(1) $$d\omega_A = -\sum_B \epsilon_B \omega_{AB} \wedge \omega_B\,, \quad \omega_{AB} + \omega_{BA} = 0\,,$$

(2) $$d\omega_{AB} = -\sum_C \epsilon_C \omega_{AC} \wedge \omega_{CB} + \frac{1}{2}\sum_{CD} K_{ABCD}\omega_C \wedge \omega_D\,,$$

(3) $$K_{ABCD} = c\epsilon_A\epsilon_B(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC})\,.$$

We restrict these forms to M^n, then

(4) $$\omega_\alpha = 0\,, \quad \alpha = n+1, \ldots, n+p\,,$$

and the Riemannian metric of M^n is written as $ds^2 = \sum_i \omega_i^2$. Since

(5) $$0 = d\omega_\alpha = -\sum_i \omega_{\alpha,i} \wedge \omega_i\,,$$

by Cartan's lemma we may write

(6) $$\omega_{\alpha,i} = \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha.$$

From these formulas, we obtain the structure equations of M^n:

(7) $$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j\,, \quad \omega_{ij} + \omega_{ji} = 0\,,$$

(8) $$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} + \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l\,,$$

(9) $$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha)\,,$$

where R_{ijkl} are the components of curvature tensor of M^n. We call

(10) $$h = \sum_{i,j,\alpha} h_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha$$

the second fundamental form of M^n. The mean curvature vector is $H = \sum_{i,\alpha} h_{ii}^\alpha e_\alpha = \sum_\alpha H^\alpha e_\alpha$, where $H^\alpha = \sum_i h_{ii}^\alpha$. We denote $|h|^2 = \sum_{i,j,\alpha}(h_{ij}^\alpha)^2$, and $|H|^2 = \sum_\alpha (H^\alpha)^2$. We call that M^n is maximal if its mean curvature field vanishes, i.e. $H = 0$.

Let $h_{ij,k}^\alpha$ and $h_{ij,kl}^\alpha$ denote the covariant derivative and the second covariant derivative of h_{ij}^α. Then we have $h_{ij,k}^\alpha = h_{ik,j}^\alpha$ and

$$h_{ij,kl}^\alpha - h_{ij,lk}^\alpha = \sum_m h_{im}^\alpha R_{mjkl} + \sum_m h_{mj}^\alpha R_{mikl} + \sum_m h_{ij}^\beta R_{\alpha\beta kl}\,,$$

where $R_{\alpha\beta kl}$ are the components of the normal curvature tensor of M^n, that is

$$R_{\alpha\beta kl} = \sum_i (h_{ik}^\alpha h_{il}^\beta - h_{ik}^\beta h_{il}^\alpha)\,.$$

If $R_{\alpha\beta kl} = 0$ at point x of M^n we say that the normal bundle connection of M^n is flat at x and it is well known [1] that $R_{\alpha\beta kl} = 0$ at point x if and only if the matrix (h_{ij}^α) are simultaneously diagonalizable at x.

3. Main results for space-like submanifolds

Lemma 3.1 (Cauchy-Swartz inequality)**.** *Let $a_1,\dots,a_n$; $b_1,\dots,b_n$ be real numbers, then*

$$\Big(\sum_i a_i b_i\Big)^2 \le \Big(\sum_i a_i^2\Big)\Big(\sum_i b_i^2\Big)$$

and the equality holds if and only if there exists a constant λ such that $a_i = \lambda b_i$ or $b_i = \lambda a_i$, $i = 1,\dots,n$.

Theorem 3.2. *If M^n is a space-like submanifold of indefinite space form $M_p^{n+p}(c)$ $(c > 0)$, S and ρ are Ricci curvature tensor and the scalar curvature of M^n, respectively, then*

$$|S|^2 \ge 2c\rho(n-1) - c^2 n(n-1)^2\,. \tag{11}$$

Moreover, $|S|^2 = 2c\rho(n-1) - c^2n(n-1)^2$ if and only if M^n is a spacelike Einstein submanifolds with $S = c(n-1)g$, where g is the Riemannian metric of M^n.

