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A COMPARISON OF TWO FEM–BASED METHODS FOR THE SOLUTION OF THE NONLINEAR OUTPUT REGULATION PROBLEM

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The regulator equation is the fundamental equation whose solution must be found in order to solve the output regulation problem. It is a system of first-order partial differential equations (PDE) combined with an algebraic equation. The classical approach to its solution is to use the Taylor series with undetermined coefficients. In this contribution, another path is followed: the equation is solved using the finite-element method which is, nevertheless, suitable to solve PDE part only. This paper presents two methods to handle the algebraic condition: the first one is based on iterative minimization of a cost functional defined as the integral of the square of the algebraic expression to be equal to zero. The second method converts the algebraic-differential equation into a singularly perturbed system of partial differential equations only. Both methods are compared and the simulation results are presented including on-line control implementation to some practically motivated laboratory models.

Keywords: nonlinear output regulation, singularly perturbed equation, gyroscope

AMS Subject Classification: 93C10

1. INTRODUCTION

The output regulation problem (ORP) is one of the most important problems of the recent control theory and its applications. As a matter of fact, it expresses the most typical practical control problem: the system output should follow a given reference while rejecting undesired disturbances. Specific feature of the output regulation is that the reference to be followed and disturbances to be rejected are generated by the known autonomous system, called as the *exogeneous* one.

The output regulation problem was extensively studied in [5, 6] for linear systems. For nonlinear systems, the problem was first studied in [7], and solutions to the output regulation of nonlinear systems have been presented in [12] using "full information" which includes the measurements of exogenous signals as well as of the system state. The necessary and sufficient conditions for the existence of a local full information solution of the classical output regulation problem are given in [4, 9, 10, 11, 12]; they basically mean that the linearized system is stabilizable and there exists a certain invariant manifold. The classical output regulation via error feedback has been solved in [1, 13] by application of system immersion technique. The plant uncertainty (parameterized by unknown constant parameters) is treated as a special case of exogenous signals and the solution, extended from the error feedback regulation, is referred to as the structurally stable regulation in [1]. For the more detailed study of the topic of the output regulation the interested reader is referred to the recent nicely written monograph [8].

The basic approach, introduced by [12], uses the solution of the so-called regulator equation. This equation is a system of partial differential equations (PDE) combined with the algebraic restriction and its solution may be obtained off-line, in advance during the regulator design, since it requires the data of the model of the plant and of the exosystem only. As a consequence, sophisticated numerical methods and unrestricted computing time are assumed to be available to handle this task. Yet, the solvability of such systems of equations remains a rather complex issue with many open problems. Moreover, the regulator equation does not fit into the usual framework of the partial differential equations – it is a first-order problem on the whole space $\mathbb{R}^\mu$ (μ being the number of independent variables which is equal to the dimension of the exosystem) with singularity in its coefficients.

The existing results concerning solvability of the regulator equation are based on the so-called geometrical approach and require a special structure of the controlled system. The simplest situation is when both the controlled plant and the exosystem are hyperbolically minimum phase (see [12]), while [10] shows that the regulator equation can be reduced to the partial differential equation part for a quite general class of systems together with algorithms for the solution of this PDE. Nonetheless, these algorithms require laborious symbolic computation and are not easy to implement as a universal applicable algorithm. This is because they are based on an undetermined power series technique [9] requiring skilled mathematician efforts during each particular system regulator design.

The aim of this paper is to further develop methods for solution of the regulator equation overcoming the drawbacks mentioned above. Namely, to create computer implemented algorithms, easily and universally applicable by almost any engineer for every system from some reasonable class without necessity to study sophisticated mathematics. The common feature of the presented methods is the use of the finite-element method (FEM) for the evaluation of the solution of the partial differential part of the regulator equation. The finite-element method is a widely used method for solution of partial differential equations arising in many areas of physics and technology. There are also numerous theoretical results concerning properties of this method. These facts motivated us to apply the finite elements to the problem described here.

As the regulator equation contains an algebraic condition, the finite elements cannot be applied directly. To overcome this difficulty two new approaches were investigated: the first approach is based on optimizing an error functional while the second one uses singular perturbations. A universal algorithm for the solution of the regulator equation based on the finite-element method and optimizing a functional was first introduced in [2]. The second method, based on the singular perturbation of differential equations theory [18] is presented in [15] for the case of full-information feedback and in [3] for the case of error-feedback. This method was verified by simulations on the gyroscope model, the results of simulations are included in this paper. Real-time results obtained on the laboratory gyroscope model can be found in [16]. The aim of this paper is to further develop and compare these two methods for the solution of the regulator equation.

The paper is organized as follows. The output regulation problem (ORP) is briefly introduced in the next section, while the Section 3 explains the crucial role of the regulator equation in solving the ORP. Section 4 collects known methods to solve RE and introduces in detail the singular perturbation based method to solve RE. Section 5 describes the laboratory model of gyroscopical platform and solves RE arising in its ORP while the subsequent section provides both simulation and experimental laboratory implementation results. Section 7 discusses implementation and numerical aspects of the methods, while the final section draws some paper conclusions.

