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Abstract. We use categories to recast the combinatorial theory of full heaps, which are certain labelled partially ordered sets that we introduced in previous work. This gives rise to a far simpler set of definitions, which we use to outline a combinatorial construction of the so-called loop algebras associated to affine untwisted Kac–Moody algebras. The finite convex subsets of full heaps are equipped with a statistic called parity, and this naturally gives rise to Kac's asymmetry function. The latter is a key ingredient in understanding the (integer) structure constants of simple Lie algebras with respect to certain Chevalley bases, which also arise naturally in the context of heaps.

Keywords: Lie algebra, Chevalley basis, heap

Classification: 17B20, 17B67, 06A07

1. Introduction

The simple Lie algebras over the complex numbers are some of the most famous examples of nonassociative algebras, and they play a key role in representation theory and mathematical physics.

There are several combinatorial approaches to the representation theory of the simple Lie algebras over $\mathbb{C}$. Two of these include Littelmann's description of representations in terms of paths [8], and the crystal basis approach of Kashiwara and the Kyoto school [7]. Both of these approaches are very versatile but can be combinatorially complicated. There is also a combinatorial approach to finding structure constants of simple Lie algebras due to Vavilov [11].

In a previous work [3], we showed how to construct simple Lie algebras in terms of their so-called minuscule representations using combinatorial structures called "full heaps" whose theory was initially developed in [3], [4]. This in turn builds on work of Stembridge [10] on minuscule elements and their heaps, and on work of Wildberger [13] on constructing minuscule representations for simply laced simple Lie algebras over $\mathbb{C}$ (although the paper [13] does not contain proofs).

This paper is a slightly expanded version of my invited talk at the Second Mile High Conference on Nonassociative Mathematics, entitled "Chevalley bases for Lie algebras and the combinatorics of Kac's asymmetry function". The first definition of full heaps, given in [3], is rather complicated, and the main original contribution of this paper is a significant simplification of this definition by recasting it in terms of certain categories.

The results of the paper [3] are easy to translate into the category-theoretic framework, so we only summarize the main results here and refer the reader to [3] for proofs and complete details. A complete treatment following the approach of this paper will appear in my planned monograph [5].

This paper is organized as follows. In Section 2, we review the necessary background from Lie theory. In Section 3, we introduce the category **Heap** of heaps, and the subcategory **Heap**(Γ) of heaps over a particular graph, Γ. Using this framework, we develop the various concepts associated with heaps, including the key notion of a "full heap". In Section 4, we show how to use full heaps to define algebras of operators. The so-called loop algebras associated to certain affine Kac–Moody algebras may be constructed in this way, along with Chevalley bases for the algebras. Finally, Section 5 explores the relationship between the simply laced and non simply laced cases in terms of folding operations on heaps.

It is known that every minuscule representation of a simple Lie algebra may be constructed using a full heap [3], [9]. However, we concentrate in this paper for reasons of space on two particular examples in Lie types D_n and B_n, and these provide interesting illustrations of the underlying concepts.

2. Preliminaries

2.1 Lie algebras. A *Lie algebra* is a vector space $\mathfrak{g}$ over a field k equipped with a bilinear map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ (the *Lie bracket*) satisfying the conditions

$$\begin{aligned} [x, x] &= 0, \\ [[x, y], z] + [[y, z], x] + [[z, x], y] &= 0, \end{aligned}$$

for all $x, y, z \in \mathfrak{g}$.

The first condition is known as *antisymmetry*, and may be regarded as a replacement for commutativity. The second condition is known as the *Jacobi identity*. It may be regarded as a replacement for associativity.

If $\mathfrak{h}$ and $\mathfrak{j}$ are subspaces of $\mathfrak{g}$, then we write $[\mathfrak{h}, \mathfrak{j}]$ to denote the subspace

$$\{[h, j] : h \in \mathfrak{h}, j \in \mathfrak{j}\}.$$

A subspace $\mathfrak{h}$ of $\mathfrak{g}$ is called a *subalgebra* of $\mathfrak{g}$ if $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$. If, furthermore, we have $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$ (or, equivalently, $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$) then $\mathfrak{h}$ is said to be an *ideal* of $\mathfrak{g}$. We write $\mathfrak{h} \leq \mathfrak{g}$ (respectively, $\mathfrak{h} \trianglelefteq \mathfrak{g}$) to mean that $\mathfrak{h}$ is a subalgebra (respectively, an ideal) of $\mathfrak{g}$.

If $\mathfrak{g}$ has no ideals other than itself and the zero ideal, then $\mathfrak{g}$ is said to be *simple*. The *derived algebra*, $\mathfrak{g}'$ of $\mathfrak{g}$, is the Lie algebra $[\mathfrak{g}, \mathfrak{g}]$, which is an ideal of $\mathfrak{g}$. If $\mathfrak{g}' = 0$, then $\mathfrak{g}$ is said to be *abelian*.

If V is any vector space over k, then the Lie algebra $\mathfrak{gl}(V)$ is the k-vector space of all k-linear maps $T : V \rightarrow V$, equipped with the Lie bracket satisfying

$$[T, U] := T \circ U - U \circ T,$$

where $\circ$ is composition of maps. Any subset S of k-linear operators from V to V generates a Lie subalgebra of $\mathfrak{gl}(V)$. The subalgebra is the smallest k-subspace of $\mathfrak{gl}(V)$ that (a) contains S and (b) is closed under the Lie bracket.

If $\mathfrak{g}$ is a Lie algebra over k, then a *derivation* of $\mathfrak{g}$ is a k-linear map $D : \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying Leibniz's law, namely

$$D([x, y]) = [D(x), y] + [x, D(y)].$$

The Jacobi identity guarantees that for each $x \in \mathfrak{g}$, the map $D_x : \mathfrak{g} \rightarrow \mathfrak{g}$ given by $D_x(y) = [x, y]$ is a derivation.