Proof. From the Gauss equation we get

$$\begin{aligned} S_{ij} &= \sum_k R_{kikj} = \sum_k \Big\{c(\delta_{kk}\delta_{ij} - \delta_{il}\delta_{jk}) - \sum_\alpha (h^\alpha_{kk}h^\alpha_{ij} - h^\alpha_{ik}h^\alpha_{jk})\Big\} \\ &= c(n-1)\delta_{ij} - \sum_\alpha H^\alpha h^\alpha_{ij} + \sum_{k,\alpha} h^\alpha_{ik}h^\alpha_{jk} \end{aligned} \tag{12}$$

So

$$\begin{aligned} |S|^2 &= \sum_{ij} S_{ij}^2 = \sum_{ij}\Big\{c(n-1)\delta_{ij} - \sum_\alpha H^\alpha h^\alpha_{ij} + \sum_{k,\alpha} h^\alpha_{ik}h^\alpha_{jk}\Big\}^2 \\ &= \sum_{ij}\Big\{c^2(n-1)^2\delta_{ij} + \Big(\sum_\alpha H^\alpha h^\alpha_{ij}\Big)^2 + \Big(\sum_{k,\alpha} h^\alpha_{ik}h^\alpha_{jk}\Big)^2 \\ &\quad - 2c(n-1)\delta_{ij}\Big(\sum_\alpha H^\alpha h^\alpha_{ij}\Big) + 2c(n-1)\delta_{ij}\Big(\sum_{k,\alpha} h^\alpha_{ik}h^\alpha_{jk}\Big) \\ &\quad - 2\Big(\sum_\alpha H^\alpha h^\alpha_{ij}\Big)\Big(\sum_{k,\alpha} h^\alpha_{ik}h^\alpha_{jk}\Big)\Big\} \\ &= c^2n(n-1)^2 + \sum_{ij}\Big(\sum_\alpha H^\alpha h^\alpha_{ij}\Big)^2 + \sum_{ij}\Big(\sum_{k,\alpha} h^\alpha_{ik}h^\alpha_{jk}\Big)^2 \\ &\quad - 2c(n-1)|H|^2 + 2c(n-1)\Big(\sum_{i,k,\alpha} h^\alpha_{ik}h^\alpha_{ik}\Big) \\ &\quad - 2\sum_{i,j}\Big(\sum_\alpha H^\alpha h^\alpha_{ij}\Big)\Big(\sum_{k,\alpha} h^\alpha_{ik}h^\alpha_{jk}\Big) \end{aligned}$$

and

$$\rho = \sum_i S_{ii} = \sum_i \Big\{ c(n-1) - \sum_\alpha H^\alpha h_{ii}^\alpha + \sum_{k,\alpha} h_{ik}^\alpha h_{ik}^\alpha \Big\}$$
$$= cn(n-1) - |H|^2 + \sum_{ij\alpha} (h_{ij}^\alpha)^2$$
$$= cn(n-1) - |H|^2 + |h|^2 , \tag{13}$$