2. THE OUTPUT REGULATION PROBLEM

At this stage some facts on nonlinear output regulation problem are recalled [9, 12]. Consider the plant

$$\begin{aligned} \dot{x} &= f(x(t)) + g(x(t))u(t) + p(x(t))w(t) \\ y(t) &= h(x(t)), \end{aligned} \tag{1}$$

where sufficient smoothness of the vector fields f, g, p and row function h is assumed. Further, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is its input, $y(t) \in \mathbb{R}$ its output and $w(t) \in \mathbb{R}^\mu$ is the state of the so-called *exogeneous system (exosystem)*. This system is supposed to be known, i. e. for the known functions $s : \mathbb{R}^\mu \to \mathbb{R}^\mu$ and $q : \mathbb{R}^\mu \to \mathbb{R}$ the exosystem is given by

$$\dot{w} = s(w), \ \ v = q(w). \tag{2}$$

As a matter of fact, the output of the exosystem $v(t)$ generates the reference signal to be followed and/or the disturbance signal to be rejected. The exosystem is assumed to be Lyapunov stable in both the forward and the backward time directions. This property is sometimes called as the *neutral stability*. Thereby, exogeneous signal is used to describe both reference to be tracked and undesired disturbance to be rejected. This leads to the so-called *output regulation problem (ORP)*, which may be tackled by various kinds of feedback compensators.

The *full information output regulation problem*[1] consists in finding the feedback compensator $u = \alpha(x, w)$ such that

1. if no exogenous signal is present then the equilibrium $x = 0$ of the controlled system is exponentially stable;

2. there exists a neighborhood $U \subset \mathbb{R}^{n+\mu}$ of $(0, 0)$ such that for each initial condition $(x(0), w(0))$ it holds $\lim_{t\to+\infty}(h(x(t)) - v(t)) = 0$.

[1]Full information means that all the states are measured, hence no observer is necessary.

The *error-feedback output regulation problem*[2] consists in finding the feedback compensator

$$e = h(x) - v,\ \dot{z} = \tilde{f}(e, z, u),\ u = \alpha(z, w) \tag{3}$$

with the following properties:

1. if no exogenous signal is present the equilibrium $x = 0$ of the controlled system (1,3) is exponentially stable;

2. there exists a neighborhood $U \subset \mathbb{R}^{n+\mu}$ of $(0, 0)$ such that for each initial condition $(x(0), v(0))$ and solution (1,3) it holds $\lim_{t\to+\infty}(h(x(t)) - v(t)) = 0$.

The involved observer contains the copy of both the controlled system and the exosystem affected by the additional loop – the *output injection*. Thus, the observer dimension is $n + \mu$ and its state is denoted by z. As mentioned above, it can be decomposed as follows

$$z = \begin{pmatrix} z_x^T & z_w^T \end{pmatrix}^T.$$

Here the z_x-part converges to the state of the controlled system and the z_w-part converges to the state of the exosystem. This approach allows to use the state feedback that was designed for the full-information case only even in the case of error feedback simply by replacing the true state (x, w) by its approximation – the state of the observer z.

At the end of this section let us briefly mention the concept of the so-called *relative degree* of the controlled system (1). For details, see [8, 14]. Assume there exists a positive integer ν such that $y(t), \dot{y}(t), \ldots, y^{(\nu-1)}(t)$ do not depend explicitly on $u(t)$ while $y^{(\nu)}(t)$ does depend on $u(t)$. Denote $\xi_1(t) = y(t)$, $\xi_2(t) = \dot{y}(t)$ etc. up to $\xi_\nu(t) = y^{(\nu-1)}(t)$. Then there exist a vector $\eta(t) \in \mathbb{R}^{n-\nu}$, a scalar function φ and a vector function Φ such that the system (1) can be reformulated as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{\nu-1} \\ \xi_\nu \\ \eta \end{pmatrix} = \begin{pmatrix} \xi_2 \\ \vdots \\ \xi_\nu \\ \varphi(\xi_1, \ldots, \xi_n u, \eta, u) \\ \Phi(\xi_1, \ldots, \xi_n u, \eta, u) \end{pmatrix}.$$

The integer ν is called the relative degree of the system (1). The dynamical system

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\eta} = \Phi(0, \ldots, 0, \tilde{\eta}, 0)$$

is usually called as the *zero dynamics*. If the zero dynamics of a given nonlinear system is asymptotically (exponentially) stable, it is called as the *(exponentially) minimum phase* system.

[2]The error-feedback means that only the tracking error (3) is measured, hence an observer must be designed to reconstruct the state of both the plant and the exosystem from the error measurements.

3. REGULATOR EQUATION AND ITS SOLUTION

The aim of this section is to introduce the so-called *Regulator Equation (RE)* and describe its role in ORP solution. As a matter of fact, the RE is the crucial relation, its solution being the essential part of the feedback compensators solving any kind of ORP.

The solution of the RE, denoted as $(\mathbf{x}, c) : \mathbb{R}^{\mu} \to \mathbb{R}^{n+1}$, provides the so-called zero-error manifold $x = \mathbf{x}(w)$ which is forward invariant under the influence of open loop control $u = c(w)$. More precisely, this manifold is the μ-dimensional manifold with the following property: if the state of the exosystem at the time t is equal to $w(t)$ then the tracking error equals to zero if the state of the plant coincides with $\mathbf{x}(w(t))$. Moreover, if the controlled input u equals to $c(w)$, the manifold $x = \mathbf{x}(w)$ is the maximal forward invariant manifold of the system (1) having that zero-error property. The above intuitively geometric description of the RE concepts immediately indicates its crucial role for the ORP problem solution. Indeed, having the RE solution, the compensator ORP should simple stabilize the zero-error manifold, i. e. to force trajectories starting away from it to converge to it.