A *Chevalley basis* for a Lie algebra $\mathfrak{g}$ over a field k of characteristic zero is a certain type of basis for $\mathfrak{g}$ all of whose structure constants are integers. In other words, if b_i and b_j are elements of a Chevalley basis B, then we have

$$b_i b_j = \sum_{k \in B} \lambda_{i,j}^k b_k,$$

where the $\lambda_{i,j}^k$ are all integers.

2.2 Dynkin diagrams. The (nonabelian) simple Lie algebras over the field $k = \mathbb{C}$ turn out to be of key importance in many areas of mathematics and physics. These algebras are classified by their Dynkin diagrams, or equivalently by their Cartan matrices. For our purposes, it is necessary to generalize the notion of a Cartan matrix.

Definition 2.2.1. Let A be an n by n matrix with integer entries. We call A a *generalized Cartan matrix* if it satisfies the following conditions:

(i) $a_{ii} = 2$ for all $1 \leq i \leq n$,
(ii) $a_{ij} \leq 0$ for $i \neq j$ and
(iii) $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$.

The generalized Cartan matrix is called *simply laced* if all its entries come from the set $\{2, 0, -1\}$.

Definition 2.2.2. The *Dynkin diagram* $\Gamma = \Gamma(A)$ associated to a generalized Cartan matrix A is a directed graph, possibly with multiple edges, and vertices indexed (for now) by the integers 1 up to n. If $i \neq j$ and $|a_{ij}| \geq |a_{ji}|$, we connect the vertices corresponding to i and j by $|a_{ij}|$ lines; this set of lines is equipped with an arrow pointing towards i if $|a_{ij}| > 1$. (There are further rules if $a_{ij}a_{ji} > 4$, but we do not need these for our purposes.)

The Dynkin diagram (up to re-indexing of the vertices) and the generalized Cartan matrix determine each other, so we may write $A = A(\Gamma)$. We call the Dynkin diagram simply laced if its associated generalized Cartan matrix is simply laced.

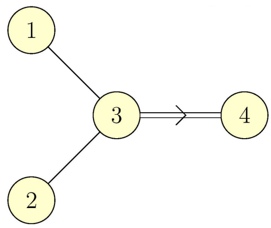
The Dynkin diagrams associated to simple Lie algebras over $\mathbb{C}$ are said to be of *finite type*. In this case, the generalized Cartan matrix is simply called the Cartan matrix. Most of the examples of Dynkin diagrams in this paper are of *affine type*. Full details of the classification of Dynkin diagrams may be found in [6, §4].

It will be convenient for some purposes to relabel the vertices of the Dynkin diagram.

Example 2.2.3. The matrix

$$A = \begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ -1 & -1 & 2 & -1 \\ 0 & 0 & -2 & 2 \end{pmatrix}$$

is a generalized Cartan matrix whose corresponding Dynkin diagram is



3. Heaps

A heap is an isomorphism class of labelled posets, depending on an underlying graph Γ and satisfying certain axioms. Heaps have a wide variety of applications in algebraic combinatorics and statistical mechanics, as explained in [12]. The algebraic and combinatorial theory of heaps mostly concentrates on the case of finite heaps, but there is a well-developed theory of infinite heaps used in the study of parallelism in computer science, where they are known as "dependence graphs" [2].

In [3], we showed how to use certain infinite heaps, called "full heaps", to construct representations of almost all untwisted affine Kac–Moody algebras. This constructs, as a special case, all the simple Lie algebras over $\mathbb{C}$ except those of types E_8, F_4 and G_2.

3.1 Heaps over graphs.

Definition 3.1.1. A *heap* is a function $\varepsilon : E \to \Gamma$ from the elements E of a partially ordered set $(E, \leq)$ to the vertices of a graph Γ, satisfying the following two properties:

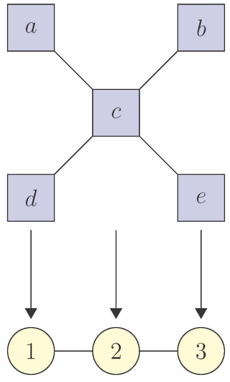
(i) For every vertex α of Γ, and for every edge $\{\alpha, \beta\}$ of Γ, the subsets $\varepsilon^{-1}(\{\alpha\})$ and $\varepsilon^{-1}(\{\alpha, \beta\})$ are chains in E. (Subsets of E of this form will be called *vertex chains* and *edge chains*, respectively.)

(ii) The partial order $\leq$ on E is the smallest partial order extending the given total orders on the vertex chains and edge chains.

Example 3.1.2. The following figure shows an example of heap $\varepsilon : E \to \Gamma$, and depicts the Hasse diagram of a partially ordered set E with 5 elements and a graph Γ with three vertices and two edges.

The vertex chains are $\varepsilon^{-1}(1) = \{a, d\}$, $\varepsilon^{-1}(2) = \{c\}$ and $\varepsilon^{-1}(3) = \{b, e\}$.

The edge chains are $\varepsilon^{-1}(\{1, 2\}) = \{a, c, d\}$ and $\varepsilon^{-1}(\{2, 3\}) = \{b, c, e\}$.



3.2 Full heaps over Dynkin diagrams. We introduced the notion of a full heap in [3], [4], but the original definition was rather complicated. In this section, we give an equivalent but simpler definition of full heaps.

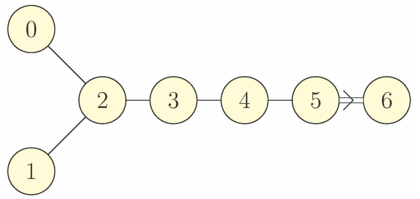
Definition 3.2.1. Let Γ be a Dynkin diagram with generalized Cartan matrix $A = (a_{ij})$. Let $\varepsilon : E \to \Gamma$ be a locally finite heap (i.e., every interval in E is finite). We call E a *full heap* over Γ if the following three properties are satisfied:

(Chains) Every vertex chain and every edge chain of E is isomorphic, as a partially ordered set, to $\mathbb{Z}$.

