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ROBUST H$^\infty$ CONTROL OF AN UNCERTAIN SYSTEM VIA A STABLE DECENTRALIZED OUTPUT FEEDBACK CONTROLLER

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This paper presents a procedure for constructing a stable decentralized output feedback controller for a class of uncertain systems in which the uncertainty is described by Integral Quadratic Constraints. The controller is constructed to solve a problem of robust H^∞ control. The proposed procedure involves solving a set of algebraic Riccati equations of the H^∞ control type which are dependent on a number of scaling parameters. By treating the off-diagonal elements of the controller transfer function matrix as uncertainties, a decentralized controller is obtained by taking the block-diagonal part of a non-decentralized stable output feedback controller which solves the robust H^∞ control problem. This approach to decentralized controller design enables the controller to exploit the coupling between the subsystems of the plant.

Keywords: robust control, decentralized control, H^∞ control

AMS Subject Classification: 93B36, 93E20, 93B50, 93B35

1. INTRODUCTION

In this paper, we present a new approach to stable decentralized output feedback robust H^∞ control for a class of uncertain systems described with uncertainty described by Integral Quadratic Constraints (IQCs); e. g., see [6]. The problem of robust decentralized control has attracted a great deal of interest in the control theory literature; e. g., see [6] – [11]. One important approach to the design of robust decentralized controllers is to treat the interconnections between subsystems as uncertainties; e. g., [8]. This approach has been very successful in many problems where the interconnections between subsystems are not well known. However in other cases, the interconnections between subsystems may be well known and it would be desirable for the controller to be able to exploit these interconnections; e. g., see [11]. Our approach to robust decentralized control falls into this later category in that it is able to exploit the interconnections between subsystems. Indeed, our main idea is that rather than treat the interconnections between subsystems in the plant model as uncertainty, we treat the off-diagonal blocks in the controller transfer function matrix as uncertainty. This enables us to replace a non-decentralized controller transfer function matrix with a corresponding block-diagonal decentralized controller transfer function matrix. However, in order to be able to treat these off diagonal blocks of the controller transfer function matrix as uncertainties, it is necessary that the controller transfer function matrix be stable. Thus, we address a problem of designing stable robust decentralized output feedback controllers. This idea of designing a controller which is robust against perturbations to the controller gain matrix is somewhat reminiscent of non-fragile controller design methods; e. g., see [9]. It is well known that the use of stable controllers is preferable to the use of unstable feedback controllers in many practical control problems; e. g., see [13, 2]. Indeed, the use of unstable controllers can lead to problems with actuator and sensor failure, sensitivity to plant uncertainties and nonlinearities and implementation problems. Also, it is well known that issues of robustness and disturbance attenuation are important in control system design. This has motivated a number researchers to consider problems of H^∞ control via the use of stable feedback controllers; e. g., see [13, 2, 3].

We consider a class of uncertain systems with structured uncertainty described by Integral Quadratic Constraints (IQCs); e. g., see [7, 6]. Indeed, our results build on the results of [7] which provide necessary and sufficient conditions for the absolute stabilization of such uncertain systems with a specified level of disturbance attenuation (but with no requirement that the output feedback controller is stable or have a decentralized structure). The key idea behind our approach is to begin with an uncertain system of the type considered in [7] and then add an additional uncertainty to form a new uncertain system. This idea was applied in the paper [4] which considered a problem of robust H^∞ control via a stable output feedback controller. The main result of this paper is to extend the results of [4] to allow for stable decentralized output feedback controllers. Indeed, a new uncertain system is constructed so that the the error introduced by replacing the stable (non-decentralized) controller by a corresponding block diagonal (decentralized) controller can be treated as an additional H^∞ norm bounded uncertainty. A similar idea is used in the paper [5] in case of a decentralized state feedback guaranteed cost control problem. This paper extends this idea to the case of decentralized output feedback H^∞ control. In this case, the additional requirement that the controller be stable needs to be imposed since the controller is dynamic rather than static.

Our main result is obtained applying the results of [7] and [4] to the new uncertain system. This gives us a procedure for constructing a stable decentralized output feedback controller solving a problem of absolute stabilization with a specified level of disturbance attenuation. This is achieved by solving three algebraic Riccati equations dependent on a set of scaling parameters. The output feedback controller obtained is of the same order of the plant. Because our approach involves the addition of new uncertainties, our results provide only sufficient conditions rather than necessary and sufficient conditions for absolute stabilization with a specified level of disturbance attenuation. However, because the new uncertainty is explicitly constructed, this can give some indication about the degree of conservatism introduced.

The remainder of the paper proceeds as follows: In Section 2 of the paper, we set up the decentralized robust H^∞ control problem under consideration. Section 3 introduces the IQCs and the notation necessary to convert the problem under consideration into a problem which can be handled using the approach of [7, 4]. This leads to our main result which is a procedure for constructing the required stable decentralized output feedback controller solving the robust H^∞ control problem under consideration. In Section 4, we present an illustrative example involving the decentralized control of a pair of vehicles.

2. PROBLEM STATEMENT

We consider an output feedback H^∞ control problem for an uncertain system of the following form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t) + \sum_{s=1}^{k} D_s \xi_s(t); \\ z(t) &= C_1 x(t) + D_{12} u(t); \\ \zeta_1(t) &= K_1 x(t) + G_1 u(t); \\ &\vdots \\ \zeta_k(t) &= K_k x(t) + G_k u(t); \\ y(t) &= C_2 x(t) + D_{21} w(t) \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the *state*, $w(t) \in \mathbb{R}^g$ is the *disturbance input*, $u(t) \in \mathbb{R}^m$ is the *control input*, $z(t) \in \mathbb{R}^q$ is the *error output*, $\zeta_1(t) \in \mathbb{R}^{h_1}, \ldots, \zeta_k(t) \in \mathbb{R}^{h_k}$ are the *uncertainty outputs*, $\xi_1(t) \in \mathbb{R}^{r_1}, \ldots, \xi_k(t) \in \mathbb{R}^{r_k}$ are the *uncertainty inputs* and $y(t) \in \mathbb{R}^l$ is the *measured output*.