The regulator equation is given by

$$\frac{\partial \mathbf{x}}{\partial w} s(w) = f(\mathbf{x}(w), c(w), w) \qquad (4)$$

$$0 = h(\mathbf{x}(w)) - q(w). \qquad (5)$$

Here, the equation (5) expresses the requirement that the tracking error is equal to zero when $x = \mathbf{x}(w)$ while the condition (4) enforces forward invariance of the zero-error manifold provided the control $u = c(w)$ is applied.[3] As already mentioned, the RE is the key ingredient of the control scheme achieving any kind of ORP solution, [8]. For example, in case of the full-information ORP, the control scheme that achieves it might be expressed in the following form:

$$u(t) = K(x(t) - \mathbf{x}(w(t))) + c(w(t)), \qquad (6)$$

where the meaning of the variables is as follows

- $u(t)$ is the control at the time t,
- $w(t)$ is the state of the exosystem at the time t,
- x is the state of the system (1),
- $\mathbf{x}(w), c(w)$ is the solution of the RE (4), (5),
- K is a vector such that $u = Kx$ is a stabilizing feedback for the approximate linearization of the system (1).

Here, the function $c(w)$ is often called as the open loop part of the controller (6). This function has to guarantee the forward time invariance of the zero error manifold. The K-dependent part of (6) is called as the closed loop. This part keeps the trajectory converging to the zero error manifold.

[3]Recall, that "forward invariant" manifold means that having initial condition in this manifold and the future state state trajectory of the plant does not leave this manifold.

4. METHODS TO SOLVE THE REGULATOR EQUATION

The previous section provided a clear motivation to develop efficient methods to solve the RE. Still, the corresponding results are quite limited and the topic is not adequately studied in the literature. As a matter of fact, the regulator equation has features that are rather untypical and make the evaluation of its solution to be an extremely challenging task:

- it is a first-order equation,
- the solution to be computed should be ideally given on the whole space $\mathbb{R}^\mu$, this implies, no boundary condition is given,
- validity of the condition $\mathbf{x}(0) = 0$ is required,
- *most importantly:* the algebraic part is present, so that standard PDE methods can not be used directly.

The purpose of this section is to recall the known methods to solve the RE and to discuss some new ideas in this respect. This is also one of the main contributions of this paper. It will be seen that main issue is how to handle the algebraic part, i. e. either to replace it by some additional PDE part (the singular perturbation method later on), or replace it by a suitable penalty functional and its minimization. These two approaches are compared mutually, as well as with a more classical Taylor expansion based method.

4.1. Taylor–expansion method

This method can be seen as a classical one. It is thoroughly presented in [8], therefore we limit ourselves to the sketch of its main ideas only. The basic idea of this method is to decompose all functions in the equations (4), (5) into the Taylor polynomial with coefficients of a sufficiently large degree. The feedforward $c(w)$ as well as the function $\mathbf{x}(w)$ are also assumed to be in form of a Taylor polynomial. Substituting the Taylor polynomials into the RE one can compute the undetermined coefficients of the Taylor expansions of the feedforward part $c(w)$ and of the zero error manifold graph $\mathbf{x}(w)$.

To further facilitate the computations, one can use the following observation: if the original system has relative degree r, then one can determine r components of the function $\mathbf{x}$ from the algebraic condition. These may be then substituted into the original equation and the number of variables to be determined decreases. However, the need for calculation of the Taylor series, especially their substitutions and computation of the coefficients is difficult to be formalized in an algorithmic way sufficient for computer implementation. Moreover, the assumptions of Theorem 3.19 in [8] are rather difficult to verify. Last but not least, Taylor based solution are approximations in an arbitrarily small neighborhood of the working equilibrium only, providing no reasonable estimate of neither neighborhood size, nor approximation error.

4.2. Minimization of the penalty

First note about the basic idea of this method can be found in [2] and its more detailed description and further improvement can be found in [17]. The main idea of this method is to convert the problem of solution of the regulator equation into an optimization problem. Such an idea allows to decouple the two tasks described above. It can be briefly introduced as follows: the differential equation in the regulator equation is solved with a fixed function $c(w)$ which is not yet the “right one” to solve algebraic part as well and therefore the error in the algebraic condition occurs. This error is evaluated throughout the domain in the exosystem state space to give certain real number called as penalty functional. This quantity is then used to set the value of the function $c(w)$ in the next iteration.

Before doing this, the system is pre-stabilized. This means, a vector $K \in \mathbb{R}^n$ is defined so that the system $f(x)+g(x)Kx$ has a stable equilibrium point at the origin. Then, one deals with the stabilized system only. The purpose of that preliminary stabilization is to guarantee the existence of the solution of the differential part of the RE for any $c(w)$. This is the direct consequence of the center manifold theorem and is also the well known fact from PDE theory. The RE is then solved on a bounded domain $\Omega_b \subset \mathbb{R}^\mu$ such that $0 \in \Omega_b$.

To be more specific, the “quality” of the variable c − the error made in the algebraic condition − is measured by the following penalty functional

$$J(c) = \int_{\Omega_b} (h(\mathbf{x}(w)) - q(w))^2 \mathrm{d}w_1 \dots \mathrm{d}w_\mu. \tag{7}$$

where $\mathbf{x}(w)$ solves the equation (4) with the function c in its right-hand side. This value is used to choose the next value of the feedforward c so that the penalty value (7) decreases. Summarizing, the algorithm can be expressed as follows:

Step 1. Choose the function c

Step 2. Compute the solution of the equation (4) with the chosen function c

Step 3. Evaluate the functional (7)

Step 4. If its value is small enough, stop. If not, choose another value of the function c and go to Step 2.