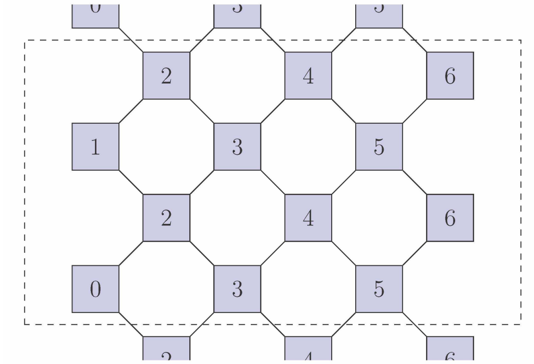
(Open map) Regarded as a map from the Hasse diagram of E to the graph Γ, the function ε is an open map. (Meaning: if $\varepsilon(x) = p$ then the neighbours of x are mapped surjectively by ε to the neighbours of p.)

(Intervals) If $\varepsilon(x) = \varepsilon(y) = p$, then $\sum_{x \leq z \leq y} a_{p,\varepsilon(z)} = 2$.

Example 3.2.2. Consider the Dynkin diagram of type $\widetilde{B}_6$, as shown below.



The next figure shows an example of a full heap $\varepsilon : E \to \widetilde{B}_6$. In this case, the partially ordered set E is infinite, but its Hasse diagram consists of a vertically repeating motif. This motif is shown in a dashed box.



3.3 Categories of heaps.

Definition 3.3.1. There is a category **Heap** whose objects are heaps, in which a morphism from a heap $\varepsilon_1 : E_1 \to \Gamma_1$ to a heap $\varepsilon_2 : E_2 \to \Gamma_2$ consists of a pair (f_E, f_Γ) such that

(i) if $x \leq y$ then $f_E(x) \leq f_E(y)$;
(ii) if a and b are adjacent, then the vertices $f_\Gamma(a)$ and $f_\Gamma(b)$ are adjacent or equal; and
(iii) the following diagram commutes:

$$\begin{array}{ccc} E_1 & \xrightarrow[\varepsilon_1]{} & \Gamma_1 \\ {\scriptstyle f_E}\downarrow & & \downarrow{\scriptstyle f_\Gamma} \\ E_2 & \xrightarrow[\varepsilon_2]{} & \Gamma_2 \end{array}$$

Definition 3.3.1 essentially says that a morphism of heaps is a poset homomorphism on the level of posets, and a graph homomorphism on the level of graphs.

Definition 3.3.2. Let Γ be a graph. The category **Heap** has a subcategory, **Heap**(Γ), whose objects are heaps over Γ. A morphism f of **Heap**(Γ) is a morphism

$$f \in \mathrm{Hom}_{\mathbf{Heap}}(A, B),$$

where A and B are objects of **Heap**(Γ), and where f is the identity on labels (i.e., $f_\Gamma = \mathrm{id}$).

Definition 3.3.3. Let $\varepsilon_1 : E_1 \to \Gamma$ and $\varepsilon_2 : E_2 \to \Gamma$ be heaps. If there is a morphism (f_E, id) in **Heap**(Γ) from E_1 to E_2 such that f_E is injective, then we call $\varepsilon_1 : E_1 \to \Gamma$ a *subheap* of $\varepsilon_2 : E_2 \to \Gamma$.

In practice, we can pretend that the maps f_E in Definition 3.3.3 are inclusions. This identifies the subheaps of $\varepsilon : E_2 \to \Gamma$ with the subsets of E_2. With the above notation, we may write $E_1 \leq E_2$ to express this.

Note that it is *not* true in general that, under the above identifications, the partial order on E_1 is obtained from restricting the partial order on E_2 to the subset E_1. For example, if E_2 is the heap E of Example 3.1.2, and $E_1 = \{a, e\}$, then a and e are comparable in E_2, but not in E_1. However, if E_1 is a convex subset of E_2, then it is the case that the partial order on E_1 is obtained by restriction. (A subset F of a partially ordered set E is *convex* if, whenever $a \leq b \leq c$ and $a, c \in F$, we have $b \in F$.)

3.4 Proper ideals.

Definition 3.4.1. Let $\varepsilon : E \to \Gamma$ be a heap.

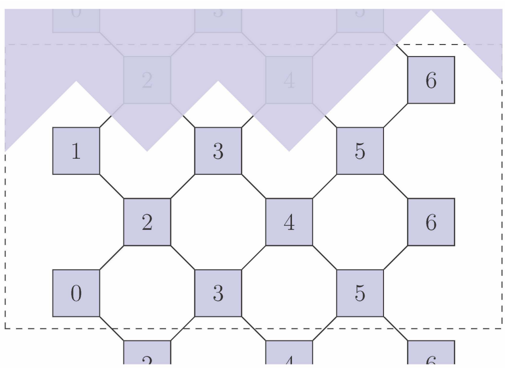
A subset F of E (or, more properly, the corresponding subheap) is called an *ideal* of E if, whenever $x \in E$ and $y \in F$ satisfy $x \leq y$, then we have $x \in F$. Dually, a subset F of E is called a *filter* of E if, whenever $x \in E$ and $y \in F$ satisfy $x \geq y$, then we have $x \in F$.

An ideal I of E is called a *proper ideal* if for any vertex chain $C = \varepsilon^{-1}(p)$ of E, we have

$$\emptyset \neq C \cap I \neq C.$$

The set of all proper ideals of E is denoted by $\mathcal{B}$.

