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CONDITIONS FOR BIMODALITY AND MULTIMODALITY OF A MIXTURE OF TWO UNIMODAL DENSITIES

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Conditions for bimodality of mixtures of two unimodal distributions are investigated in some special cases. Based on general characterizations, explicit criteria for the parameters are derived for mixtures of two Cauchy, logistic, Student, gamma, log-normal, Gumbel and other distributions.

Keywords: bimodality, unimodality, multimodality, mixture of two unimodal distributions

AMS Subject Classification: 62E10, 60E05

1. INTRODUCTION

In some practical situations, one has to deal with a mixture of two distributions. It is important to know whether the resulting distribution is unimodal or bimodal. For example, the efficiency of some statistical methods could be affected if the density has two local maxima instead of one. Some theoretical properties hold only for unimodal distributions as well, see [8]. Intuitively, the shape of a mixture density varies depending on the values of its parameters. Therefore, one would like to have a characterization of the parametric space with respect to modality.

There are several papers devoted to the modality issue. However, most of the authors focus solely on a special case of a mixture of two normal densities, see for example [1, 3, 5, 6, 10]. A precise specification when bimodality occurs is derived in [9].

A more general case of a mixture of two unimodal densities is considered in [7], where the component densities are assumed to satisfy only certain weak prerequisites. The necessary and sufficient conditions for their mixture to be unimodal are provided. However, these constraints are rather complex and they are expressed only in a quite nonintuitive form, cf. Theorem 1 in our Section 2. Moreover, the shape of a mixture distribution is not described in a non-unimodal case.

In this paper, we consider a distribution with a density g which is a mixture of two unimodal distributions with densities f_1, f_2, i. e.

$$g(x) = pf_1(x) + (1-p)f_2(x), \; x \in \mathbb{R}, \; p \in (0,1).$$

The component densities f_1, f_2 are of some known parametric form, for example Cauchy, logistic, log-normal, gamma etc. Values p and $1-p$ are their weights and we refer to p as a mixture proportion.

We follow the main ideas from [7] and we add the necessary and sufficient conditions for bimodality of a mixture of two unimodal distributions in Section 2. A generalization for a multimodality situation is provided as well. Results for particular parametric choices of the densities f_1 and f_2 are derived and summarized in Section 3. Some final remarks and discussion on the obtained results are given in Section 4.

2. GENERAL CRITERIA FOR UNIMODALITY AND BIMODALITY

Modality properties of a mixture of two general unimodal distributions are investigated in this section. We restrict ourselves only to distributions with continuous densities because such formulations are sufficient for the applications in Section 3.

First, we introduce the following result published in [7].

Theorem 1. Let f_1, f_2 be continuous unimodal densities. Let f_1 have a unique mode at a point M_1 and let f_2 have a unique mode at a point M_2, where $M_1 < M_2$. Assume that f_1, f_2 are differentiable on the interval (M_1, M_2).

Let $E = \{x \in (M_1, M_2) : f_1'(x) \neq 0 \text{ or } f_2'(x) \neq 0\}$ and for $x \in E$ define

$$\phi(x) = \left| \frac{f_1'(x)}{f_2'(x)} \right|,$$

where we set $\phi(x) = \infty$ if $f_1'(x) \neq 0$ and $f_2'(x) = 0$.

A mixture with the density $g = pf_1 + (1-p)f_2$ is unimodal for all $p \in (0,1)$ if and only if the function $\phi : E \to [0, \infty]$ is nondecreasing.

If ϕ is not nondecreasing, let $p(f_1, f_2)$ stand for a set of all $p \in (0,1)$ such that the mixture $g = pf_1 + (1-p)f_2$ is not unimodal. Then $p(f_1, f_2)$ is equal to the union of all open intervals $\big((1+\phi(x))^{-1}, (1+\phi(y))^{-1}\big)$ over all pairs $x, y \in E$ such that $x < y$ and $\phi(y) < \phi(x)$.

Theorem 1 provides the conditions for unimodality of a mixture of two general unimodal distributions. However, it does not describe the situation when a mixture is not unimodal. Generally, a non-unimodal mixture of two unimodal densities does not have to be necessarily bimodal as it is shown in the following example.

Example 1. Consider unimodal densities f_1, f_2 defined as

$$f_1(x) = \begin{cases} \frac{1}{29}(x+6) & x \in [-6,0], \\ \frac{1}{29}(-2x+6) & x \in (0,1], \\ \frac{1}{29}(-x+5) & x \in (1,2], \\ \frac{1}{29}(-2x+7) & x \in (2,3], \\ \frac{1}{29}(-x+4) & x \in (3,4], \\ 0 & \text{otherwise}, \end{cases} \qquad f_2(x) = \begin{cases} \frac{1}{29}x & x \in [0,1], \\ \frac{1}{29}(2x-1) & x \in (1,2], \\ \frac{1}{29}(x+1) & x \in (2,3], \\ \frac{1}{29}(2x-2) & x \in (3,4], \\ \frac{1}{29}(10-x) & x \in (4,10], \\ 0 & \text{otherwise}. \end{cases}$$

It can be easily checked that f_1, f_2 satisfy $f_2(x) = f_1(4-x)$. Figure 1 shows their mixture $g = \big(1/2\big)\, f_1 + \big(1/2\big)\, f_2$. Since g has three local maxima, it is neither unimodal nor bimodal.

Note that it would be possible to define in a similar way differentiable f_1, f_2 with the same property, i.e. with a "three-modal" mixture g.

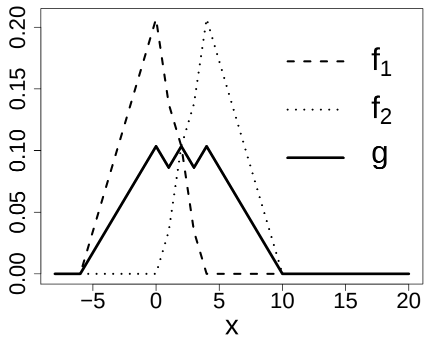

Fig. 1. Mixture of two unimodal densities with three local maxima.