To precise Step 4: generally, one cannot expect that the value of the error functional is driven to zero in a finite number of steps. Rather, one has to apply the feedforward even if the error of the minimization was nonzero. As shown in [17], this minimization error is closely related to the tracking error. So, the demand for precision of the tracking imposes a condition for stopping the algorithm.

The equation (4) is solved using the finite-element method. Convergence of this method is analyzed in [17] in detail. An illustrative example is presented there and computational issues are discussed there as well.

4.3. Singular perturbation method

This alternative approach uses the theory of singularly perturbed differential equations. It was presented first in [15, 16] without any detailed mathematical treatment. Here, its convergence analysis and further theoretical aspects will be analyzed. This is also a new contribution of this paper.

The main idea of the singular perturbation approach is to consider the algebraic equation as the limit case of the equation

$$\varepsilon\dot{c} = h(x(t)) - q(w(t)) \tag{8}$$

for $\varepsilon \to 0+$. If the algebraic condition is replaced by the equation (8), one can formulate the following center manifold for the perturbed system:

$$\frac{\partial \mathbf{x}}{\partial w}s(w) = f(\mathbf{x}(w), c(w), w) \tag{9}$$

$$\varepsilon\frac{\partial c}{\partial w}s(w) = h(\mathbf{x}(w)) - q(w). \tag{10}$$

This is a system of purely differential equations, it can also be solved using a standard software for solution of PDE, in particular, using FEM methods and FEMLAB package. Again, one seeks the solution on the bounded domain Ω_b as described in the previous section.

The question of solvability of this set of equations is not well answered yet, rather, it is a future research topic. There is a result about solvability of the singularly perturbed equations which unfortunately does not fit well onto this problem. The main theorem together with its proof can be found in [20]. This theorem deals with the system

$$\begin{aligned}\dot{x}_\varepsilon &= f(x_\varepsilon, y_\varepsilon, t)\\ \varepsilon\dot{y}_\varepsilon &= g(x_\varepsilon, y_\varepsilon, t).\end{aligned}$$

If $\varepsilon \to 0+$ then the above system approaches the algebraic-differential equation

$$\begin{aligned}\dot{x} &= f(x, y, t)\\ 0 &= g(x, y, t).\end{aligned}$$

However, this sole fact is not sufficient to guarantee convergence $x_\varepsilon \to x$. This convergence is guaranteed if, among all, the condition

$$\frac{\partial g}{\partial y}(x, y, t) < 0$$

holds. Satisfying this is a major problem as the most natural choice (presented above) is not admissible anymore. The theory of singular perturbations of differential equations is still a developing theory so there is a hope for a more suitable result to appear. However, one can derive a condition guaranteeing existence of a solution of the system of equations (9) for a (fixed) $\varepsilon > 0$.

Let us assume in the following text that the system has already been converted into an exactly linearized form. In this form, $h(x(w)) = x_1$, $\dot{x}_1 = x_2, \ldots, \dot{x}_{\nu-1} = x_\nu$, $\dot{x}_\nu = \Phi(x, c)$ where $\nu \leq n$ and $\Phi(0, c) \neq 0$ if $c \neq 0$. The following notation will be useful in the sequel: let the matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$ be defined as follows: the elements of the matrix A are given as $a_{ij} = \frac{\partial \varphi_i}{\partial x_j}(0,0,0)$, $i, j = 1, \ldots, n$, the matrix B is defined by $B = \varphi(0,0,0)$. (Note that for elements the following holds: $b_\nu c = \Phi(0, c)$, $b_j = 0$ for $j < \nu$.) Moreover, define the function $\zeta : \mathbb{R}^n \times R \times \mathbb{R}^\mu$ by $\zeta(x, c, w) = \varphi(x, c, w) - Ax - Bc$. For the purpose of the following proposition, define also $H = (1, 0, \ldots, 0) \in \mathbb{R}^n$ and the matrix

$$\tilde{A} = \begin{pmatrix} A & B \\ H & 0 \end{pmatrix}.$$

Note that the matrix $\tilde{A}$ has the following structure (with $s = n - \nu$):

$$\tilde{A} = \begin{pmatrix} 0 & 1 & \ldots & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & \ldots & 0 & 0 \\ a_{\nu,1} & a_{\nu,2} & \ldots & a_{\nu,\nu} & a_{\nu,\nu+1} & \ldots & a_{\nu,n} & b_\nu \\ \alpha_{1,1} & \alpha_{1,2} & \ldots & \alpha_{1,\nu} & \beta_{1,1} & \ldots & \beta_{1,s} & \gamma_1 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{s,1} & \alpha_{s,2} & \ldots & \alpha_{s,\nu} & \beta_{s,1} & \ldots & \beta_{s,s} & \gamma_s \\ 1 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

The main tool for investigation of solvability of this equation is the theorem from [18]:

Lemma. Let $f \in L^2(\Omega_b)$, β, γ be functions defined on Ω_b such that $\mathrm{div}\beta - \gamma > 0$ (here, $\mathrm{div}\beta$ means the divergence of the vector function β). Then the equation

$$\beta \nabla \mathbf{x}(w) + \gamma \mathbf{x}(w) = f(w)$$

has a solution $\mathbf{x} \in L^2(\Omega_b)$. Moreover, there exists a positive constant k independent of the function f such that $\|\mathbf{x}\| \leq k\|f\|$.