Example 3.4.2. The following figure shows a proper ideal of the full heap in Example 3.2.2. The elements of the ideal are drawn as unshaded.



Definition 3.4.3. Let R_Γ be the $\mathbb{R}$-vector space with basis $\{\alpha_i : i \in \Gamma\}$ indexed by the vertices of Γ. If L is a finite convex subheap of E, then we define the *content* of L, $\chi(L) \in R_\Gamma$, to be the vector

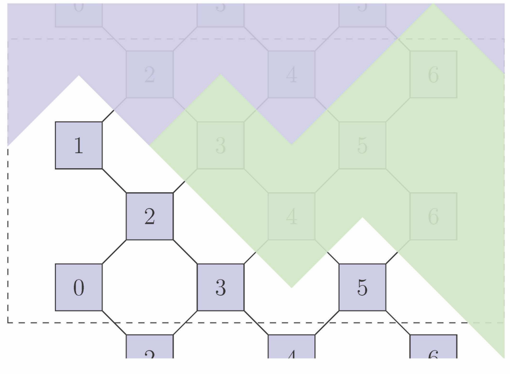
$$\sum_{p \in \Gamma} |\varepsilon^{-1}(p) \cap L| \alpha_p.$$

If $\alpha \in R_\Gamma$ then we define $\mathcal{L}_\alpha = \mathcal{L}_\alpha(E)$ to be the set of all finite convex subheaps L of E with $\chi(L) = \alpha$. We may abbreviate $\mathcal{L}_{\alpha_i}$ to $\mathcal{L}_i$.

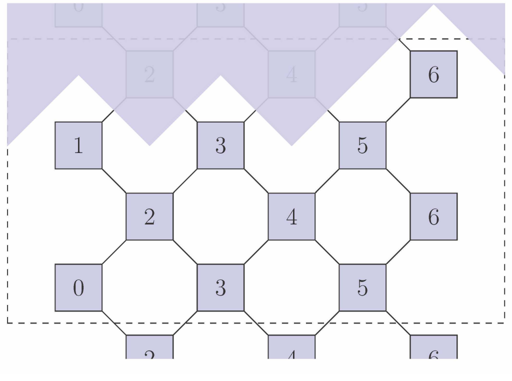
Definition 3.4.4. Let $I \in \mathcal{B}$ and let L be a finite convex subheap of E with $\chi(L) = \alpha$ (in other words, $L \in \mathcal{L}_\alpha(E)$). Then

(a) we write $L \succ I$ (or $I \prec_\alpha I'$) to mean that both $I' := I \cup L \in \mathcal{B}$ and $I \cap L = \emptyset$, and
(b) we write $L \prec I$ (or $I' \prec_\alpha I$) to mean that both $L \leq I$ and $I \backslash L \in \mathcal{B}$.

Example 3.4.5. Consider the full heap $\varepsilon : E \rightarrow \Gamma$ of Example 3.2.2. Let I be the proper ideal



of E, and let J be the proper ideal



of E. Note that we have $I \subset J$, and that the set $L = J \backslash I$ contains 5 elements: one labelled 3, one labelled 4, one labelled 5 and two labelled 6. In this case, we have $\chi(L) = \alpha$, where $\alpha = \alpha_3 + \alpha_4 + \alpha_5 + 2\alpha_6$. We express this by saying that $I \prec_\alpha J$.

Given a vector β with positive integral coefficients and a proper ideal $I_1 \in \mathcal{B}$, there may not exist a proper ideal I_2 with $I_1 \prec_\beta I_2$. (For example if, in the notation of Example 3.4.5, we have $I_1 = I$ and $\beta = \alpha_4 + \alpha_5$, no such proper ideal I_2 exists.) However, if such an I_2 exists, it is necessarily unique. There is an analogous statement in the case where $I_2 \prec_\beta I_1$.

4. Algebras of operators on heaps

Using the formalism developed in Section 3, we can now introduce certain linear operators associated to full heaps. This algebra of linear operators can then be equipped with a Lie bracket, which allows heap-theoretic techniques to be applied to Lie algebras.

4.1 Raising and lowering operators.

Definition 4.1.1. Let $\varepsilon : E \to \Gamma$ be a full heap. Let $\mathcal{B}$ be the set of proper ideals of E, and let V_E be the k-vector space with $\mathcal{B}$ as a basis. Let L be a finite convex subheap of E. We define linear operators X_L, Y_L and H_L on V_E as follows:

$$X_L(v_I) = \begin{cases} v_{I \cup L} & \text{if } L \succ I, \\ 0 & \text{otherwise,} \end{cases}$$

$$Y_L(v_I) = \begin{cases} v_{I \setminus L} & \text{if } L \prec I, \\ 0 & \text{otherwise,} \end{cases}$$

$$H_L(v_I) = \begin{cases} v_I & \text{if } L \prec I \text{ and } L \not\succ I, \\ -v_I & \text{if } L \succ I \text{ and } L \not\prec I, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 4.1.2. Let $\varepsilon : E \to \Gamma$ be a full heap. For each vertex p of Γ, we define linear operators on V_E given by:

$$X_p = \sum_{L \in \mathcal{L}_p(E)} X_L;$$

$$Y_p = \sum_{L \in \mathcal{L}_p(E)} Y_L;$$

$$H_p = \sum_{L \in \mathcal{L}_p(E)} H_L.$$

It may appear at first that the sums in Definition 4.1.2 are infinite. However, on any given basis element of V_E, all but at most one of the terms act as zero, even if Γ has infinitely many vertices, so the definition is sound.

4.2 Loop algebras. The operators above can be regarded as generators for a Lie subalgebra $\mathfrak{g}_E$ of $\mathfrak{gl}(V_E)$. The latter Lie algebra turns out to be an interesting one.