Obviously, we need to describe the shape of a mixture in a non-unimodality situation. The following Theorem 2 claims that under some additional assumptions, the resulting mixture is always either unimodal or bimodal.

Theorem 2. Let f_1, f_2 be unimodal densities. Assume that f_1 has a unique mode at a point M_1 and f_2 has a unique mode at a point M_2, where $M_1 < M_2$. Let f_1 and f_2 be differentiable on some open interval I, $[M_1, M_2] \subset I$.

Let the function $\phi(x) = |f_1'(x)/f_2'(x)|$ be continuous on (M_1, M_2) and let $\lim_{x\to M_1+} \phi(x) = 0$ and $\lim_{x\to M_2-} \phi(x) = \infty$. Assume that there exist points x_1, x_2 such that $M_1 < x_1 < x_2 < M_2$ and the function ϕ is increasing on the interval (M_1, x_1), decreasing on (x_1, x_2) and again increasing on (x_2, M_2).

A mixture with the density $g = pf_1 + (1-p)f_2$ is bimodal if and only if $p \in (p_1, p_2)$ where
$$\frac{1}{p_i} = 1 + \phi(x_i), \quad i = 1, 2.$$

The density g is unimodal if $p \in (0, p_1] \cup [p_2, 1)$.

P r o o f. Recall that $p(f_1, f_2)$ is the set of all mixture proportions $p \in (0,1)$ such that the corresponding density $g = pf_1 + (1-p)f_2$ is not unimodal. We show that g is bimodal for all $p \in p(f_1, f_2)$.

In view of Theorem 1, the set $p(f_1, f_2)$ is equal to the union of all open intervals $((1+\phi(x))^{-1}, (1+\phi(y))^{-1})$ over all pairs $x, y \in E$ such that $x < y$ and $\phi(y) < \phi(x)$. Since the function ϕ is continuous on (M_1, M_2) and it is decreasing only on the interval (x_1, x_2) and increasing otherwise, the set $p(f_1, f_2)$ is equal to the interval (p_1, p_2), where $p_i = (1 + \phi(x_i))^{-1}$, $i = 1, 2$.

Let $p \in (p_1, p_2)$. The corresponding mixture g is clearly nondecreasing on the interval $(-\infty, M_1]$ and nonincreasing on $[M_2, \infty)$. Since M_1 is the only mode of the density f_1, there exists $\varepsilon_1 > 0$ such that f_1 is increasing on $(M_1 - \varepsilon_1, M_1)$. Therefore, g is increasing on $(M_1 - \varepsilon_1, M_1)$ as well. Similarly, there exists $\varepsilon_2 > 0$ such that g is decreasing on $(M_2, M_2 + \varepsilon_2)$. Hence, the density g reaches its local extremes only on the interval $[M_1, M_2]$.

If $y \in [M_1, M_2]$ is a local extreme of g then necessarily $g'(y) = 0$. Let us look for all the points y such that $g'(y) = 0$. The condition

$$g'(y) = pf_1'(y) + (1 - p)f_2'(y) = 0$$

can be equivalently rewritten using the definition of ϕ as $1/p = \phi(y) + 1$. According to our assumptions, the function ϕ is continuous on the interval (M_1, M_2), $\lim_{x \to M_1+} \phi(x) = 0$, and $\lim_{x \to M_2-} \phi(x) = \infty$. Furthermore, ϕ has a local maximum at the point x_1 and a local minimum at the point x_2. Therefore, for any fixed $q \in (\phi(x_2), \phi(x_1))$ there exist exactly three points y_1, y_2, y_3 such that $M_1 < y_1 < x_1 < y_2 < x_2 < y_3 < M_2$ and $q = \phi(y_i)$, $i = 1, 2, 3$. We consider $p \in (p_1, p_2)$ where $p_i = (1 + \phi(x_i))^{-1}$, $i = 1, 2$. Therefore, $1/p - 1 \in (\phi(x_2), \phi(x_1))$ and there exist exactly three different points $y_1 < y_2 < y_3$ such that $1/p = \phi(y_i) + 1$, $i = 1, 2, 3$. As we have shown above, this is equivalent to $g'(y_i) = 0$, $i = 1, 2, 3$.

Since the density g is not unimodal for the chosen p, it has to have at least two local maxima. We have shown that the function g has exactly three stationary points and therefore, it has two local maxima and one local minimum. Hence, g is bimodal. □

Theorem 2 ensures that a non-unimodal mixture with more than two local maxima cannot occur under the specified assumptions. This means that the properties required for f_1, f_2 and ϕ eliminate cases as the one presented in Example 1. In particular, the desired course of the function ϕ is important.

Undoubtedly, it would be possible to work with some weaker assumptions in Theorem 2. For example, the assumption about differentiability of f_1, f_2 on I, $[M_1, M_2] \subset I$ could be weakened, see Remark 1. However, the presented form is sufficient for the applications in the next section.

A discrete analogue of Theorem 1 is derived in [7] and it would be possible to formulate an analogue of Theorem 2 for a mixture of two unimodal discrete distributions as well. In this paper, we deal with continuous distributions and therefore, this topic is not elaborated here. Some results for a mixture of two Poisson and two binomial distributions can be found in [4].

Theorem 2 can be generalized to describe multimodality of a mixture of two unimodal densities. The given proof indicates that a K-modal, $K \geq 2$, mixture can occur if the function ϕ has "enough" (at least $K - 1$) local maxima. This is formulated more precisely in the following theorem.