This lemma deals with linear scalar equations, however, the equation (9) is nonlinear and the solution is a vector of functions. The following remedy was found: first, the equation (9) is rewritten into the form

$$\frac{\partial \xi}{\partial w} s(w) = \tilde{A}\xi + \tilde{\zeta}(\xi(w)) \tag{11}$$

where

$$\xi(w) = \begin{pmatrix} \mathbf{x}(w) \\ c(w) \end{pmatrix} \quad \text{and} \quad \tilde{\zeta}(\xi(w)) = \zeta(\mathbf{x}(w), c(w), w).$$

Assume the matrix $\tilde{A}$ is diagonalizable, this means, there exists a nonsingular matrix T and a diagonal matrix D so that

$$\tilde{A} = T^{-1}DT.$$

Let $\eta = T\xi$. Then the equation (11) turns into

$$\frac{\partial \eta}{\partial w} s(w) = D\eta + \tilde{\zeta}(T^{-1}\eta(w)).$$

Using $\eta^{(0)}\text{Ł}^2(\Omega_b)$ one can define the sequence $\eta^{(k)},\ k \in N$ by

$$\frac{\partial \eta^{(k)}}{\partial w} s(w) = D\eta^{(k)} + \tilde{\zeta}(T^{-1}\eta^{(k-1)}(w)). \tag{12}$$

Note that the system of equations (12) is in fact a system of independent equations for $\eta_i^{(k)},\ i = 1, \ldots, n+1$. Moreover, one can prove that the condition from the lemma is satisfied if the element d_{ii} is real nonzero as the divergence term in the condition from the lemma is zero. This holds thanks to neutral stability of the exosystem which implies zero eigenvalues of its Jacobi matrix.

The system (12) has a solution $\eta^{(k)}$ for every $\eta^{(k-1)}$. Now it is to prove convergence of this series to a fixed point of the equation (12). This is done by the Banach fixed point theorem. First, observe that there exists a positive constant C so that

$$\|\eta^{(k)}\| \leq C\|\tilde{\zeta}(T^{-1}\eta^{(k-1)}\|.$$

Let $\eta^{(0)}, \bar{\eta}^{(0)} \in L^2(\Omega_b)$. These functions give rise to two sequences $\eta^{(k)}, \bar{\eta}^{(k)}$. For the difference holds

$$\|\eta^{(k)} - \bar{\eta}^{(k)}\| \leq C\|\tilde{\zeta}(T^{-1}\eta^{(k-1)} - \tilde{\zeta}(T^{-1}\bar{\eta}^{(k-1)}\|.$$

Assume now the function $\tilde{\zeta}$ is Lipschitz. Then there exists a constant κ such that $\|\tilde{\zeta}(T^{-1}\eta^{(k-1)} - \tilde{\zeta}(T^{-1}\bar{\eta}^{(k-1)}\| \leq \kappa\|\eta^{(k-1)} - \bar{\eta}^{(k-1)}\|$. Now one observes that the mapping $\eta^{(k-1)} \mapsto \eta^{(k)}$ defined by the equation (12) is a contraction if $C\kappa < 1$. This can be summarized in the following theorem:

Theorem. The equation (9) has a solution $\mathbf{x}(w)$ if the following conditions are satisfied:

- the matrix $\tilde{A}$ is diagonalizable with nonzero real eigenvalues,
- $C\kappa < 1$.

(the symbols $\tilde{A}, C, \kappa$ being defined above).

Corollary. Let the matrix $\tilde{A}$ be diagonalizable with no zero eigenvalues and let $C\kappa < 1$. Then the equation (9) has a solution in $L^2(\Omega)$.

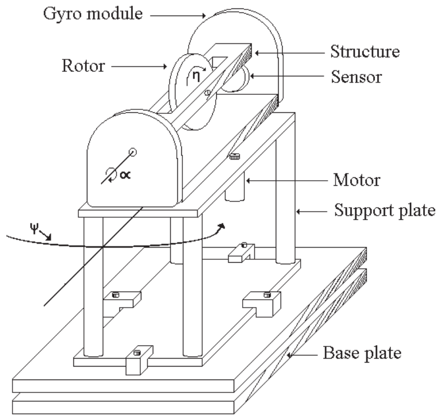

Fig. 1. The Gyroscope.

5. THE GYROSCOPE CASE STUDY: THE REGULATOR EQUATION AND ITS SOLUTION

The first and the second method described in Section 4 were demonstrated on the model of the inverted pendulum on a cart system. (The system is described in [4, 10].) The results of these mathematical simulations are contained in [2, 10, 17] and mutually compared in [17]. The present paper aims to demonstrate the third method and further discuss its properties.

In order to do so, this third method was used to control the gyroscope platform (described thoroughly in [19]) which is a nonminimum-phase fourth-order system with a two-dimensional zero dynamics. The results are thoroughly presented in [3, 15]. The latter paper deals with the so-called error-output regulation problem. This means the only information available was the tracking error. The state value of the controlled system as well as of the exosystem was gained from this variable. These results are also presented here. The system in question is a gyroscope of two axes, shown schematically in Figure 1, which is a lab experiment developed by Quanser Inc. For more information, see www.quanser.com. The gyroscope consists basically of the following components: a support plate holding the gyro module with a rotor which rotates at a constant speed, its movement being produced by a DC motor, sensors for the angles α and ψ, and a data acquisition card connecting the gyroscope to a computer.