So

$$|S|^2 = c^2 n(n-1)^2 + \sum_{ij} \Big(\sum_\alpha H^\alpha h_{ij}^\alpha \Big)^2 + \sum_{ij} \Big(\sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha \Big)^2$$
$$- 2c(n-1)|H|^2 + 2c(n-1)\big(\rho + |H|^2 - cn(n-1)\big)$$
$$- 2 \sum_{i,j} \Big(\sum_\alpha H^\alpha h_{ij}^\alpha \Big) \Big(\sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha \Big)$$
$$= 2c\rho(n-1) - c^2 n(n-1)^2 + \sum_{ij} \Big(\sum_\alpha H^\alpha h_{ij}^\alpha \Big)^2 + \sum_{ij} \Big(\sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha \Big)^2$$
$$- 2 \sum_{i,j} \Big(\sum_\alpha H^\alpha h_{ij}^\alpha \Big) \Big(\sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha \Big)$$
$$\geq 2c\rho(n-1) - c^2 n(n-1)^2 + \sum_{ij} \Big(\sum_\alpha H^\alpha h_{ij}^\alpha \Big)^2 + \sum_{ij} \Big(\sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha \Big)^2$$
$$- 2 \Big(\sum_{ij} \Big(\sum_\alpha H^\alpha h_{ij}^\alpha \Big)^2 \Big)^{1/2} \Big(\sum_{ij} \Big(\sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha \Big)^2 \Big)^{1/2}$$
$$= 2c\rho(n-1) - c^2 n(n-1)^2 + \Big\{ \Big(\sum_{ij} \Big(\sum_\alpha H^\alpha h_{ij}^\alpha \Big)^2 \Big)^{1/2}$$
$$- \Big(\sum_{ij} \Big(\sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha \Big)^2 \Big)^{1/2} \Big\}^2 \geq 2c\rho(n-1) - c^2 n(n-1)^2 . \tag{14}$$

The first inequality has used Lemma 3.1.

So we have

$$|S|^2 \geq 2c\rho(n-1) - c^2 n(n-1)^2 .$$

Now we will prove the second part of this theorem.

If M^n is a spacelike Einstein submanifold with $S = c(n-1)g$, then we have the following equations:

$$|S|^2 = c^2 n(n-1)^2 , \quad \text{and} \quad \rho = cn(n-1) ,$$

i.e.

$$|S|^2 = 2c\rho(n-1) - c^2 n(n-1)^2 .$$

Conversely, if the Eq. (14) becomes an equality, then all the inequality of Eq. (14) will become equality. From the Lemma 3.1, there exist a constant λ such that

$$\sum_\alpha H^\alpha h_{ij}^\alpha = \lambda \sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha$$

or

$$\lambda \sum_\alpha H^\alpha h_{ij}^\alpha = \sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha \quad \text{for all} \quad i,j \in \{1,\dots,n\} \tag{15}$$

and

$$\sum_{ij} \Big(\sum_\alpha H^\alpha h_{ij}^\alpha\Big)^2 = \sum_{ij} \Big(\sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha\Big)^2 . \tag{16}$$

(I) If $\lambda = 0$, we know that

$$\sum_\alpha H^\alpha h_{ij}^\alpha = 0 \quad \text{or} \quad \sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha = 0 \quad \text{for all} \quad i,j \in \{1,\dots,n\} . \tag{17}$$

then

$$H = 0 \quad \text{or} \quad \sum_{i,k,\alpha} [h_{ik}^\alpha]^2 = 0 \tag{18}$$

If $H = 0$, then M^n is maximal. From the Eq. (14), we have the following equations:

$$\begin{aligned} |S|^2 &= 2c\rho(n-1) - c^2 n(n-1)^2 + \sum_{ij} \Big(\sum_\alpha H^\alpha h_{ij}^\alpha\Big)^2 + \sum_{ij} \Big(\sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha\Big)^2 \\ &\quad - 2\Big(\sum_{ij} \Big(\sum_\alpha H^\alpha h_{ij}^\alpha\Big)^2\Big)^{1/2} \Big(\sum_{ij} \Big(\sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha\Big)^2\Big)^{1/2} \\ &= 2c\rho(n-1) - c^2 n(n-1)^2 + \sum_{ij} \Big(\sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha\Big)^2 . \end{aligned} \tag{19}$$

We have that

$$\sum_{ij} \Big(\sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha\Big)^2 = 0 , \tag{20}$$

for $i,j \in \{1,\dots,n\}$. From this equation, we get

$$\sum_{k,\alpha} h_{ik}^\alpha h_{ik}^\alpha = 0 \quad \text{for} \quad i = 1,\dots,n . \tag{21}$$

So $h_{ij}^\alpha = 0$, for $i,j \in \{1,\dots,n\}$ and $\alpha \in \{n+1,\dots,n+p\}$, i.e. M^n is totally geodesic.