Theorem 3. Let f_1, f_2 be the same as in Theorem 2. Let $\phi = |f_1'/f_2'|$ be continuous on (M_1, M_2) and let $\lim_{x \to M_1+} \phi(x) = 0$ and $\lim_{x \to M_2-} \phi(x) = \infty$. Assume that there exist $K \in \mathbb{N}$, $K \geq 1$, and points $x_0, \ldots, x_{2K+1}$, $x_0 = M_1$ and $x_{2K+1} = M_2$, such that $x_i < x_{i+1}$, $i = 0, \ldots, 2K$, and the function ϕ is increasing on all the intervals (x_{2i}, x_{2i+1}), $i = 0, \ldots, K$, separately, and decreasing on (x_{2i-1}, x_{2i}), $i = 1, \ldots, K$, separately. Define

$$\begin{aligned} u_1 &= \min\{\phi(x_{2i-1}), i = 1, \ldots, K\}, & v_1 &= \max\{\phi(x_{2i-1}), i = 1, \ldots, K\}, \\ u_2 &= \max\{\phi(x_{2i}), i = 1, \ldots, K\}, & v_2 &= \min\{\phi(x_{2i}), i = 1, \ldots, K\} \end{aligned}$$

and let $u_2 < u_1$. Define p_1, p_2, q_1, q_2 as

$$\frac{1}{p_i} = 1 + u_i, \quad \frac{1}{q_i} = 1 + v_i \quad i = 1, 2.$$

A mixture with the density $g = pf_1 + (1-p)f_2$ is $K+1$-modal if and only if $p \in (p_1, p_2)$. If $p \in (q_1, q_2) \smallsetminus (p_1, p_2)$ then g is $k+1$-modal for some $1 \leq k < K$ and g is unimodal if $p \in (0, q_1] \cup [q_2, 1)$.

Proof. We have $v_2 \leq u_2 < u_1 \leq v_1$ and therefore, $0 < q_1 \leq p_1 < p_2 \leq q_2 < 1$. According to our assumptions and in view of Theorem 1, a mixture $g = pf_1 + (1-p)f_2$ is not unimodal if and only if $p \in p(f_1, f_2)$ where

$$p(f_1, f_2) = \bigcup_{i=1}^{K} \Big(\frac{1}{\phi(x_{2i-1}) + 1}, \frac{1}{\phi(x_{2i}) + 1} \Big).$$

We assume $u_2 < u_1$ and thus, $p(f_1, f_2)$ is equal to an open interval (q_1, q_2). Hence, g is unimodal for $p \in (0, q_1] \cup [q_2, 1)$.

Let $p \in (p_1, p_2)$. It is shown in the proof of Theorem 2 that g can reach its local extremes only on $[M_1, M_2]$ and the condition $g'(y) = 0$ is equivalent to $1/p = \phi(y)+1$. Notice that $(p_1, p_2) = \bigcap_{i=1}^{K} \big((\phi(x_{2i-1}) + 1)^{-1}, (\phi(x_{2i}) + 1)^{-1} \big)$ holds. If $p \in (p_1, p_2)$ then $1/p \in (\phi(x_{2i}) + 1, \phi(x_{2i-1}) + 1)$ for all $i = 1, \ldots, K$. Similarly as in the proof of Theorem 2 there exist exactly $2K+1$ points $y_1, \ldots, y_{2K+1}$ such that $y_i \in (x_{i-1}, x_i)$ and $1/p = \phi(y_i) + 1$. This means that $g'(y_i) = 0$, $i = 1, \ldots, 2K+1$, and g has $2K+1$ stationary points on (M_1, M_2). It is easy to see that the condition $g'(y) > 0$ is equivalent to $1 + \phi(y) < 1/p$ and $g'(y) < 0$ is equivalent to $1 + \phi(y) > 1/p$. Fix $i = 1, \ldots, K$ and study g at y_{2i}. The function ϕ is decreasing in y_{2i} because $y_{2i} \in (x_{2i-1}, x_{2i})$. Hence, there exists $\varepsilon > 0$ such that $\phi(y) > \phi(y_{2i})$ for all $y \in (y_{2i} - \varepsilon, y_{2i})$ and $\phi(y) < \phi(y_{2i})$ for all $y \in (y_{2i}, y_{2i} + \varepsilon)$. Remind that $\phi(y_{2i}) + 1 = 1/p$ and thus, $1 + \phi(y) > 1/p$ for $y \in (y_{2i} - \varepsilon, y_{2i})$ and $1 + \phi(y) < 1/p$ for $y \in (y_{2i}, y_{2i} + \varepsilon)$. This means that g is decreasing on $(y_{2i} - \varepsilon, y_{2i})$ and increasing on $(y_{2i}, y_{2i} + \varepsilon)$. Hence, g has a local minimum at y_{2i}. One would analogously prove that g reaches its local maximum at y_{2i+1}, $i = 0, \ldots, K$. A mixture g has $K+1$ local maxima and K local minima for $p \in (p_1, p_2)$ and therefore, it is $K+1$-modal.

If $p \in (q_1, q_2) \smallsetminus (p_1, p_2)$ then $1/p$ lies in exactly k open intervals $(\phi(x_{2i}) + 1, \phi(x_{2i-1}) + 1)$ for some $1 \leq k < K$. Similarly as in the previous situation, the function g has $2k+1$ stationary points: $k+1$ local maxima and k local minima. This means that g is $k+1$-modal. □

The assumption $u_2 < u_1$ in Theorem 3 is important. If $u_2 > u_1$ then a $K+1$-modal mixture cannot occur even though the function ϕ has K local maxima. In this case one can obtain only a k-modal mixture for some $2 \leq k \leq K$ or a unimodal mixture.

Remark 1. Let f_1, f_2 be continuous unimodal densities with unique modes at M_1 and M_2 respectively and let $M_1 < M_2$. Assume that f_1, f_2 are both differentiable on $(M_1, M_2) \smallsetminus I$, where I is a finite set and let the function $\phi(x) = -f_1'(x)/f_2'(x)$ be defined and continuous for $x \in (M_1, M_2) \smallsetminus I$.