The equations describing this system, obtained from the dynamics of the system and the physical parameters, are as follows

$$\begin{aligned} a_1\ddot{\alpha} + a_2\dot{\psi}\cos\alpha + a_3\dot{\psi}^2\sin\alpha\cos\alpha &= a_4\tan\alpha \\ b_1\ddot{\psi} + b_2\cos^2\alpha\ddot{\psi} + b_3\sin^2\alpha\ddot{\psi} + b_4\,\dot{\alpha}\cos\alpha & \\ +b_5\,\dot{\psi}\dot{\alpha}\sin\alpha\cos\alpha &= b_6 u + b_7\dot{\psi}. \end{aligned} \tag{13}$$

Here, the angle α defines the angular position of the structure with the rotor with respect to the gyro module, angle ψ is located between the gyro module and the support plate, and the control input u is the voltage applied to the DC motor. The constants a_i, b_i were found by identification. Their values are in the following table:

a_1 0.0054435	a_2 0.4717409	a_3 −0.0004879	a_4 2.4610918
b_1 0.002	b_2 0.0008470	b_3 0.001335	b_4 −0.4717
b_5 0.0009758	b_6 0.1126816997	b_7 −0.01044	

Converting the equations (13) into a system of first order and introducing the notation

$$\psi = x_1,\ \dot{\psi} = x_2,\ \alpha = x_3,\ \dot{\alpha} = x_4$$

one obtains

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{a_4}{a_1}\tan x_1 - \frac{a_2}{a_1}x_4\cos x_1 - \frac{a_3}{a_1}x_4^2\sin x_1\cos x_1 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \frac{b_6 u + b_7 x_4 - b_4 x_2\cos x_1 - b_5 x_4 x_2 \sin x_1 \cos x_1}{b_1 + b_2\cos^2 x_1 + b_3 \sin^2 x_1}. \end{aligned} \tag{14}$$

The position of the plate (denoted by x_3) is considered as the output. Thus the first two equations express the zero dynamics. The system is not minimum phase. Our task is to design a control so that the output tracks the trajectory $x_3 = k\sin t$, $k \in \mathbb{R}$. This corresponds to the case if the exosystem were

$$\begin{aligned} \dot{w}_1 &= w_2 \\ \dot{w}_2 &= -w_1. \end{aligned} \tag{15}$$

The algebraic condition then reads

$$0 = w_1 - x_3. \tag{16}$$

The regulator equation derived from the mathematical description of the gyroscope is (the argument is omitted)

$$\begin{aligned} \omega\left(w_2\frac{\partial \mathbf{x}_1}{\partial w_1} - w_1\frac{\partial \mathbf{x}_1}{\partial w_2}\right) &= \mathbf{x}_2 \\ \omega\left(w_2\frac{\partial \mathbf{x}_2}{\partial w_1} - w_1\frac{\partial \mathbf{x}_2}{\partial w_2}\right) &= \frac{a_4}{a_1}\tan \mathbf{x}_1 - \frac{a_2}{a_1}\mathbf{x}_4\cos \mathbf{x}_1 - \frac{a_3}{a_1}\mathbf{x}_4^2\sin \mathbf{x}_1\cos \mathbf{x}_1 \\ \omega\left(w_2\frac{\partial \mathbf{x}_3}{\partial w_1} - w_1\frac{\partial \mathbf{x}_3}{\partial w_2}\right) &= \mathbf{x}_4 \\ \omega\left(w_2\frac{\partial \mathbf{x}_4}{\partial w_1} - w_1\frac{\partial \mathbf{x}_4}{\partial w_2}\right) &= \frac{b_6 \mathrm{c} + b_7 \mathbf{x}_4 - b_4 \mathbf{x}_2\cos \mathbf{x}_1 - b_5 \mathbf{x}_4 \mathbf{x}_2 \sin \mathbf{x}_1 \cos \mathbf{x}_1}{b_1 + b_2\cos^2 \mathbf{x}_1 + b_3 \sin^2 \mathbf{x}_1} \end{aligned} \tag{17}$$

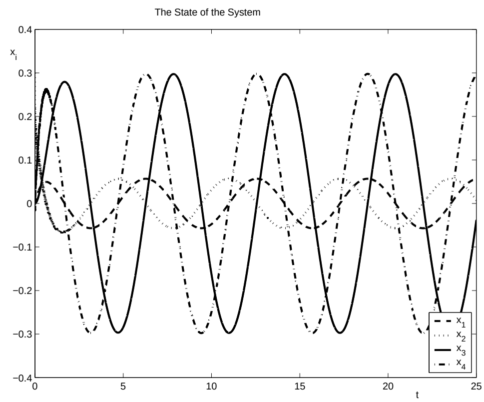

Fig. 2. State of the Gyroscope – Simulations.

together with the condition (16). According to (6) the control $u(t)$ is defined as $\tilde{u}(t) = -K(x(t) - \mathbf{x}(w(t))) + c(w(t))$ such that the feedback vector K stabilizes the linearization of the gyroscope. (The value $x(t)$ is to be replaced by the state of the observer in the practical realization, see below.) As already mentioned, the stabilizing feedback is to be defined prior to solution of the regulator equation. The results described below were obtained, similarly as in [15], with

$$K = (-58.3795 \quad -1.1314 \quad -1 \quad 7.6697).$$

The condition (16) is replaced by a perturbed equation $\varepsilon \dot{c} = w_1 - x_3$. This implies that the algebraic condition is replaced by the equation

$$\varepsilon \omega \left(w_2 \frac{\partial c}{\partial w_1} - w_1 \frac{\partial c}{\partial w_2} \right) = \mathbf{x}_3 - w_1.$$

This is solved together with the system (17) using the Finite-Element Method.