The uncertainty in this system is described by a set of equations of the form

$$\begin{aligned} \xi_1(t) &= \phi_1(t, \zeta_1(\cdot)|_0^t) \\ \xi_2(t) &= \phi_2(t, \zeta_2(\cdot)|_0^t) \\ &\vdots \\ \xi_k(t) &= \phi_k(t, \zeta_k(\cdot)|_0^t) \end{aligned} \tag{2}$$

where the following Integral Quadratic Constraint is satisfied.

Definition 1. (Integral Quadratic Constraint; see Savkin and Petersen [7], Petersen et al. [6]) An uncertainty of the form (2) is an admissible uncertainty for the system (1) if the following conditions hold: Given any locally square integrable control input $u(\cdot)$ and locally square integrable disturbance input $w(\cdot)$, and any corresponding solution to the system (1), (2), let $(0, t_*)$ be the interval on which this solution exists. Then there exist constants $d_1 \geq 0, \ldots, d_k \geq 0$ and a sequence $\{t_i\}_{i=1}^{\infty}$ such that $t_i \to t_*$, $t_i \geq 0$ and

$$\int_0^{t_i} \|\xi_s(t)\|^2 \, \mathrm{d}t \leq \int_0^{t_i} \|\zeta_s(t)\|^2 \, \mathrm{d}t + d_s \quad \forall i \quad \forall s = 1, \ldots, k. \tag{3}$$

Here $\|\cdot\|$ denotes the standard Euclidean norm and $\boldsymbol{L}_2[0,\infty)$ denotes the Hilbert space of square integrable vector valued functions defined on $[0,\infty)$. Note that t_i and $t_\star$ may be equal to infinity. The class of all such admissible uncertainties $\xi(\cdot) = [\xi_1(\cdot),\ldots,\xi_k(\cdot)]$ is denoted Ξ.

It is assumed that the measured output vector $y(t) \in \mathbb{R}^l$ has been decomposed into p components as follows

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} \tag{4}$$

where $y_i \in \mathbb{R}^{l_i}$ for $i = 1, 2, \ldots, p$ and $l = \sum_{i=1}^{p} l_i$. Also, it is assumed that the control input vector $u(t) \in \mathbb{R}^m$ has been decomposed into p components as follows

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \end{bmatrix} \tag{5}$$

where $u_i \in \mathbb{R}^{m_i}$ for $i = 1, 2, \ldots, p$ and $m = \sum_{i=1}^{p} m_i$. Each of the components u_i is regarded as the control input vector corresponding to the measured output vector component y_i although no assumptions are made concerning the structure of the system matrices A, B_2 and C_2.

For the uncertain system (1), (3), we consider a problem of absolute stabilization with a specified level of disturbance attenuation. The class of controllers considered are stable decentralized output feedback controllers of the form

$$\begin{aligned} \dot{x}_{ci}(t) &= A_{ci}x_{ci}(t) + B_{ci}y_i(t); \\ u_i(t) &= C_{ci}x_{ci}(t) \end{aligned} \tag{6}$$

for all $i = 1, 2, \ldots, p$ where each A_{ci} is a Hurwitz matrix. Each local feedback controller has a transfer function $H_{ii}(s) = C_{ci}(sI - A_{ci})^{-1}B_{ci}$. The control law (6) is a special case of the general output feedback controller

$$\begin{aligned} \dot{x}_c(t) &= A_c x_c(t) + B_c y(t); \\ u(t) &= C_c x_c(t) \end{aligned} \tag{7}$$

where A_c is a Hurwitz matrix and such that controller transfer function matrix

$$H(s) = C_c(sI - A_c)^{-1}B_c \tag{8}$$

has a block-diagonal structure.

Definition 2. The uncertain system (1), (3) is said to be *absolutely stabilizable with disturbance attenuation* γ via the output feedback controller (7) if there exists constants $c_1 > 0$ and $c_2 > 0$ such that the following conditions hold:

1. For any initial condition $[x(0), x_c(0)]$, any admissible uncertainty inputs $\xi(\cdot)$ and any disturbance input $w(\cdot) \in \boldsymbol{L}_2[0,\infty)$, then

$$[x(\cdot), x_c(\cdot), u(\cdot), \xi_1(\cdot), \ldots, \xi_k(\cdot)] \in \boldsymbol{L}_2[0,\infty)$$

(hence, $t_* = \infty$) and

$$\|x(\cdot)\|_2^2 + \|x_c(\cdot)\|_2^2 + \|u(\cdot)\|_2^2 + \sum_{s=1}^{k} \|\xi_s(\cdot)\|_2^2$$
$$\leq c_1 \Big[\|x(0)\|^2 + \|x_c(0)\|^2 + \|w(\cdot)\|_2^2 + \sum_{s=1}^{k} d_s \Big]. \tag{9}$$

2. The following H^∞ norm bound condition is satisfied: If $x(0) = 0$ and $x_c(0) = 0$, then
$$J \triangleq \sup_{w(\cdot)\in \boldsymbol{L}_2[0,\infty)} \sup_{\xi(\cdot)\in\Xi} \frac{\|z(\cdot)\|_2^2 - c_2 \sum_{s=1}^{k} d_s}{\|w(\cdot)\|_2^2} < \gamma^2. \tag{10}$$

   Here, $\|q(\cdot)\|_2$ denotes the $\boldsymbol{L}_2[0,\infty)$ norm of a function $q(\cdot)$. That is, $\|q(\cdot)\|_2^2 \triangleq \int_0^\infty \|q(t)\|^2 \, \mathrm{d}t$.