Theorem 4.2.1. *Let $\varepsilon : E \to \Gamma$ be a full heap over a Dynkin diagram Γ associated to a Kac–Moody algebra of untwisted affine type. Then the Lie algebra $\mathfrak{g}_E$ (over $\mathbb{C}$) is isomorphic to the loop algebra of the Kac–Moody algebra.*

Proof: This follows from [3, Theorem 7.10]. □

Remark 4.2.2. Loop algebras are so called because of their connections with smooth maps from the circle S^1 to the simple Lie algebra $\mathfrak{g}$. They have no direct connection to the loops that are familiar in the theory of nonassociative algebras.

Remark 4.2.3. An alternative approach to proving the faithfulness of the representation of the loop algebra in Theorem 4.2.1 is to use the fact, proved in [6, Lemma 8.6], that loop algebras have no nontrivial ideals with finite dimensional quotients.

The only known examples of full heaps over *finite* graphs are heaps over the Dynkin diagrams of affine Kac–Moody algebras. Only two of these examples involve twisted algebras, and we expect the theorem to be extendable to these cases.

The derived Kac–Moody algebra, that is, the derived algebra $\mathfrak{g}'$ of the Kac–Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$, may be constructed from the (infinite dimensional) loop algebra by passing to a universal central extension, which increases the dimension by one. (This construction is described in detail in [6, §7].) This derived algebra can be described directly by generators and relations, using the *Serre presentation*. This is given as follows.

Theorem 4.2.4. *Let A be a symmetrizable generalized Cartan matrix. The derived Kac–Moody algebra $\mathfrak{g} = \mathfrak{g}'(A)$ corresponding to A is the Lie algebra over $\mathbb{C}$ generated by elements $\{e_i, f_i, h_i : i \in \Gamma\}$ indexed by the vertices of the Dynkin diagram Γ, subject to the defining relations*

$$\begin{aligned} [h_i, h_j] &= 0, \\ [h_i, e_j] &= A_{ij} e_j, \\ [h_i, f_j] &= -A_{ij} f_j, \\ [e_i, f_j] &= \delta_{ij} h_i, \\ \underbrace{[e_i, [e_i, \cdots [e_i,}_{1-A_{ij} \text{ times}} e_j] \cdots]] &= 0, \\ \underbrace{[f_i, [f_i, \cdots [f_i,}_{1-A_{ij} \text{ times}} f_j] \cdots]] &= 0, \end{aligned}$$

where δ is the Kronecker delta.

PROOF: This is a special case of [6, Theorem 9.11]. □

The loop algebra may be obtained from the algebra $\mathfrak{g}'(A)$ of Theorem 4.2.4 by quotienting out the one-dimensional centre. Furthermore, the images of the generators e_i, f_i and h_i in the quotient may be respectively identified with the operators X_i, Y_i and H_i of Definition 4.1.2. It is therefore possible to obtain a presentation of the loop algebra by adding relations to the Serre presentation.

The Kac–Moody algebra itself may also be reconstructed from its derived algebra $\mathfrak{g}'(A)$ by adding an extra derivation, which has the effect of increasing the dimension by one. This derivation may also be described in terms of the full heap [3, §7], although the universal central extension apparently cannot.

One main reason loop algebras are of interest is because of their connection with the simple Lie algebras over $\mathbb{C}$. The latter are some of the most well-known nonassociative structures. They were classified around 1900 into four infinite families (A_n, B_n, C_n, D_n) together with five exceptional cases (E_6, E_7, E_8, F_4 and G_2). Of course, part of the work involved in the classification involves constructing the algebras.

It turns out that a loop algebra $\mathfrak{g}$ associated to an untwisted Kac–Moody algebra is of the form

$$\mathfrak{g} \cong \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} \mathfrak{g}_0,$$

where t is an indeterminate and $\mathfrak{g}_0$ is a simple Lie algebra over $\mathbb{C}$.

The Lie bracket on $\mathfrak{g}$ is given by

$$[P \otimes x, Q \otimes y] := PQ \otimes [x, y].$$

In these cases, there is a distinguished vertex called 0, and the algebra $\mathfrak{g}_0$ is the subalgebra of $\mathfrak{g}$ generated by the operators X_p, Y_p and H_p for $p \neq 0$.

The full heaps approach can be used to construct all but three of these algebras directly, the exceptions being types E_8, F_4 and G_2.

The heaps covered by Theorem 4.2.1 can be shown to be in natural bijection with the *minuscule representations* for the corresponding simple Lie algebras. In type A, this result follows from the classification of full heaps over Dynkin diagrams of type affine A_l given in Z.S. McGregor-Dorsey in his M.A. thesis [9]. We plan to publish separately a conceptual general proof of this bijection, but we remark that it is also possible to establish the bijection in types other than A by using a brute force argument.

If $\varepsilon : E \rightarrow \Gamma$ is a full heap over an untwisted affine Kac–Moody algebra, then it turns out that the automorphism group of E (in the category $\mathbf{Heap}(\Gamma)$) is infinite cyclic. The indeterminate t mentioned above, and its inverse, correspond to the upward and downward shifts suggested by the dashed boxes.

More precisely, if ϕ is the generator of the aforementioned infinite cyclic group corresponding to an upward shift, and if I is a proper ideal of E, then $\phi(I)$ is a proper ideal of E containing I. Furthermore, the finite convex subheap $L = \phi(I) \backslash I$ has content δ, where δ is nothing other than the lowest positive *imaginary root* of the Kac–Moody algebra.

4.3 Chevalley bases from full heaps. A natural question is to try to describe explicitly the subset of $\mathfrak{gl}(V_E)$ corresponding to the Lie algebra generated by the operators X_p, Y_p and H_p. One might guess that such a basis might include elements of the form

$$\sum_{L \in \mathcal{L}_\alpha(E)} X_L$$

for suitable α, but in general these elements do not even lie in the algebra.