If a mixture $g = pf_1 + (1-p)f_2$ has a sharp local minimum at a point $y \in (M_1, M_2)$ then we can find $\varepsilon > 0$ such that g' exists on $(y-\varepsilon, y+\varepsilon) \smallsetminus \{y\}$ and $g'(x) < 0$ for $x \in (y-\varepsilon, y)$ and $g'(x) > 0$ for $x \in (y, y+\varepsilon)$. This is equivalent to $1+\phi(x) > 1/p$ for $x \in (y-\varepsilon, y)$ and $1+\phi(x) < 1/p$ for $x \in (y, y+\varepsilon)$, see Proof of Theorem 3. If ϕ is nondecreasing on $(M_1, M_2) \smallsetminus I$ then a sharp local minimum of g on (M_1, M_2) cannot exist and we can show that g is unimodal for all $p \in (0,1)$. On the other hand, if there exists a point $y \in (M_1, M_2)$ such that ϕ is continuous, decreasing at y, $\phi(y) = 1/p - 1$ or ϕ has a discontinuity at y and $\lim_{x \to y-} \phi(x) > 1/p - 1 > \lim_{x \to y+} \phi(x)$ then g has a sharp local minimum at y. This gives a clue how to modify Theorem 2 and Theorem 3 with weaker assumptions. However, one needs to be careful with functions which reach their local minimum on an interval.

3. SPECIAL CASES

In this section, we apply the general results from Theorem 1 and Theorem 2 to mixtures of some known and widely used unimodal parametric distributions. The obtained criteria for unimodality and bimodality are formulated in the following paragraphs. A detailed derivation is given as a proof only for a mixture of two Cauchy distributions. In the other cases, one can proceed in an analogous way and thus, the assertions are presented without a proof.

3.1. Cauchy distribution

For $a \in \mathbb{R}$, $b > 0$, let $\mathsf{C}(a,b)$ denote the Cauchy distribution with a density $f(x) = (b/\pi) \cdot [b^2 + (x-a)^2]^{-1}$, $x \in \mathbb{R}$. This distribution is unimodal with a unique mode at a. The following assertion describes the shape of a mixture of two Cauchy distributions.

Proposition 1. Let $a_1, a_2 \in \mathbb{R}$, $b_1, b_2 > 0$. Let f_1, f_2 be densities of $\mathsf{C}(a_1, b_1)$ and $\mathsf{C}(a_2, b_2)$ respectively and consider their mixture $g = pf_1 + (1-p)f_2$, $p \in (0,1)$. Define $a = |a_1 - a_2|/b_1$ and $b = b_2/b_1$.

1. If $4a^2 + 3a^4 - 16b^2 + 4a^2b^2 \leq 0$ then g is unimodal for all $p \in (0,1)$.
2. If $4a^2 + 3a^4 - 16b^2 + 4a^2b^2 > 0$ then there exist roots x_1, x_2 of the equation

$$-3ax^4 + (4b^2 + 6a^2 - 4)x^3 - 3a(b^2 + a^2 - 3)x^2 - 6a^2x + a^3 + ab^2 = 0$$

such that $0 < x_1 < a/2 < x_2 < a$. Define

$$\frac{1}{p_i} = 1 + \frac{1}{b}\frac{x_i[b^2 + (x_i - a)^2]^2}{(a - x_i)(1 + x_i^2)^2}, \quad i = 1, 2.$$

Then $0 < p_1 < p_2 < 1$ and g is bimodal if and only if $p \in (p_1, p_2)$. Otherwise, g is unimodal.

Proof. The modality is invariant with respect to a location and scale and therefore, the mixture $g = pf_1 + (1-p)f_2$ has the same modality properties as a mixture $\tilde{g} = p\tilde{f}_1 + (1-p)\tilde{f}_2$, where $\tilde{f}_1$, $\tilde{f}_2$ are densities of $\mathsf{C}(0,1)$ and $\mathsf{C}(a,b)$ respectively. Let us study the shape of $\tilde{g}$ using Theorem 1. For $x \in (0,a)$ we have

$$\phi(x) = \frac{1}{b}\frac{x[b^2+(x-a)^2]^2}{(a-x)(1+x^2)^2} \quad \text{and} \quad \phi'(x) = \frac{[b^2+(x-a)^2]R(x)}{b(a-x)^2(1+x^2)^3},$$

where $R(x) = -3ax^4+(4b^2+6a^2-4)x^3-3a(b^2+a^2-3)x^2-6a^2x+a^3+ab^2$. The sign of $\phi'(x)$ is the same as the sign of the fourth degree polynomial $R(x)$. This polynomial has a negative coefficient standing by the term x^4 and $R(0) = a(a^2+b^2) > 0$, $R(a) = ab(a^2+b^2) > 0$. This implies that R has always two roots lying outside the interval $[0,a]$. Furthermore, R is decreasing at 0, increasing at a and it has a local minimum at $a/2$. The number of the roots of R on $(0,a)$ depends on the value of $R(a/2) = -4a^2-3a^4+16b^2-4a^2b^2$. If $R(a/2) \geq 0$ then R is non-negative on $(0,a)$ and thus, ϕ is non-decreasing. In view of Theorem 1, $\tilde{g}$ is unimodal for all $p \in (0,1)$.