Assumption 1. The uncertain system (1), (3) will be assumed to satisfy the following conditions throughout the paper:

(i) The pair (A, C_1) is observable.

(ii) The pair (A, B_1) is controllable.

3. THE MAIN RESULTS

The main idea behind our approach is to design a non-decentralized stable output feedback controller with transfer function matrix $H(s)$ using the methodology described in [4]. Then, a decentralized controller is obtained by taking only the block-diagonal part of the transfer function matrix $H(s)$. The ignored blocks of the transfer function matrix $H(s)$ are treated as additional uncertainties which are added to the uncertainties in the original uncertain system (1), (3).

Consider a stable output feedback controller for the uncertain system (1), (3) of the form (7) with transfer function matrix $H(s)$ as in (8). Also, suppose $H(s)$ is partitioned to be compatible with u in (5) and y in (4) as follows:

$$H(s) = \begin{bmatrix} H_{11}(s) & H_{12}(s) & \ldots & H_{1p}(s) \\ H_{21}(s) & H_{22}(s) & \ldots & H_{2p}(s) \\ \vdots & & \ddots & \vdots \\ H_{p1}(s) & H_{p2}(s) & \ldots & H_{pp}(s) \end{bmatrix}. \tag{11}$$

We then construct the corresponding stable decentralized output feedback controller

$$u(s) = \tilde{H}(s) y(s)$$

where

$$\tilde{H}(s) = \begin{bmatrix} H_{11}(s) & 0 & \dots & 0 \\ 0 & H_{22}(s) & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & H_{pp}(s) \end{bmatrix}. \quad (12)$$

Such a controller could also be described in state space form as in (6).

We now define a sequence of uncertainty transfer function matrices obtained from the blocks of the transfer function matrix $H(s)$ which are not included in the transfer function matrix $\tilde{H}(s)$:

$$\begin{aligned} \Delta_1(s) &= \begin{bmatrix} H_{12}(s) & H_{13}(s) & \dots & H_{1p}(s) \end{bmatrix}; \\ \Delta_2(s) &= \begin{bmatrix} H_{21}(s) & H_{23}(s) & \dots & H_{2p}(s) \end{bmatrix}; \\ &\vdots \\ \Delta_p(s) &= \begin{bmatrix} H_{p1}(s) & H_{p2}(s) & \dots & H_{p(p-1)}(s) \end{bmatrix}. \end{aligned} \quad (13)$$

Note that because the transfer function matrix $H(s)$ is assumed to be stable, each of the above transfer function matrices will also be stable. Also, we define

$$\begin{aligned} \tilde{\xi}_1(s) &= -\Delta_1(s)\tilde{\zeta}_1(s); \\ \tilde{\xi}_2(s) &= -\Delta_2(s)\tilde{\zeta}_2(s); \\ &\vdots \\ \tilde{\xi}_p(s) &= -\Delta_p(s)\tilde{\zeta}_p(s); \end{aligned} \quad (14)$$

where

$$\begin{aligned} \tilde{\zeta}_1 &= \begin{bmatrix} y_2' & y_3' & \dots & y_p' \end{bmatrix}' = C_{1,1}x + H_1 w; \\ \tilde{\zeta}_2 &= \begin{bmatrix} y_1' & y_3' & \dots & y_p' \end{bmatrix}' = C_{1,2}x + H_2 w; \\ &\vdots \\ \tilde{\zeta}_p &= \begin{bmatrix} y_1' & y_2' & \dots & y_{p-1}' \end{bmatrix}' = C_{1,p}x + H_p w; \end{aligned} \quad (15)$$

the matrices $C_{1,1}, C_{1,2}, \dots, C_{1,p-1}$ are corresponding sub-matrices of the matrix C_2, and the matrices $H_1, H_2, \dots, H_p$ are corresponding sub-matrices of the matrix D_{21}.

Then for the decentralized controller, we can write

$$u(s) = \tilde{H}(s)y(s) = H(s)y(s) + \sum_{i=1}^{p} J_i \tilde{\xi}_i(s) \quad (16)$$

where

$$\begin{aligned} J_1 &= \begin{bmatrix} I_{m_1 \times m_1} \\ 0_{\tilde{m}_1 \times m_1} \end{bmatrix}; \\ J_2 &= \begin{bmatrix} 0_{\bar{m}_1 \times m_2} \\ I_{m_2 \times m_2} \\ 0_{\tilde{m}_2 \times m_2} \end{bmatrix}; \\ &\vdots \\ J_p &= \begin{bmatrix} 0_{\bar{m}_p \times m_p} \\ I_{m_p \times m_p} \end{bmatrix}. \end{aligned} \quad (17)$$

Here $\bar{m}_i = \sum_{j=1}^{i} m_j$ and $\tilde{m}_i = m - \bar{m}_i$ for $i = 1, 2, \ldots, p$.

Now it follows from the above construction that if we apply the decentralized stable output feedback control $u(s) = \tilde{H}(s)y(s)$ to the uncertain system (1), (3), we obtain the same closed loop system as if we apply the controller $u(s) = H(s)y(s)$ to the following uncertain system:

$$\begin{array}{rcl} \dot{x}(t) &=& Ax(t) + B_1 w(t) + B_2 u(t) + \sum_{s=1}^{k} D_s \xi_s(t) + \sum_{i=1}^{p} B_2 J_i \tilde{\xi}_i(t); \\ z(t) &=& C_1 x(t) + \sum_{i=1}^{p} D_{12} J_i \tilde{\xi}_i(t) + D_{12} u(t); \\ \zeta_1(t) &=& K_1 x(t) + \sum_{i=1}^{p} G_1 J_i \tilde{\xi}_i(t) + G_1 u(t); \\ && \vdots \\ \zeta_k(t) &=& K_k x(t) + \sum_{i=1}^{p} G_k J_i \tilde{\xi}_i(t) + G_k u(t); \\ y(t) &=& C_2 x(t) + D_{21} w(t). \end{array} \tag{18}$$