However, it does turn out to be possible to construct a Chevalley basis all of whose elements are of the form

$$\sum_{L \in \mathcal{L}_\alpha(E)} \pm X_L$$

for suitable choices of signs. In order to explain the sign choices, we need to introduce the notion of the parity of a heap. We first tackle the simply laced case, which uses an idea due to Wildberger [13].

Definition 4.3.1. Let $\varepsilon : E \to \Gamma$ be a full heap, and assume that Γ is simply laced (i.e., has no multiple edges). Fix an orientation of Γ, and write $p \to p'$ if there is an arrow from vertex p to vertex p'.

Let F be a finite convex subheap of E, and let $\kappa(F)$ be the number of pairs $(\alpha, \beta) \in F \times F$ such that both

(i) $\alpha > \beta$ and
(ii) either $\varepsilon(\alpha) = \varepsilon(\beta)$ or $\varepsilon(\alpha) \to \varepsilon(\beta)$.

The *parity*, $\varepsilon(F)$, of the heap F is defined to be $(-1)^{\kappa(F)}$.

Theorem 4.3.2. *Let $\varepsilon : E \to \Gamma$ be a full heap over the Dynkin diagram of a simply laced untwisted affine Kac–Moody algebra.*

A Chevalley basis for the image of the simple Lie algebra $\mathfrak{g}_0$ is given by the union of the following three sets:

(i) $\{H_p : p \neq 0\}$;
(ii) $\{X_\alpha = \sum_{L \in \mathcal{L}_\alpha(E)} \varepsilon(L) X_L : \alpha \in \Delta^+\}$;
(iii) $\{Y_\alpha = \sum_{L \in \mathcal{L}_\alpha(E)} \varepsilon(L) Y_L : \alpha \in \Delta^+\}$.

The indexing set Δ^+ is the set of positive roots for the simple Lie algebra.

A basis for the image of the loop algebra is obtained from the above basis by composing with periodic upward and downward shifts (*according to the interpretation of the indeterminate t*).

Proof: This is proved in [3, Theorem 6.7]. □

What is surprising about the previous basis is not so much that the structure constants are integers, but that the linear span of the basis is closed under the Lie bracket. The proof of this relies on some interesting properties of the parity function.

Lemma 4.3.3. *Let $\varepsilon : E \to \Gamma$ be a full heap over a simply laced untwisted affine Lie algebra. Let α and β be positive roots for the corresponding simple Lie algebra $\mathfrak{g}_0$ with the property that $\alpha + \beta$ is also a positive root.*

Suppose that I and J are proper ideals of E such that $I \prec_\alpha J$. Then either

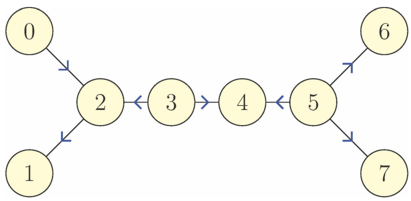
(i) *there exists $I' \in \mathcal{B}(E)$ such that $I' \prec_\beta I$, or*
(ii) *there exists $J' \in \mathcal{B}(E)$ such that $J' \succ_\beta J$,*

but not both.

Furthermore, given any convex finite subheap $L \in \mathcal{L}_{\alpha+\beta}(E)$, there is a unique way to decompose $L = L' \cup L''$ as a disjoint union of finite convex subheaps $L' \in \mathcal{L}_\alpha(E)$ and $L'' \in \mathcal{L}_\beta(E)$ such that either L' is an ideal of L and L'' is a filter of L, or vice versa.

Proof: See [3, Corollary 5.5]. □

Example 4.3.4. Type $\widetilde{D}_7$ is an example of a simply laced untwisted affine Dynkin diagram. Here is an example of an orientation on the diagram.



In the root system of type D_7, which corresponds to a simple Lie algebra, it can be shown that

$$\alpha = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_7$$

and

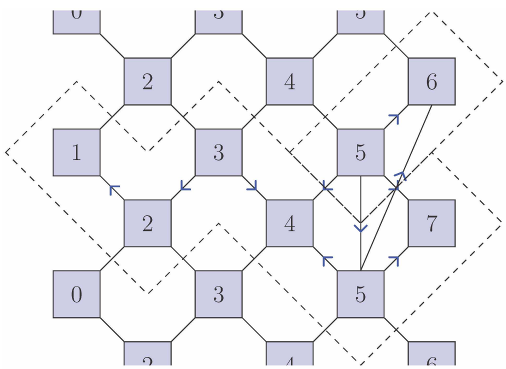
$$\beta = \alpha_5 + \alpha_6$$

are roots; furthermore,

$$\gamma = \alpha + \beta = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_5 + \alpha_6 + \alpha_7$$

is also a root.

The next figure shows a full heap $\varepsilon : E \to \widetilde{D}_7$, and a finite convex subheap L of E in which L has an ideal L' (the lower dashed region) of content α and a filter L'' (the upper dashed region) of content β.



Note that the reason for the downward pointing arrow between the two elements labelled 5 is part (ii) of Definition 4.3.1.

There are two downward pointing arrows in the L' region. This corresponds to the fact that $\varepsilon(L') = (-1)^2 = 1$.

There are no downward pointing arrows in the L'' region, so $\varepsilon(L'') = (-1)^0 = 1$.

There are five downward pointing arrows overall, so $\varepsilon(L) = (-1)^5 = -1$.

There are three downward pointing arrows crossing from the L'' to the L' region. This means that $\varepsilon(L')\varepsilon(L'')\varepsilon(L) = (-1)^3 = -1$.