If $R(a/2) < 0$ then R has two distinct roots $x_1 < a/2 < x_2$ on $(0,a)$. In this case, the derivative ϕ' is negative on (x_1,x_2) and positive on the intervals $(0,x_1)$ and (x_2,a). The function ϕ is increasing on $(0,x_1)$, decreasing on (x_1,x_2) and again increasing on (x_2,a). In view of Theorem 2, $\tilde{g}$ is bimodal if and only if $p \in (p_1,p_2)$, where $p_i = [1+\phi(x_i)]^{-1}$, $i = 1,2$. It can be easily verified that $p_1, p_2 \in (0,1)$ and $p_1 < p_2$ since ϕ is decreasing on (x_1,x_2). Hence, the interval (p_1,p_2) is well-defined if $-4a^2-3a^4+16b^2-4a^2b^2 < 0$. □

The inequality $4a^2+3a^4-16b^2+4a^2b^2 > 0$ holds if and only if

$$a > \sqrt{\frac{2}{3}}\sqrt{\sqrt{1+b^4+14b^2}-1-b^2}.$$

Note that the expression $\sqrt{1+b^4+14b^2}-1-b^2$ is positive for every $b > 0$. For a special case $b = 1$ the inequality simplifies to $a > 2/\sqrt{3}$ and one can even obtain an explicit formula for the roots x_1, x_2. The following assertion follows from Proposition 1.

Proposition 2. Let $a_1, a_2 \in \mathbb{R}$, $b > 0$ and define $a = |a_1 - a_2|/b$. Let f_1, f_2 be densities of $\mathsf{C}(a_1,b)$ and $\mathsf{C}(a_2,b)$ respectively. If $a > 2/\sqrt{3}$ then define

$$x_1 = \frac{1}{2}\left[a - \sqrt{4+a^2-\frac{4\sqrt{4+a^2}}{\sqrt{3}}}\right], \quad x_2 = \frac{1}{2}\left[a + \sqrt{4+a^2-\frac{4\sqrt{4+a^2}}{\sqrt{3}}}\right]$$

and

$$\frac{1}{p_i} = 1 + \frac{x_i[1+(x_i-a)^2]^2}{(a-x_i)(1+x_i^2)^2}, \quad i = 1,2.$$

Then x_1, x_2 are real numbers satisfying $0 < x_1 < x_2 < a$ and $0 < p_1 < p_2 < 1$.

A mixture $g = pf_1 + (1-p)f_2$ is bimodal if and only if $a > 2/\sqrt{3}$ and $p \in (p_1,p_2)$ simultaneously. In all other cases, g is unimodal.

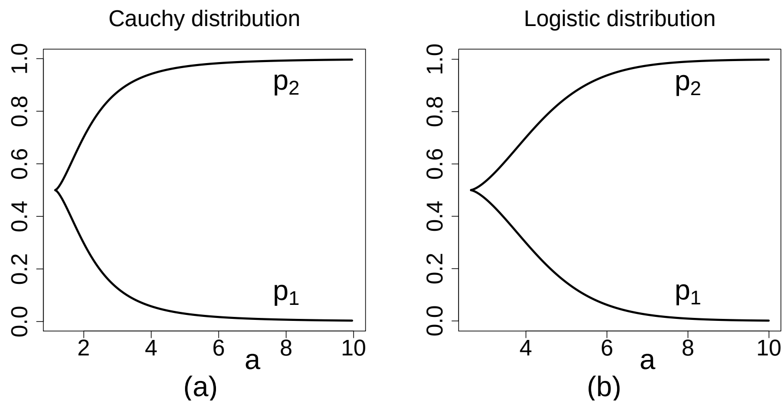

Fig. 2. Boundaries p_1 and p_2 for (a) the Cauchy distribution and (b) the logistic distribution.

Figure 2 (a) illustrates the dependence of the values p_1, p_2 on the parameter $a = |a_1 - a_2|/b > 2/\sqrt{3}$ for a mixture of $\mathsf{C}(a_1, b)$ and $\mathsf{C}(a_2, b)$. As expected, the width of this interval increases with an increasing a. Since the Cauchy distribution is symmetric around its mode and we consider the common scale b, the intervals (p_1, p_2) are always symmetric around $1/2$.

3.2. Logistic distribution

For $a \in \mathbb{R}$ and $b > 0$, let $\mathsf{Logist}(a, b)$ denote the logistic distribution with the density $f(x) = 1/b\mathrm{e}^{-(x-a)/b}[1 + \mathrm{e}^{-(x-a)/b}]^{-2}$, $x \in \mathbb{R}$. This distribution is unimodal with a unique mode at a. The following statement informs about the modality of a mixture of two logistic distributions with a common scale.

Proposition 3. Let $a_1, a_2 \in \mathbb{R}$, $b > 0$. Define $a = |a_2 - a_1|/b$. Let f_1, f_2 be densities of $\mathsf{Logist}(a_1, b)$ and $\mathsf{Logist}(a_2, b)$ respectively. If $a > 2\ln(2 + \sqrt{3})$ then define

$$y_1 = \frac{1}{4}(1 + \mathrm{e}^a - \sqrt{\mathrm{e}^{2a} - 14\mathrm{e}^a + 1}), \quad y_2 = \frac{1}{4}(1 + \mathrm{e}^a + \sqrt{\mathrm{e}^{2a} - 14\mathrm{e}^a + 1})$$

and

$$\frac{1}{p_i} = 1 + \frac{\mathrm{e}^{-a}(y_i - 1)(\mathrm{e}^a + y_i)^3}{(\mathrm{e}^a - y_i)(1 + y_i)^3}, \quad i = 1, 2.$$

Then $0 < p_1 < p_2 < 1$.

A mixture $g = pf_1 + (1 - p)f_2$ is bimodal if and only if $a > 2\ln(2 + \sqrt{3})$ and $p \in (p_1, p_2)$ simultaneously. In all other cases, g is unimodal.

Figure 2 (b) shows graphically the dependence of the boundaries p_1, p_2 on the parameter $a = |a_2 - a_1|/b$.