Also, $\tilde{\zeta}_1, \ldots \tilde{\zeta}_p$ are defined as in (15), and the additional uncertainty inputs $\tilde{\xi}_i(t)$ are related to the additional uncertainty outputs $\tilde{\zeta}_i(t))$ according to the equations (13), (14). Now for a given stable output feedback controller transfer function matrix $H(s)$, we define the positive constants β_i so that

$$\beta_i \quad \geq \quad \|\Delta_i(s)\|_\infty^2;$$

for $i = 1, 2, \ldots, p$. Here $\|\cdot\|_\infty$ denotes the H^∞ norm and the $\Delta_i(s)$ are defined as in (13). From these inequalities, we can conclude that the uncertainty inputs $\tilde{\xi}_i(t)$ satisfy the following IQCs of the form (3):

$$\int_0^{t_i} (\beta_s \|\tilde{\zeta}_s(t)\|^2 - \|\tilde{\xi}_s(t)\|^2)\, \mathrm{d}t \quad \geq \quad -\tilde{d}_s \tag{19}$$

for all i and for $s = 1, 2, \ldots, p$. Here the $\tilde{d}_s$ are any positive constants. Our main result is obtained by applying the main results of [4] to the uncertain system (18), (15), (3), (19). Note that since the equivalence between the use of a stable decentralized output feedback controller on the uncertain system (1), (3) and the use of a general stable output feedback controller on the uncertain system (18), (15), (3), (19) only holds for the specific realization of the additional uncertainties defined by (13), (14), then we will obtain only sufficient condition for absolute stabilization via a stable decentralized output feedback controller.

In order to construct a stable decentralized output feedback controller for the uncertain system (18), (15), (3), (19), we must use a slight extension of the results of [4]. We note that the results of [4] depend on the results of [7] and as in [7], we consider a corresponding system dependent on a set of scaling parameters $\tau_1, \ldots, \tau_{\tilde{k}}$

where $\tilde{k} = k + p$:

$$\begin{array}{rcl} \dot{x}(t) &=& Ax(t) + \bar{B}_1\bar{w}(t) + B_2u(t); \\ \bar{z}(t) &=& \bar{C}_1x(t) + \bar{D}_{11}\bar{w}(t) + \bar{D}_{12}u(t); \\ y(t) &=& C_2x(t) + \bar{D}_{21}\bar{w}(t). \end{array} \tag{20}$$

Here

$$\begin{array}{rcl} \bar{C}_1 &=& \left[\begin{array}{c} C_1 \\ \sqrt{\tau_1}K_1 \\ \vdots \\ \sqrt{\tau_k}K_k \\ \sqrt{\tau_{k+1}\beta_1}C_{1,1} \\ \vdots \\ \sqrt{\tau_{k+p}\beta_p}C_{1,p} \end{array}\right]; \bar{D}_{12} = \left[\begin{array}{c} D_{12} \\ \sqrt{\tau_1}G_1 \\ \vdots \\ \sqrt{\tau_k}G_k \\ 0 \\ \vdots \\ 0 \end{array}\right]; \\ \bar{D}_{11} &=& \left[\begin{array}{ccccccc} 0 & 0 & \dots & 0 & \frac{1}{\sqrt{\tau_{k+1}}}D_{12}J_1 & \dots & \frac{1}{\sqrt{\tau_{k+p}}}D_{12}J_p \\ 0 & 0 & \dots & 0 & \sqrt{\frac{\tau_1}{\tau_{k+1}}}G_1J_1 & \dots & \sqrt{\frac{\tau_1}{\tau_{k+p}}}G_1J_p \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & \sqrt{\frac{\tau_k}{\tau_{k+1}}}G_kJ_1 & \dots & \sqrt{\frac{\tau_k}{\tau_{k+p}}}G_kJ_p \\ \frac{\sqrt{\tau_{k+1}}}{\gamma}H_1 & 0 & \dots & 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \frac{\sqrt{\tau_{k+p}}}{\gamma}H_p & 0 & \dots & 0 & 0 & & 0 \end{array}\right]; \\ \bar{B}_1 &=& \left[\begin{array}{ccc} \gamma^{-1}B_1 & \bar{B}_{1,1} & \bar{B}_{1,2} \end{array}\right]; \\ \bar{B}_{1,1} &=& \left[\begin{array}{ccc} \sqrt{\tau_1}^{-1}D_1 & \dots & \sqrt{\tau_k}^{-1}D_k \end{array}\right]; \\ \bar{B}_{1,2} &=& \left[\begin{array}{ccc} \sqrt{\tau_{k+1}}^{-1}B_2J_1 & \dots & \sqrt{\tau_{k+p}}^{-1}B_2J_p \end{array}\right]; \\ \bar{D}_{21} &=& \left[\begin{array}{cccc} \gamma^{-1}D_{21} & 0 & 0 & 0 \end{array}\right]. \end{array} \tag{21}$$

The results of [4,7] involve solving the H^∞ control problem corresponding to the system (20) and the H^∞ norm bound condition

$$\bar{J} \triangleq \sup_{\bar{w}(\cdot)\in \boldsymbol{L}_2[0,\infty),x(0)=0,x_c(0)=0} \frac{\|\bar{z}(\cdot)\|_2^2}{\|\bar{w}(\cdot)\|_2^2} < 1. \tag{22}$$