The quantity $\varepsilon(L')\varepsilon(L'')\varepsilon(L)$ in Example 4.3.4 (which involves the region-crossing downward pointing arrows) turns out to be important. Furthermore, it may be defined solely in terms of the roots α and β by defining

$$\varepsilon(\alpha_p, \alpha_q) = \begin{cases} -1 & \text{if } p = q \text{ or there is an arrow from } p \text{ to } q, \\ 1 & \text{otherwise,} \end{cases}$$

and extending by bimultiplicativity to the root lattice (that is, the $\mathbb{Z}$-span of the α_p). We then have

$$\varepsilon(L')\varepsilon(L'')\varepsilon(L) = \varepsilon(\beta, \alpha).$$

This latter quantity may also be written as $\varepsilon(L'', L')$; it is important in this case that L is the disjoint union of L' and L''.

Remark 4.3.5. The function ε on pairs of roots is called *Kac's asymmetry function*, and it appears in [6, §7.8]. The function has the key property that if α, β and $\alpha + \beta$ are positive roots, then $\varepsilon(\beta, \alpha) = -\varepsilon(\alpha, \beta)$.

Several identities involving the operators X_α and Y_α can be expressed easily in terms of Kac's asymmetry function. In order to do this concisely, it is convenient to define the operator E_α to be X_α if $\alpha > 0$, and $-Y_{-\alpha}$ if $\alpha < 0$. The next result is a special case of [6, (7.8.5)], and a complete statement and proof in the context of full heaps may be found in [3, Proposition 5.4].

Proposition 4.3.6. *Let $\varepsilon : E \to \Gamma$ be a full heap over a simply laced untwisted affine Lie algebra. Let α and β be roots for the corresponding simple Lie algebra $\mathfrak{g}_0$, and suppose that $\alpha \neq -\beta$. Then we have*

$$[E_\alpha, E_\beta] = \begin{cases} \varepsilon(\alpha, \beta) E_{\alpha+\beta} & \text{if } \alpha + \beta \text{ is a root,} \\ 0 & \text{otherwise.} \end{cases}$$

□

The full heap in Example 4.3.4 gives rise to a representation of the simple Lie algebra of type D_7 on the space V_E. This representation turns out to be isomorphic to a countably infinite direct sum of isomorphic irreducible representations. Furthermore, the irreducible representation arising is one of the *spin representations*, of dimension $2^{7-1} = 64$. The other spin representation can be obtained by exchanging the roles of the labels 6 and 7 in the heap (in other words, twisting by a graph automorphism).

The traditional construction of the spin representations involves Clifford algebras [1, §13.5], but the construction here shows that it is not necessary to involve Clifford algebras.

Remark 4.3.7. The full heaps over the Dynkin diagram of type $\widetilde{E}_6$ give rise to the two 27-dimensional representations of the simple Lie algebra of type E_6. This should have something to do with the 27-dimensional exceptional Jordan algebra, also known as the Albert algebra. It would be very interesting to have an explicit connection between heaps and the Albert algebra.

5. The non simply laced case

Kac also introduced a version of the asymmetry function for non simply laced Lie algebras. In this case, the definition is much less transparent, but we now explain how the theory of heaps may be used to explain why the more general definition is still natural.

5.1 Folding.

Definition 5.1.1. Let $\varepsilon : E \to \Gamma$ be a full heap, and let $\mu : \Gamma \to \Gamma$ be a graph automorphism of order 2 with the property that for any vertex p of Γ, p and $\mu(p)$ are not adjacent vertices.

We then construct a new graph $\overline{\Gamma}$, whose vertices are the orbits of vertices of Γ under the action of $\{\mathrm{id}, \mu\}$.

The vertices $\bar{p}$ and $\bar{q}$ are joined if p and q are adjacent. If q is adjacent to both the distinct vertices p and $\mu(p)$, we install a double edge between q and p with an arrow pointing towards $\bar{p}$. (It is possible for this procedure to result in a double edge with two arrows in opposite directions.)

We say that Γ *folds* to $\overline{\Gamma}$. We denote the corresponding projection map by π and call it a *folding*.

Proposition 5.1.2. *If $\varepsilon : E \to \Gamma$ is a full heap over a simply laced untwisted affine Dynkin diagram and $\pi : \Gamma \to \overline{\Gamma}$ is a folding such that*

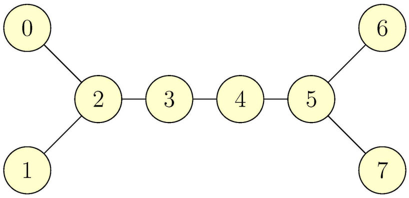
$$\pi \circ \varepsilon : E \to \overline{\Gamma}$$

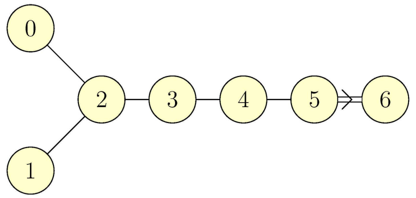
is a heap, then in fact $\pi \circ \varepsilon$ is a full heap.

Proof: This is proved in [3, Proposition 6.1]. □

All known full heaps over non simply laced Dynkin diagrams can be constructed from simply laced examples using Proposition 5.1.2.

Example 5.1.3. The next two figures show Dynkin diagrams of types $\widetilde{D}_7$ and $\widetilde{B}_6$ respectively.