6. THE GYROSCOPE CASE STUDY: SIMULATIONS, LABORATORY IMPLEMENTATION AND EXPERIMENTS

There was a possibility of experimental application of feedback compensators on the laboratory model of the gyroscope, thus the experiments were carried out using this model. As none of both angular velocities was measurable on the model, an observer had to be designed as described above. More about the observer can be found in [3, 16].

The results of simulations are shown first. Figure 2 shows all the states of the system while Figure 3 shows the tracking error. One sees that the tracking error approaches zero quite quickly.

Results obtained from the laboratory experiment are thoroughly described in [16]. Their major feature is that not only the velocities but also the angles and the states

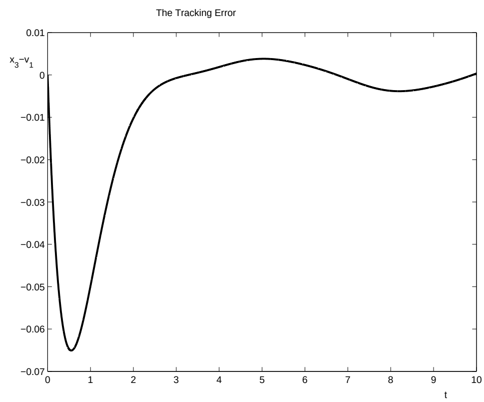

Fig. 3. Tracking Error – Simulations.

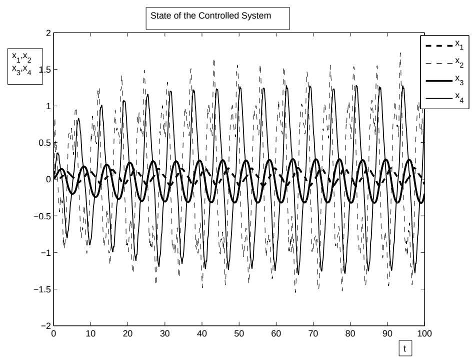

Fig. 4. State of the Gyroscope – Laboratory Experiment.

of the exosystem were estimated by the observer. Its sole input was the tracking error. The way how the problem was posed corresponds to the "error-feedback" setting. The tracking error (denoted by $e(x)$ in the previous section) can be seen in Figure 5, matching the reference with the output is shown in Figure 6. In the real case, the error is much smaller than the value of the reference signal (about 5% of the amplitude of the reference signal). This is the main result of this paper as it shows that the proposed method is capable of controlling the real plant even in presence of disturbances that were not taken into account at the design stage of the method. In other words this is a verification of some robustness of the developed approach. The control signal is seen in Figure 7.

Presence of the observer is a key point. The velocities can be obtained e. g. via differentiating the outputs, however, this leads to a significant error as shown in [15], which is not present when observer is used, cf. [3, 16].

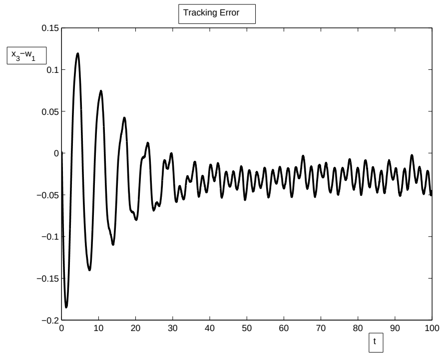

Fig. 5. Tracking Error – Laboratory Experiment.

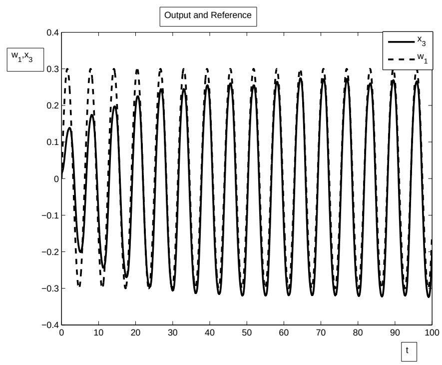

Fig. 6. Output and Reference – Laboratory Experiment.

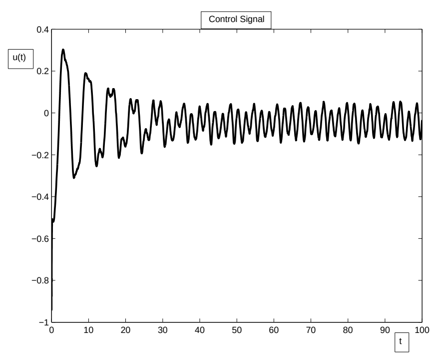

Fig. 7. Control – Laboratory Experiment.

7. IMPLEMENTATION ASPECTS AND METHODS COMPARISON

The systems of partial differential equations were solved using the software package FEMLAB 2.3. (Today, its successor is called Comsol Multiphysics). It is a powerful solver that easily enables to define the problem. Moreover, even the peculiar features of the regulator equation (first order, no boundary conditions) could be handled. The mesh on which the problem was solved was generated by the Femlab's built-in mesh generator. In order to guarantee the condition $\mathbf{x}_i = 0$, the mesh was refined in the neighborhood of the origin.