Here,

$$\bar{w}(\cdot) = \left[\begin{array}{c} \gamma w(\cdot) \\ \sqrt{\tau_1}\xi_1(\cdot) \\ \vdots \\ \sqrt{\tau_k}\xi_k(\cdot) \\ \sqrt{\tau_{k+1}}\tilde{\xi}_1(\cdot) \\ \vdots \\ \sqrt{\tau_{k+p}}\tilde{\xi}_p(\cdot) \end{array}\right]; \quad \bar{z}(\cdot) = \left[\begin{array}{c} z(\cdot) \\ \sqrt{\tau_1}\zeta_1(\cdot) \\ \vdots \\ \sqrt{\tau_k}\zeta_k(\cdot) \\ \sqrt{\tau_{k+1}}\tilde{\zeta}_1(\cdot) \\ \vdots \\ \sqrt{\tau_{k+p}}\tilde{\zeta}_p(\cdot) \end{array}\right]$$

In order to solve this H^∞ problem, we will convert it into a standard H^∞ control problem by removing the $\bar{D}_{11}$ term using standard loop shifting ideas; e. g., see Section 5.5 of [1]. In order to achieve this, we restrict attention to parameters $\tau_1 \dots \tau_{\tilde{k}}$ such that $\bar{D}_{11}'\bar{D}_{11} < I$:

Assumption 2. The constants $\tau_1 > 0, \dots, \tau_{\tilde{k}} > 0$ are assumed to be chosen such that $\bar{D}_{11}'\bar{D}_{11} < I$.

Now define $\quad \Phi = I - \bar{D}_{11}'\bar{D}_{11} > 0; \quad \bar{\Phi} = I - \bar{D}_{11}\bar{D}_{11}' > 0$

Also, define transformed inputs and outputs as

$$\begin{aligned}\hat{w} &\triangleq \Phi^{\frac{1}{2}}\bar{w} - \Phi^{-\frac{1}{2}}\bar{D}_{11}'\left[\bar{C}_1 x + \bar{D}_{12}u\right];\\ \hat{z} &\triangleq \bar{\Phi}^{-\frac{1}{2}}\left[\bar{C}_1 x + \bar{D}_{12}u\right].\end{aligned}$$

Hence, $$\bar{w} = \Phi^{-\frac{1}{2}}\hat{w} + \Phi^{-1}\bar{D}_{11}'\left[\bar{C}_1 x + \bar{D}_{12}u\right].$$

Now, using these definitions, it is straightforward to verify that

$$\|\bar{w}(t)\|^2 - \|\bar{z}(t)\|^2 \equiv \|\hat{w}(t)\|^2 - \|\hat{z}(t)\|^2.$$

Therefore, the H^∞ norm bound condition (22) will hold if and only if

$$\hat{J} \triangleq \sup_{\hat{w}(\cdot)\in \boldsymbol{L}_2[0,\infty), x(0)=0, x_c(0)=0} \frac{\|\hat{z}(\cdot)\|_2^2}{\|\hat{w}(\cdot)\|_2^2} < 1. \tag{23}$$

Also, we can re-write the state equations (20) as

$$\begin{aligned}\dot{x}(t) &= \hat{A}x(t) + \hat{B}_1\hat{w}(t) + B_2 u(t);\\ \hat{z}(t) &= \hat{C}_1 x(t) + \hat{D}_{12}u(t);\\ y(t) &= \hat{C}_2 x(t) + \hat{D}_{21}\hat{w}(t) + \hat{D}_{22}u(t)\end{aligned} \tag{24}$$

where

$$\begin{aligned}\hat{A} &\triangleq A + \bar{B}_1\bar{D}_{11}'\bar{\Phi}^{-1}\bar{C}_1;\\ \hat{B}_1 &\triangleq \bar{B}_1\Phi^{-\frac{1}{2}}\\ \hat{B}_2 &\triangleq B_2 + \bar{B}_1\bar{D}_{11}'\bar{\Phi}^{-1}\bar{D}_{12};\\ \hat{C}_1 &\triangleq \bar{\Phi}^{-\frac{1}{2}}\bar{C}_1;\\ \hat{D}_{12} &\triangleq \bar{\Phi}^{-\frac{1}{2}}\bar{D}_{12};\\ \hat{C}_2 &\triangleq C_2 + \bar{D}_{21}\bar{D}_{11}'\bar{\Phi}^{-1}\bar{C}_1;\\ \hat{D}_{21} &\triangleq \bar{D}_{21}\Phi^{-\frac{1}{2}};\\ \hat{D}_{22} &\triangleq \bar{D}_{21}\bar{D}_{11}'\bar{\Phi}^{-1}\bar{D}_{12};\\ \hat{E}_1 &= \hat{D}_{12}'\hat{D}_{12}.\end{aligned} \tag{25}$$

The main idea behind the result of [4] is to force the controller to be stable by introducing some extra uncertainty into the origin uncertain system. This is done in such a way so that the controller must not only achieve absolute stabilization with disturbance attenuation γ when applied to the original uncertain system but also the controller must achieve internal stability when applied to a "null" system; i. e., the controller itself must be stable. In order to define the required new uncertain system, we consider a state feedback version of the problem of absolute stabilization with disturbance attenuation (which corresponds to a state feedback version of the H^∞ problem (24), (23)). The solution to this problem is given in terms of the existence of solutions to a parameter dependent algebraic Riccati equation. The Riccati equation under consideration is defined as follows: Let $\tau_1 > 0, \ldots, \tau_{\tilde{k}} > 0$ and $\beta_1 > 0, \ldots, \beta_p > 0$, be given constants and consider the algebraic Riccati equation

$$\begin{aligned}(\hat{A} - \hat{B}_2\hat{E}_1^{-1}\hat{D}_{12}'\hat{C}_1)'X + X(\hat{A} - \hat{B}_2\hat{E}_1^{-1}\hat{D}_{12}'\hat{C}_1)\\ +X(\hat{B}_1\hat{B}_1' - \hat{B}_2\hat{E}_1^{-1}\hat{B}_2')X\\ +\hat{C}_1'(I - \hat{D}_{12}\hat{E}_1^{-1}\hat{D}_{12}')\hat{C}_1 = 0;\end{aligned} \tag{26}$$

Assumption 3. The uncertain system (24), (3), (19) will be assumed to be such that $\hat{E}_1 > 0$ for any $\tau_1 > 0, \ldots, \tau_{\tilde{k}} > 0$.