If $\sum_{i,k,\alpha} [h_{ik}^\alpha]^2 = 0$,so $h_{ij}^\alpha = 0$, for $i,j \in \{1,\dots,n\}$ and $\alpha \in \{n+1,\dots,n+p\}$, i.e. M^n is totally geodesic.

From the Eq. (12), we know that

$$S_{ij} = c(n-1)\delta_{ij} . \tag{22}$$

(II) If $\lambda \neq 0$, from the equation $\sum_\alpha H^\alpha h_{ij}^\alpha = \lambda \sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha$, and equation (16), we have the following equation:

$$(\lambda^2 - 1)\Big[\sum_{ij}\Big(\sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha\Big)^2\Big] = 0\,, \tag{23}$$

then $\sum_{ij}\big(\sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha\big)^2 = 0$ or $\lambda^2 = 1$.

If $\sum_{ij}\big(\sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha\big)^2 = 0$, then $\big(\sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha\big)^2 = 0$ for all i, j. So $h_{ij}^\alpha = 0$, for $i, j \in \{1, \ldots, n\}$ and $\alpha \in \{n+1, \ldots, n+p\}$, i.e. M^n is totally geodesic.

If $\lambda^2 = 1$, then $\lambda = 1$ or $\lambda = -1$. If $\lambda = -1$, then $\sum_\alpha H^\alpha h_{ij}^\alpha = -\sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha$, so we have that $H^2 + |h|^2 = 0$, i.e. $h = 0$. If $\lambda = 1$, then $\sum_\alpha H^\alpha h_{ij}^\alpha = \sum_{k,\alpha} h_{ik}^\alpha h_{jk}^\alpha$.

From equation (12), we have the following equation:

$$S_{ij} = c(n-1)\delta_{ij}\,. \tag{24}$$

□

Remark 3.3. When $p = 1$, i.e. M^n is a space-like hypersurface, the inequality given in [5].

Corollary 3.4. *If M^n is a maximal space-like submanifold of indefinite space form $M_p^{n+p}(c)(c > 0)$, S and ρ are Ricci curvature tensor and the scalar curvature of M^n, respectively, then*

$$|S|^2 = 2c\rho(n-1) - c^2 n(n-1)^2 \tag{25}$$

if and only if M^n is totally geodesic.

Proof. If M^n is totally geodesic, then from equations (12) and (13),

$$|S|^2 = c^2 n(n-1)^2\,, \quad \text{and} \quad \rho = cn(n-1)\,,$$

i.e.

$$|S|^2 = 2c\rho(n-1) - c^2 n(n-1)^2\,.$$

Conversely, from equations $H = 0$, (19), (20) and (21), we know that M^n is totally geodesic. □

Theorem 3.5. *If M^n is a complete spacelike submanifold with flat normal bundle and with positive sectional curvature immersed in indefinite space form $M_p^{n+p}(c)$, ($c > 0$, $p \geq 2$, $n \geq 2$), S and ρ are Ricci curvature tensor and the scalar curvature of M^n, respectively, then*

$$|S|^2 = 2c\rho(n-1) - c^2 n(n-1)^2 \tag{26}$$

if and only if M^n is totally geodesic.

Proof. If M^n is totally geodesic, then from equations (12) and (13),

$$|S|^2 = c^2 n(n-1)^2\,, \quad \text{and} \quad \rho = cn(n-1)\,,$$

i.e.

$$|S|^2 = 2c\rho(n-1) - c^2 n(n-1)^2\,.$$

Conversely, from case (I) and case (II) in the proof of Theorem 3.2, we will prove that M^n must be geodesic under the conditions: $\lambda = 1$ and

$$\sum_{\alpha} H^{\alpha} h_{ij}^{\alpha} = \sum_{k,\alpha} h_{ik}^{\alpha} h_{jk}^{\alpha}\,, \tag{27}$$

for $i, j \in \{1, \dots, n\}$.