Femlab offers its own postprocessing functions to use the solution further. They were used to obtain simulations. This was necessary as this set of functions includes an interpolation function or computation of the integral defining the error functional. However, to use the results without Femlab, the results were interpolated on the nodes of a pre-defined grid (which may be different from the grid used for evaluation of the solution) so that standard Matlab interpolation can be used.

Both the newly developed methods proved to be viable for practical implementation. They possess some features that are common to them but they differ in some others. The main difference is the speed of computations. The method based on the minimization of the error functional requires a successive evaluation of the regulator equation. Moreover, the optimization method also involves some extra expenses. The method based on singular-perturbation does not suffer from this drawback as the solution of the equation (9) is evaluated once. On the other hand, the variable c is a part of the solution which implies higher demand for memory. This might be an issue if a fine mesh is required. A problem is also a not fully investigated convergence of this method. However, as in practice one has to solve the perturbed equation only, the Theorem gives a useful condition guaranteeing existence of solution of the perturbed regulator equation.

The estimate of the tracking error when using the feedforward based on the numerical approximation of the solution of the regulator equation is contained in [17]. Even if this article deals with the optimization-based method only, the estimate is applicable for the singular perturbation-based one as well as evaluating the error functional for the solution of (9) can be carried out analogously in this case. The effect of the discretization of the equations solved by the FEM solver is discussed in the cited article as well, this analysis is also valid for both methods. A feature that deserves a special note here is the high tolerance to coarse discretization of the domain Ω_b.

8. CONCLUSION

Several methods to solve the regulator equation arising in the output regulation problem were presented and discussed. The detailed mathematical analysis of one of them was performed, namely, of the singular perturbation based method. Further implementation aspects of this method are discussed and compared to other known presented methods. Finally, the singular perturbation method was also tested on laboratory system via experiment on a laboratory gyroscopical system to show that it can be potentially applicable for real-time control design.

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REFERENCES

[1] C. I. Byrnes, F. Delli Priscoli, A. Isidori, and W. Kang: Structurally stable output regulation of nonlinear systems. Automatica *33* (1997), 369–285.

[2] S. Čelikovský and B. Rehák: Output regulation problem with nonhyperbolic zero dynamics: a FEMLAB-based approach. In: Proc. 2nd IFAC Symposium on System, Structure and Control 2004, Oaxaca 2004, pp. 700–705.

[3] S. Čelikovský and B. Rehák: FEMLAB-based error feedback design for the output regulation problem. In: Proc. Asian Control Conference 2006. Bandung 2006.

[4] S. Devasia: Nonlinear inversion-based output tracking. IEEE Trans. Automat. Control *41* (1996), 930–942.

[5] B. A. Francis and W. M. Wonham: The internal model principle of control theory. Automatica *12* (1976), 457–465.

[6] B. A. Francis: The linear multivariable regulator problem. SIAM J. Control Optim. *15* (1977), 486–505.

[7] J. S. A. Hepburn and W. M. Wonham: Error feedback and internal models on differentiable manifolds. IEEE Trans. Automat. Control *29* (1981), 397–403.

[8] J. Huang: Nonlinear Output Regulation: Theory and Applications. SIAM, New York 2004.

[9] J. Huang: Output regulation of nonlinear systems with nonhyperbolic zero dynamics. IEEE Trans. Automat. Control *40* (1995), 1497–1500.

[10] J. Huang: On the solvability of the regulator equations for a class of nonlinear systems. IEEE Trans. Automat. Control *48* (2003), 880–885.

[11] J. Huang and W. J. Rugh: On a nonlinear multivariable servomechanism problem. Automatica *26* (1990), 963–972.

[12] A. Isidori and C. I. Byrnes: Output regulation of nonlinear systems. IEEE Trans. Automat. Control *35* (1990), 131–140.

[13] A. Isidori: Nonlinear Control Systems. Third edition. Springer, Berlin – Heidelberg – New York 1995.

[14] H. K. Khalil: Nonlinear Systems. Pearson Education Inc., Upper Saddle River 2000.

[15] B. Rehák, J. Orozco-Mora, S. Čelikovský, and J. Ruiz-León: FEMLAB-based output regulation of nonhyperbolically nonminimum phase system and its real-time implementation. In: Proc. 16th IFAC World Congress, Prague, IFAC 2005.

[16] B. Rehák, J. Orozco-Mora, S. Čelikovský, and J. Ruiz-León: Real-time error-feedback output regulation of nonhyperbolically nonminimum phase system. In: Proc. American Control Conference 2007, New York 2007.

[17] B. Rehák and S. Čelikovský: Numerical method for the solution of the regulator equation with application to nonlinear tracking. Automatica *44* (2008), 1358–1365.

[18] H.-G. Roos, M. Stynes, and L. Tobiska: Numerical Methods for Singularly Perturbed Differential Equations. Springer, Berlin 1996.

[19] J. Ruiz-León, J. L. Orozco-Mora, and D. Henrion: Real-time H_2 and H_∞ control of a gyroscope using Polynomial Toolbox 2.5. In: Latin-American Conference on Automatic Control CLCA 2002, Guadalajara 2002.

[20] A. B. Vasil'eva and B. F. Butuzov: Asymptotic Expansions of Solutions of Singularly Perturbed Equations (in Russian). Nauka, Moscow 1973.

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