We now present a result relating the Riccati equation (26) to the problem of absolute stabilization with disturbance attenuation via state feedback. The proof of this theorem follows along similar lines to the proof of a corresponding result given in [4].

Lemma 1. Let $\beta_1 > 0, \ldots, \beta_p > 0$ be given constants and suppose the uncertain system (24), (3), (19) satisfies Assumptions 1 and 3 and is absolutely stabilizable with disturbance attenuation γ via a controller of the form (7) (but which is not necessarily stable). Then, there exist constants $\tau_1 > 0, \ldots, \tau_{\tilde{k}} > 0$ satisfying Assumption 2 and such that the Riccati equation (26) has a solution $X > 0$. Furthermore, the uncertain system (24), (3), (19) is absolutely stabilizable with disturbance attenuation γ via the state feedback controller

$$u(t) = Kx(t) \tag{27}$$

where

$$K = -\hat{E}_1^{-1}(\hat{B}_2'X + \hat{D}_{12}'\hat{C}_1). \tag{28}$$

We now suppose that constants $\beta_1 > 0, \ldots, \beta_p > 0$, $\tau_1 > 0, \ldots, \tau_{\tilde{k}} > 0$ have been found such that Assumption 2 is satisfied and the Riccati equation (26) has a solution $X > 0$ and we will use the corresponding state feedback gain matrix K defined in (28) to define a new uncertain system as follows

$$\begin{aligned}\dot{x}(t) &= \tilde{A}x(t) + B_1w(t) + \tilde{B}_2u(t) + \sum_{s=1}^{k+1} D_s\xi_s(t) + \sum_{i=1}^{p} B_2J_i\tilde{\xi}_i(t);\\ z(t) &= \tilde{C}_1x(t) + J\xi_{k+1} + \sum_{i=1}^{p} D_{12}J_i\tilde{\xi}_i(t) + \tilde{D}_{12}u(t);\end{aligned}$$

$$
\begin{array}{rcl}
\zeta_1(t) & = & \tilde{K}_1 x(t) + F_1 \xi_{k+1} + \sum_{i=1}^{p} G_1 J_i \tilde{\xi}_i(t) + \tilde{G}_1 u(t); \\
 & \vdots & \\
\zeta_k(t) & = & \tilde{K}_k x(t) + F_k \xi_{k+1} + \sum_{i=1}^{p} G_k J_i \tilde{\xi}_i(t) + \tilde{G}_k u(t); \\
\zeta_{k+1}(t) & = & \tilde{K}_{k+1} x(t) + \tilde{G}_{k+1} u(t); \\
\tilde{\zeta}_1(t) & = & C_{1,1} x(t) + H_1 w; \\
\tilde{\zeta}_2(t) & = & C_{1,2} x(t) + H_2 w; \\
 & \vdots & \\
\tilde{\zeta}_p(t) & = & C_{1,p-1} x(t) + H_p w; \\
y(t) & = & C_2 x(t) + D_{21} w(t)
\end{array}
\tag{29}
$$

where

$$
\begin{array}{rcl}
\tilde{A} & = & A + \frac{1}{2} B_2 K; \quad \tilde{B}_2 = \frac{1}{2} B_2; \quad D_{k+1} = B_2; \\
\tilde{C}_1 & = & C_1 + \frac{1}{2} D_{12} K; \quad J = D_{12}; \quad \tilde{D}_{12} = \frac{1}{2} D_{12}; \\
\tilde{K}_1 & = & K_1 + \frac{1}{2} G_1 K; \quad F_1 = G_1; \quad \tilde{G}_1 = \frac{1}{2} G_1; \\
 & \vdots & \\
\tilde{K}_k & = & K_k + \frac{1}{2} G_k K; \quad F_k = G_k; \quad \tilde{G}_k = \frac{1}{2} G_k; \\
\tilde{K}_{k+1} & = & \frac{1}{2} K; \quad \tilde{G}_{k+1} = -\frac{1}{2} I_{m\times m}.
\end{array}
\tag{30}
$$

Also, we extend the IQC (3) to include the additional uncertainty input ξ_{k+1}:

$$
\int_0^{t_i} \|\xi_s(t)\|^2 \mathrm{d}t \le \int_0^{t_i} \|\zeta_s(t)\|^2 \, \mathrm{d}t + d_s \quad \forall i \ \ \forall s = 1, \ldots, k+1. \tag{31}
$$

Here d_{k+1} is any positive constant. We consider two special cases of the uncertainty input ξ_{k+1}.

Case 1. $\xi_{k+1}(t) \equiv \zeta_{k+1}(t) = \frac{1}{2} K x(t) - \frac{1}{2} u(t)$. In this case, it is clear that this uncertainty input satisfies the IQC (31). Also, it is straightforward to verify that with this value of $\xi_{k+1}(t)$ the system (29) becomes

$$
\begin{array}{rcl}
\dot{x}(t) & = & (A + B_2 K) x(t) + B_1 w(t) + \sum_{s=1}^{k} D_s \xi_s(t) + \sum_{i=1}^{p} B_2 J_i \tilde{\xi}_i(t); \\
z(t) & = & (C_1 + D_{12} K) x(t) + \sum_{i=1}^{p} D_{12} J_i \tilde{\xi}_i(t);
\end{array}
$$