One can be interested in a mixture of two logistics with different scales as well, i. e. a mixture of $\mathsf{Logist}(a_1, b_1)$ and $\mathsf{Logist}(a_2, b_2)$ for $b_1 \neq b_2$. It is possible to proceed in an analogous way as in the proof of Proposition 1 and study equivalently a mixture of $\mathsf{Logist}(0,1)$ and $\mathsf{Logist}(a,b)$ where $a = |a_1 - a_2|/b_1$ and $b = b_2/b_1$. At one point one needs to investigate the sign of a function

$$\begin{aligned} R(y) =& 1 - b - 4y + y^2 + by^2 + 4b\mathrm{e}^a y^b - e^{2a} y^{2b} - b\mathrm{e}^{2a} y^{2b} - 4b\mathrm{e}^a y^{2+b} \\ &+ 4\mathrm{e}^{2a} y^{1+2b} - \mathrm{e}^{2a} y^{2+2b} + b\mathrm{e}^{2a} y^{2+2b} \end{aligned}$$

on the interval $(\mathrm{e}^{-a/b}, 1)$. However, this is not easy in general because b does not have to be an integer and the course of R could be complicated. Hence, we leave this problem unsolved.

3.3. Student distribution

The Student t-distribution with n degrees of freedom is another important unimodal distribution with a unique mode at the point 0. For $n = 1$ it simplifies to the Cauchy distribution $\mathsf{C}(0,1)$ and it tends to the normal distribution $\mathsf{N}(0,1)$ as n approaches infinity.

Proposition 4. Let $c \in \mathbb{R}$, $c > 0$ and $n \in \mathbb{N}, n \geq 1$. Let f be the density of the Student t-distribution with n degrees of freedom. If $c > 2\sqrt{n/(2+n)}$ then the equation

$$-(n+2)x^4 + 2c(n+2)x^3 - [c^2(2+n) - n(5+n)]x^2 - nc(5+n)x + n(c^2+n) = 0 \quad (1)$$

has two distinct real roots $x_1 < x_2$ on the interval $(0, c)$. Define

$$\frac{1}{p_i} = 1 + \left(\frac{n + x_i^2}{n + (x_i - c)^2}\right)^{-\frac{n+3}{2}} \frac{x_i}{c - x_i}, \quad i = 1, 2.$$

Then $0 < p_1 < p_2 < 1$.

A mixture with the density $g(x) = pf(x) + (1-p)f(x-c)$ is bimodal if and only if $c > 2\sqrt{n/(2+n)}$ and $p \in (p_1, p_2)$ simultaneously. In all other cases, g is unimodal.

The boundaries p_1, p_2 depending on the shift $c > 2\sqrt{n/(2+n)}$ are plotted for several choices of n in Figure 3. Note that for $n = 1$ we get the condition $c > 2/\sqrt{3}$ and the obtained curve is the same as in Figure 2 (a) for the Cauchy distribution. On the other hand, the expression $2\sqrt{n/(2+n)}$ tends to 2 as n approaches infinity and this corresponds to the known condition for a mixture of two normals $\mathsf{N}(0,1)$ and $\mathsf{N}(c,1)$, see [9].

One could also study a mixture of two shifted Student distributions with different degrees of freedom. However, we would need to investigate roots of a fifth-degree polynomial equation depending on three parameters instead of considering the equation (1). This becomes quite complicated in general and thus, we do not handle this problem here.

3.4. Laplace distribution

Let $\mathsf{Laplace}(a,b)$ denote the Laplace distribution with the density $f(x) = 1/(2b) \cdot \cdot \exp\{-|x-a|/b\}$, $x \in \mathbb{R}$. This distribution is unimodal with a unique mode at a.

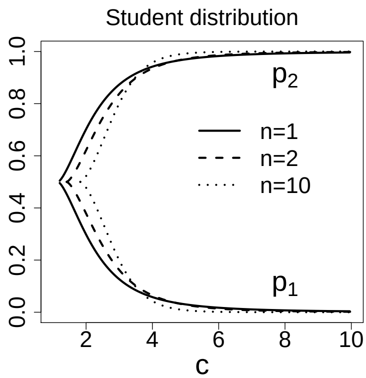

Fig. 3. Boundaries p_1, p_2 for the Student distribution.

Proposition 5. Let $a > 0$, $b > 0$. Let f_1, f_2 be densities of $\mathsf{Laplace}(0,1)$ and $\mathsf{Laplace}(a,b)$ respectively. A mixture $g = pf_1 + (1-p)f_2$ is bimodal if and only if

$$p \in \left(\frac{1}{1 + b^2 \mathrm{e}^{\frac{a}{b}}}, \frac{1}{1 + b^2 \mathrm{e}^{-a}} \right).$$

In all other cases, g is unimodal.

Remark that the assumptions of Theorem 2 are not satisfied for f_1, f_2 from Proposition 5 and the assertion is derived directly from Theorem 1. The function ϕ is decreasing on the whole interval $(0,a)$ and therefore, for any $a > 0, b > 0$ there exists a mixture proportion $p \in (0,1)$ for which the corresponding mixture $g = pf_1 + (1-p)f_2$ is bimodal.

The dependence of the boundaries p_1, p_2 on the parameter a for various choices of $b > 0$ is illustrated in Figure 4 (a). The intervals (p_1, p_2) are symmetric around $1/2$ for $b = 1$ and asymmetric otherwise.

3.5. Gamma distribution

For $a > 0$, $q > 0$, let $\mathsf{G}(a,q)$ denote the Gamma distribution with the density $f(x) = \left[a^q/\Gamma(q)\right] \mathrm{e}^{-ax} x^{q-1}$, $x > 0$, and $f(x) = 0$ otherwise. The parameter q is referred to as a shape and a is a rate. For $q > 1$, $\mathsf{G}(a,q)$ is unimodal with a unique mode at $(q-1)/a$.

First, we look at the modality properties of a mixture of two Gamma distribution with a common shape and different rates.

Proposition 6. Let $q > 0$, $a > b > 0$ and let f_1, f_2 be densities of $\mathsf{G}(a,q)$ and $\mathsf{G}(b,q)$ respectively.