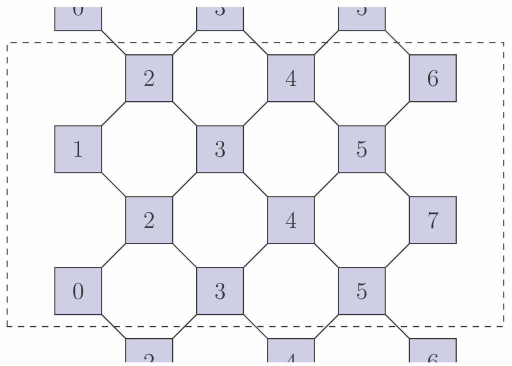




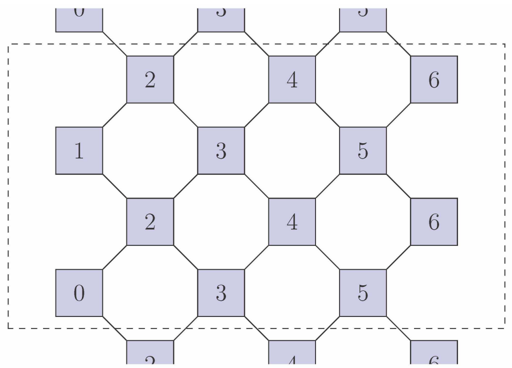
The diagram Γ of type $\widetilde{D}_7$ has an automorphism that exchanges vertices 6 and 7 and fixes all other vertices. Under this automorphism, the diagram of type $\widetilde{D}_7$ folds to the diagram $\overline{\Gamma}$ of type $\widetilde{B}_6$. There is a double arrow pointing towards $\pi(6)$ because 5 is adjacent to the distinct vertices 6 and $\mu(6)$.

The next example shows the effect of the folding in Example 5.1.3 on the level of full heaps.

Example 5.1.4. The following heap is a full heap over the Dynkin diagram of type $\widetilde{D_7}$, with vertices numbered as in Example 5.1.3.



Under the automorphism of Example 5.1.3, this heap folds to the full heap over $\widetilde{B}_6$ shown below.



5.2 Parity.

Definition 5.2.1. Suppose that $\varepsilon : E \to \Gamma$ is a full heap over a simply laced untwisted affine Dynkin diagram and that $\pi \circ \varepsilon : E \to \overline{\Gamma}$ is a folding. We denote these heaps by E and $\pi(E)$ respectively, where π is the induced morphism of heaps.

Suppose also that Γ has an orientation compatible with π; that is, if there is an arrow from p to q, then there is an arrow from $\mu(p)$ to $\mu(q)$. (This means that $\overline{\Gamma}$ inherits an orientation from that of Γ.)

If L is a finite convex subheap of $\pi(E)$, then the *parity*, $\varepsilon(L)$ of L is defined to be $\varepsilon(\pi^{-1}(L))$.

Remark 5.2.2. One would think that parity in the non simply laced case could be defined more directly, but the obvious thing to try does not work.

Using Definition 5.2.1, we obtain an analogue of Theorem 4.3.2, which describes a Chevalley basis for non simply laced simple Lie algebras and loop algebras. More precisely, Theorem 4.3.2 can be generalized by dropping the "simply laced" requirement. This result is also proved in [3, Theorem 6.7].

The relation between roots and ideals and filters of finite convex subheaps is more complicated than in the simply laced case. For example, the simple Lie algebra of type B_6 has positive roots

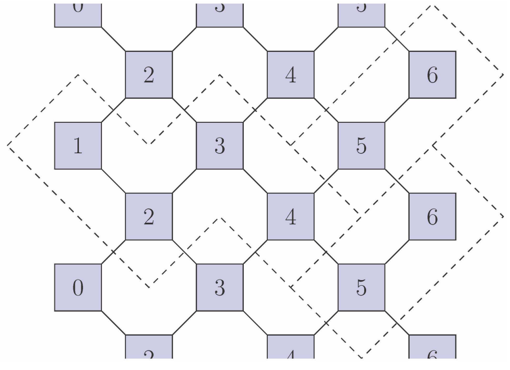
$$\alpha = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$$

and

$$\beta = \alpha_5 + \alpha_6.$$

Not only is $\alpha + \beta$ a root, but so is $\alpha + 2\beta$; this is behaviour we do not see in simply laced cases.

Example 5.2.3. On the level of heaps, we find that a heap L of content $\alpha + 2\beta$ has an ideal of content β *and* a filter of content β. This is shown in the following figure.



Definition 5.2.4. Suppose that the finite convex subheap L has an ideal L' of content α and a filter L'' of content β, and furthermore, that L is the disjoint union of L' and L''. As in the simply laced case, we may define

$$\varepsilon(L'', L') = \varepsilon(L')\varepsilon(L'')\varepsilon(L).$$

This induces an asymmetry function $\varepsilon(\beta, \alpha) = \varepsilon(L'', L')$ on the root lattice.

Unlike the situation of the simply laced case, it is somewhat awkward to define this asymmetry function solely in terms of the roots. The concept of folding heaps casts some light on what is going on here.

Using the more general asymmetry function, one can state an analogue of Proposition 4.3.6 in the non simply laced case. The next result is a special case of [6, (7.9.3)]. The full details of the correspondence with Kac's results may be found in [3, Proposition 6.5].

Proposition 5.2.5. *Let $\varepsilon : E \to \Gamma$ be a full heap over a non simply laced untwisted affine Lie algebra. Let α and β be roots for the corresponding simple Lie algebra $\mathfrak{g}_0$, suppose that $\alpha \neq -\beta$, and let E_α and E_β be the operators of Proposition* 4.3.6. *Let p be the largest integer such that $\alpha - p\beta$ is a root. Then we have*

$$[E_\alpha, E_\beta] = \begin{cases} (p+1)\varepsilon(\alpha,\beta)E_{\alpha+\beta} & \text{if } \alpha+\beta \text{ is a root,} \\ 0 & \text{otherwise.} \end{cases}$$

□

The full heap in Example 5.2.3 gives rise to a representation of the simple Lie algebra of type B_6 on the space V_E. As in Example 4.3.4, this representation is isomorphic to a countably infinite direct sum of isomorphic irreducible representations. Again, the irreducible representation arising is the (unique) *spin representation*, of dimension $2^6 = 64$. As in type D_l, the traditional construction of the spin representations in type B_l involves Clifford algebras.

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