$$
\begin{aligned}
\zeta_1(t) &= (K_1 + G_1K)x(t) + \sum_{i=1}^{p} G_1J_i\tilde{\xi}_i(t);\\
&\vdots\\
\zeta_k(t) &= (K_k + G_kK)x(t) + \sum_{i=1}^{p} G_kJ_i\tilde{\xi}_i(t);\\
\tilde{\zeta}_1(t) &= C_{1,1}x(t) + H_1w(t);\\
\tilde{\zeta}_2(t) &= C_{1,2}x(t) + H_2w(t);\\
&\vdots\\
\tilde{\zeta}_p(t) &= C_{1,p}x(t) + H_pw(t);\\
y(t) &= C_2x(t) + D_{21}w(t)
\end{aligned} \tag{32}
$$

where the IQC (3) is satisfied. However, the uncertain system (32), (3), (19) is the closed loop uncertain system obtained when the state feedback control law (27), (28) is applied to the uncertain system (18), (15), (3), (19). Thus, according to the construction of K and Lemma 1, this uncertain system will be absolutely stable with disturbance attenuation γ. It should also be noted that for the system (32), the control input $u(t)$ (which is the output of the controller) does not affect the system.

Case 2. $\xi_{k+1}(t) \equiv -\zeta_{k+1}(t) = -\frac{1}{2}Kx(t) + \frac{1}{2}u(t)$. In this case, it is clear that this uncertainty input satisfies the IQC (31). Also, it is straightforward to verify that with this value of $\xi_{k+1}(t)$ the system (29) reduces to the original system (18), (15).

In order to obtain our main result, we will follow the approach taken in [4] and apply the results of [7] to the uncertain system (29), (19), (31). Indeed, if the uncertain system (29), (19), (31) is absolutely stabilizable with disturbance attenuation γ via an output feedback controller of the form (7) (not necessarily stable) then it follows from Case 1 above that for the corresponding value of the additional uncertainty, this is equivalent to the open loop situation illustrated in Figure 1.

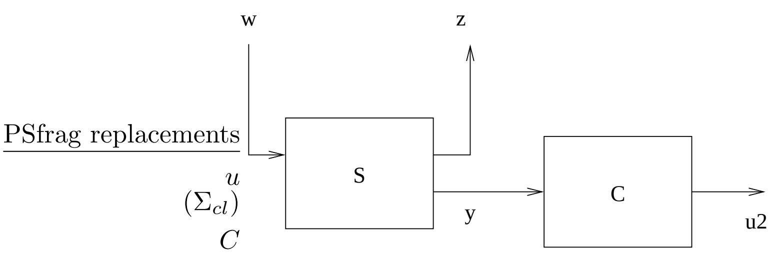

Fig. 1. Block diagram corresponding to Case 1.

In this block diagram the block (Σ_{cl}) refers to the closed loop uncertain system defined by (32), (19), (31) and the block C refers to the output feedback controller of the form (7). Since definition of absolute stabilizability with disturbance attenuation γ requires the stability of the entire closed loop system, it follows that the output feedback controller must in fact be stable.

It follows from Case 2 above that for the corresponding value of additional uncertainty, when the controller (7) is applied to the uncertain system (29), (19), (31), this is equivalent to the situation shown in Figure 2.

In this block diagram the block (Σ) refers to the uncertain system defined by (18), (15), (3), (19) and the block C refers to the output feedback controller of the form (7). From this, we can conclude that the output feedback controller (7) solves the problem of absolute stabilizability with disturbance attenuation γ.

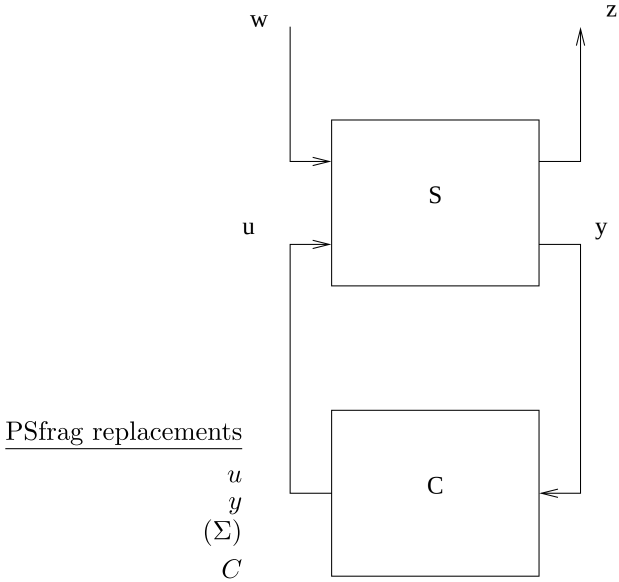

Fig. 2. Block diagram corresponding to Case 2.

Combining the conclusions from both cases, we can conclude that the output feedback controller of the form (7) obtained by applying results of [7] to the uncertain system (29), (19), (31) is in fact a stable output feedback controller which solves the problem absolute stabilizability with disturbance attenuation γ for the uncertain system (18), (15), (3), (19). This leads us to the following result which is stated in terms of a pair of algebraic Riccati equations. The Riccati equations under consideration are defined as follows: Let $\tilde{\tau}_1 > 0, \ldots, \tilde{\tau}_{\bar{k}} > 0$ be given constants where $\bar{k} = \tilde{k} + 1$. Consider the algebraic Riccati equations

$$\begin{aligned}(\check{A} - \check{B}_2\check{E}_1^{-1}\check{D}_{12}'\check{C}_1)'\check{X} + \check{X}(\check{A} - \check{B}_2\check{E}_1^{-1}\check{D}_{12}'\check{C}_1)\\ +\check{X}(\check{B}_1\check{B}_1' - \check{B}_2\check{E}_1^{-1}\check{B}_2')\check{X}\\ +\check{C}_1'(I - \check{D}_{12}\check{E}_1^{-1}\check{D}_{12}')\check{C}_1 = 0;\end{aligned} \quad (33)$$