If $q > (a+b)^2(a-b)^{-2}$ define $D = (q-1)[q(a-b)^2 - (a+b)^2]$,

$$x_1 = \frac{(a+b)(q-1) - \sqrt{D}}{2ab}, \quad x_2 = \frac{(a+b)(q-1) + \sqrt{D}}{2ab}$$

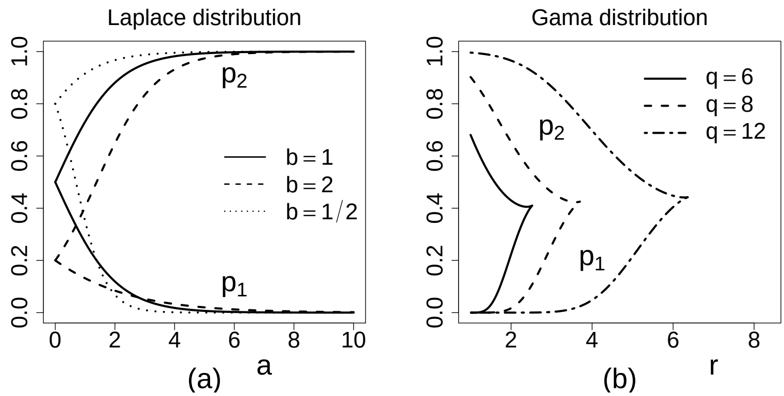

Fig. 4. Boundaries p_1, p_2 for (a) the Laplace distribution and (b) the Gamma distribution.

and

$$\frac{1}{p_i} = 1 + \left(\frac{a}{b}\right)^q \mathrm{e}^{(b-a)x} \frac{ax_i - q + 1}{q - 1 - bx_i}, \quad i = 1, 2.$$

Then $x_1, x_2 \in \mathbb{R}$, $(q-1)/a < x_1 < x_2 < (q-1)/b$ and $0 < p_1 < p_2 < 1$.

A mixture $g = pf_1 + (1-p)f_2$ is bimodal if and only if $q > (a+b)^2(a-b)^{-2}$ and $p \in (p_1, p_2)$ simultaneously. In all other cases, g is unimodal.

The following assertion describes the modality of a mixture of two Gamma distribution with a common rate and different shapes.

Proposition 7. Let $a > 0$ and $q > r > 1$. Let f_1, f_2 be densities of $\mathsf{G}(a, r)$ and $\mathsf{G}(a, q)$ respectively. If r, q are such that $q > 2$ and $r < q - 2\sqrt{q-1} + 1$ define $D = (q - r + 1)^2 - 4(q-1)$,

$$x_1 = \frac{q + r - 3 - \sqrt{D}}{2}, \quad x_2 = \frac{q + r - 3 + \sqrt{D}}{2}$$

and

$$\frac{1}{p_i} = 1 + \frac{\Gamma(q)}{\Gamma(r)} x_i^{r-q} \frac{x_i - r + 1}{q - 1 - x_i}, \quad i = 1, 2.$$

Then $x_1, x_2 \in \mathbb{R}$, $r - 1 < x_1 < x_2 < q - 1$ and $0 < p_1 < p_2 < 1$.

A mixture with the density $g = pf_1 + (1-p)f_2$ is bimodal if and only if $q > 2$, $r < q - 2\sqrt{q-1} + 1$ and $p \in (p_1, p_2)$ simultaneously. If $q \le 2$ or $q - r \le 2\sqrt{q-1} - 1$ or $p \notin (p_1, p_2)$ then g is unimodal.

The boundaries p_1, p_2 are plotted depending on the parameter r for various choices of q in Figure 4 (b). Unlike for Cauchy or logistic distribution in Figure 2, the intervals (p_1, p_2) are not symmetric around $1/2$. Note that a mixture of $\mathsf{G}(a, r)$ and $\mathsf{G}(a, q)$ has the same modality properties as a mixture of $\mathsf{G}(1, r)$ and $\mathsf{G}(1, q)$.

This is why the conditions for bimodality in Proposition 7 do not depend on the parameter a.

An important special case of the Gamma distribution is the χ^2 distribution. Criteria for bimodality of a mixture of two χ^2 distributions with n and m degrees of freedom can be easily obtained from Proposition 7 applied to $\mathsf{G}(1/2, n/2)$ and $\mathsf{G}(1/2, m/2)$.

One could also consider the most general case of a mixture of $\mathsf{G}(a, r)$ and $\mathsf{G}(b, q)$, $a \neq b$, $r \neq q$. However, this situation is quite complex and it is not easy to solve it in general. Hence, we do not deal with this problem here.

3.6. Other distributions

This section briefly presents results for some other distributions, namely for Gumbel, log-normal, Rayleigh and Maxwell. All these distributions are unimodal and asymmetric around their modes. The asymmetry implies that the obtained intervals (p_1, p_2) are not symmetric around $1/2$ similarly as we have seen in Figure 4 (b) for the Gamma distribution.

The Gumbel distribution is known as a distribution of extreme values. Its density $f(x) = \exp\{-x - \mathrm{e}^{-x}\}$, $x \in \mathbb{R}$, is unimodal with a unique mode at 0.

Proposition 8. Let $c > 0$ and let f be the density of the Gumbel distribution. If $c > 2\ln(1+\sqrt{2})$ then define

$$y_1 = \frac{\mathrm{e}^c + 1 - \sqrt{\mathrm{e}^{2c} - 6\mathrm{e}^c + 1}}{4}, \quad y_2 = \frac{\mathrm{e}^c + 1 + \sqrt{\mathrm{e}^{2c} - 6\mathrm{e}^c + 1}}{4}$$

and

$$\frac{1}{p_i} = 1 + \frac{1}{\mathrm{e}^c} \frac{y_i - 1}{\mathrm{e}^c - y_i} \exp\left\{\frac{1}{y_i}(\mathrm{e}^c - 1)\right\}, \quad i = 1, 2.$$

Then $0 < p_1 < p_2 < 1$.