If $H = 0$, from Corollary 3.4, we know that M^n is totally geodesic. Now we suppose $H \neq 0$, and choose $e_{n+1} = \frac{H}{|H|}$. Then, it follows that

$$H = \sum_i h_{ii}^{n+1} e_{n+1}\,, \quad \text{and} \quad H^{\alpha} = \sum_i h_{ii}^{\alpha} = 0\,, \quad \alpha > n+1\,. \tag{28}$$

Since the normal bundle of M^n is flat, we choose $e_1, \cdots, e_n$ such that

$$h_{ij}^{\alpha} = \lambda_i^{\alpha} \delta_{ij}, \quad \text{for} \quad \alpha = n+1, \dots, n+p\,.$$

From equation (27), we have the following equations:

$$|H|^2 = |H^{n+1}|^2 = |h|^2\,. \tag{29}$$

Taking the covariant derivative of (29), we obtain

$$H^{n+1} H_k^{n+1} = \sum_{ij\alpha} h_{ij}^{\alpha} h_{ij,k}^{\alpha} \tag{30}$$

and by Lemma 3.1, we have

$$|H|^2 |\nabla H|^2 \leq |h|^2 |\nabla h|^2\,. \tag{31}$$

Then the Laplacian of $|h|^2$ is given by:

$$\begin{aligned} \frac{1}{2}\triangle |h|^2 = \frac{1}{2}\triangle |H|^2 &= |\nabla h|^2 + \sum_{ij\alpha} h_{ij}^{\alpha} \triangle h_{ij}^{\alpha} \\ &= |\nabla h|^2 + \sum_i \lambda^{n+1}(H^{n+1}) + \frac{1}{2} R_{ijij} (\lambda_i^{\alpha} - \lambda_j^{\alpha})^2 \end{aligned} \tag{32}$$

We define an operator $\Box$ acting on any function f by:

$$\Box f = \sum_{ij} (H^{n+1}\delta_{ij} - h_{ij}^{n+1}) f_{,ij}$$

Since $(H^{n+1}\delta_{ij} - h_{ij}^{n+1})$ is trace free, it follows from [2] that $\Box$ is self-adjoint relative to L^2-inner product of M^n, i.e.,

$$\int_{M^n} f \Box g = \int_{M^n} g \Box f\,.$$

Thus we have

$$\begin{aligned} \Box H^{n+1} &= \sum_{ij} (H^{n+1}\delta_{ij} - h_{ij}^{n+1}) H_{ij}^{n+1} \\ &= \frac{1}{2}\triangle |H|^2 - |\nabla H|^2 - \sum_i \lambda^{n+1}(H^{n+1}) \end{aligned} \tag{33}$$

From equations (30),(31),(32),(33),

$$\Box H^{n+1} \geq \frac{1}{2} R_{ijij} (\lambda_i^\alpha - \lambda_j^\alpha)^2 \,. \tag{34}$$

Because $S_{ij} = c(n-1)\delta_{ij}$, we see by the Bonnet-Myers theorem that M^n is bounded and hence compact.

Since $\Box$ is self-adjoint, we have

$$0 \geq \int_{M^n} \frac{1}{2} R_{ijij} (\lambda_i^\alpha - \lambda_j^\alpha)^2 \,. \tag{35}$$

Then, by hypothesis $R_{ijij} > 0$, so $\lambda_i^\alpha = \lambda_j^\alpha$ for $\alpha \in \{n+1, \ldots, n+p\}$ and $i, j \in \{1, \ldots, n\}$.

From equation (27), we have

$$(n-1)(\lambda_1^{n+1})^2 = (\lambda_1^{n+2})^2 + \cdots + (\lambda_1^{n+p})^2. \tag{36}$$

From equation (28), we have

$$n\lambda_1^{n+2} = \cdots = n\lambda_1^{n+p} = 0 \,, \tag{37}$$

then we have

$$(n-1)(\lambda_1^{n+1})^2 = 0, \tag{38}$$

so $\lambda_1^{n+1} = \lambda_1^{n+2} = \cdots = \lambda_1^{n+p} = 0$, i.e. M^n is a totally geodesic submanifold. $\Box$

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