A mixture $g(x) = pf(x) + (1-p)f(x-c)$ is bimodal if and only if $c > 2\ln(1+\sqrt{2})$ and $p \in (p_1, p_2)$ simultaneously. In all other cases, g is unimodal.

Let $\mathsf{LN}(a, S)$ stand for the log-normal distribution with parameters $a \in \mathbb{R}$ and $S > 0$. It is a single-tailed unimodal distribution with the density $f(x) = 1/\big(\sqrt{2\pi}Sx\big) \cdot \exp\big\{-(\ln x - a)^2/(2S^2)\big\}$, $x > 0$, and $f(x) = 0$ otherwise. It has a unique mode at $\mathrm{e}^{a - S^2}$.

Proposition 9. Let $a, b \in \mathbb{R}$, $S > 0$ and assume $a < b$. Let f_1, f_2 be densities of $\mathsf{LN}(a, S)$ and $\mathsf{LN}(b, S)$ respectively. If $b - a > 2S$ define

$$y_1 = \frac{a + b - 2S^2 - \sqrt{(b-a)^2 - 4S^2}}{2}, \quad y_2 = \frac{a + b - 2S^2 + \sqrt{(b-a)^2 - 4S^2}}{2}$$

and

$$\frac{1}{p_i} = 1 + \frac{y_i + S^2 - a}{b - S^2 - y_i} \exp\left\{\frac{b^2 - a^2 - 2y_i(b-a)}{2S^2}\right\}, \quad i = 1, 2.$$

Then $0 < p_1 < p_2 < 1$.

A mixture $g = pf_1 + (1-p)f_2$ is bimodal if and only if $b - a > 2S$ and $p \in (p_1, p_2)$ simultaneously. In all other cases, g is unimodal.

For $b > 0$, denote by $\mathsf{Rayleigh}(b)$ the Rayleigh distribution with the density $f(x) = \left(x/b^2\right) \exp\left\{-x^2/(2b^2)\right\}$, $x > 0$, and $f(x) = 0$ otherwise. This distribution is unimodal with a unique mode at b.

Proposition 10. Let $c > b > 0$ and let f_1, f_2 be densities of $\mathsf{Rayleigh}(b)$ and $\mathsf{Rayleigh}(c)$ respectively. If $b^2/c^2 < 5 - 2\sqrt{6}$ define

$$y_1 = \frac{b^2 + c^2 - \sqrt{b^4 - 10b^2c^2 + c^4}}{2}, \quad y_2 = \frac{b^2 + c^2 + \sqrt{b^4 - 10b^2c^2 + c^4}}{2}$$

and

$$\frac{1}{p_i} = 1 + \frac{c^4}{b^4}\frac{y_i - b^2}{c^2 - y_i} \exp\left\{\frac{y_i}{2}\left(\frac{1}{c^2} - \frac{1}{b^2}\right)\right\}, \ i = 1, 2.$$

Then $0 < p_1 < p_2 < 1$.

A mixture $g = pf_1 + (1 - p)f_2$ is bimodal if and only if $b^2/c^2 < 5 - 2\sqrt{6}$ and $p \in (p_1, p_2)$ simultaneously. In all other cases, g is unimodal.

For $a > 0$ denote by $\mathsf{Maxwell}(a)$ the Maxwell distribution with a density $f(x) = 2/\left(a^3\sqrt{2\pi}\right)x^2 \exp\left\{-x^2/(2a^2)\right\}$, $x > 0$, and $f(x) = 0$ otherwise. This distribution is unimodal with a unique mode at $\sqrt{2}a$.

Proposition 11. Let $0 < a < b$ and let f_1 and f_2 be densities of $\mathsf{Maxwell}(a)$ and $\mathsf{Maxwell}(b)$ respectively. If $a/b < \sqrt{2} - 1$ define

$$y_1 = a^2 + b^2 - \sqrt{a^4 - 6a^2b^2 + b^4}, \quad y_2 = a^2 + b^2 + \sqrt{a^4 - 6a^2b^2 + b^4}$$

and

$$\frac{1}{p_i} = 1 + \frac{b^5}{a^5}\frac{y_i - 2a^2}{2b^2 - y_i} \exp\left\{-\frac{y_i}{2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)\right\}, \ i = 1, 2.$$

Then $0 < p_1 < p_2 < 1$.

A mixture $g = pf_1 + (1 - p)f_2$ is bimodal if and only if $a/b < \sqrt{2} - 1$ and $p \in (p_1, p_2)$ simultaneously. In all other cases, g is unimodal.

4. CONCLUDING REMARKS

In this paper, we investigate modality properties of a mixture of two unimodal distributions. Some general results for bimodality and multimodality are presented and based on them we derive explicit unimodality and bimodality conditions for parameters of mixtures in some special cases. The considered distributions are well-known and widely used and their mixtures often occur in practice. Using the derived formulas, it is simple to decide whether a studied mixture distribution is unimodal or bimodal if the parameters are known. However, other applications are possible as well. For example, if a bimodal mixture of two given logistics is needed, one can choose appropriate weights based on our results. Alternatively, the lower bound for the shift between the two component densities of a bimodal mixture of two shifted Student t-distributions can be easily obtained. Explicit modality conditions for a mixture of two unimodal distributions can be useful in some other fields of statistics as well. For instance, a connection between the modality of a mixture of two normal distributions and the shape of the failure rate is investigated in [2]. A similar study could be done also for mixtures of other distributions using the criteria derived in this paper.

There are several topics which can be considered for a further research. For instance, mixtures of two unimodal distributions of different parametric forms are not investigated in this paper. Furthermore, some of the most general situations are left unsolved and some work could be done for a discrete distributions as well. Finally, it would be interesting to study modality in a multivariate case.

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