$$\begin{aligned}(\check{A} - \check{B}_1\check{D}_{21}'\check{E}_2^{-1}\check{C}_2)\check{Y} + \check{Y}(\check{A} - \check{B}_1\check{D}_{21}'\check{E}_2^{-1}\check{C}_2)'\\ +\check{Y}(\check{C}_1'\check{C}_1 - \check{C}_2'\check{E}_2^{-1}\check{C}_2)\check{Y}\\ +\check{B}_1(I - \check{D}_{21}'\check{E}_2^{-1}\check{D}_{21})\check{B}_1' = 0\end{aligned} \quad (34)$$

where

$$
\begin{aligned}
\breve{A} &= \tilde{A} + \breve{B}_1 \breve{D}_{11}' \left(I_{\tilde{q}\times\tilde{q}} - \breve{D}_{11}\breve{D}_{11}'\right)^{-1} \breve{C}_1; \\
\breve{B}_2 &= \tilde{B}_2 + \breve{B}_1 \breve{D}_{11}' \left(I_{\tilde{q}\times\tilde{q}} - \breve{D}_{11}\breve{D}_{11}'\right)^{-1} \breve{D}_{12}; \\
\breve{C}_2 &= C_2 + \breve{D}_{21} \breve{D}_{11}' \left(I_{\tilde{q}\times\tilde{q}} - \breve{D}_{11}\breve{D}_{11}'\right)^{-1} \breve{C}_1; \\
\breve{D}_{22} &= \breve{D}_{21} \breve{D}_{11}' \left(I_{\tilde{q}\times\tilde{q}} - \breve{D}_{11}\breve{D}_{11}'\right)^{-1} \breve{D}_{12}; \\
\breve{B}_1 &= \breve{B}_1 \left(I_{\tilde{p}\times\tilde{p}} - \breve{D}_{11}'\breve{D}_{11}\right)^{-\frac{1}{2}};\ \breve{D}_{21} = \breve{D}_{21} \left(I_{\tilde{p}\times\tilde{p}} - \breve{D}_{11}'\breve{D}_{11}\right)^{-\frac{1}{2}}; \\
\breve{C}_1 &= \left(I_{\tilde{q}\times\tilde{q}} - \breve{D}_{11}\breve{D}_{11}'\right)^{-\frac{1}{2}} \breve{C}_1;\ \breve{D}_{12} = \left(I_{\tilde{q}\times\tilde{q}} - \breve{D}_{11}\breve{D}_{11}'\right)^{-\frac{1}{2}} \breve{D}_{12}; \\
\breve{E}_1 &= \breve{D}_{12}'\breve{D}_{12};\ \breve{E}_2 = \breve{D}_{21}\breve{D}_{21}'; \breve{B}_1 = \left[\begin{array}{ccc} \gamma^{-1}B_1 & \breve{B}_{1,1} & \breve{B}_{1,2} \end{array}\right]; \\
\breve{B}_{1,1} &= \left[\begin{array}{ccc} \sqrt{\tilde{\tau}_1}^{-1} D_1 & \ldots & \sqrt{\tilde{\tau}_{k+1}}^{-1} D_{k+1} \end{array}\right]; \\
\breve{B}_{1,2} &= \left[\begin{array}{ccc} \sqrt{\tilde{\tau}_{k+2}}^{-1} B_2 J_1 & \ldots & \sqrt{\tilde{\tau}_{k+p+1}}^{-1} B_2 J_p \end{array}\right]; \\
\breve{D}_{21} &= \left[\begin{array}{cccc} \gamma^{-1}D_{21} & 0 & 0 & 0 \end{array}\right];
\end{aligned}
\tag{35}
$$

$$
\breve{C}_1 = \left[\begin{array}{c} \tilde{C}_1 \\ \sqrt{\tilde{\tau}_1}\tilde{K}_1 \\ \vdots \\ \sqrt{\tilde{\tau}_{k+1}}\tilde{K}_{k+1} \\ \sqrt{\tilde{\tau}_{k+2}\beta_1} C_{1,1} \\ \vdots \\ \sqrt{\tilde{\tau}_{k+p+1}\beta_p} C_{1,p} \end{array}\right]; \breve{D}_{12} = \left[\begin{array}{c} \tilde{D}_{12} \\ \sqrt{\tilde{\tau}_1}\tilde{G}_1 \\ \vdots \\ \sqrt{\tilde{\tau}_{k+1}}\tilde{G}_{k+1} \\ 0 \\ \vdots \\ 0 \end{array}\right];
\tag{36}
$$

$$
\breve{D}_{11} = \left[\begin{array}{ccccccc}
0 & 0 & \ldots & 0 & \frac{1}{\sqrt{\tilde{\tau}_{k+1}}} J & \frac{1}{\sqrt{\tilde{\tau}_{k+2}}} D_{12} J_1 & \ldots & \frac{1}{\sqrt{\tilde{\tau}_{k+p+1}}} D_{12} J_p \\
0 & 0 & \ldots & 0 & \sqrt{\frac{\tilde{\tau}_1}{\tilde{\tau}_{k+1}}} F_1 & \sqrt{\frac{\tilde{\tau}_1}{\tilde{\tau}_{k+2}}} G_1 J_1 & \ldots & \sqrt{\frac{\tilde{\tau}_1}{\tilde{\tau}_{k+p+1}}} G_1 J_p \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \ldots & 0 & \sqrt{\frac{\tilde{\tau}_k}{\tilde{\tau}_{k+1}}} F_k & \sqrt{\frac{\tilde{\tau}_k}{\tilde{\tau}_{k+2}}} G_k J_1 & \ldots & \sqrt{\frac{\tilde{\tau}_k}{\tilde{\tau}_{k+p+1}}} G_k J_p \\
\frac{\sqrt{\tilde{\tau}_{k+2}}}{\gamma} H_1 & 0 & \ldots & 0 & 0 & 0 & & 0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \\
\frac{\sqrt{\tilde{\tau}_{k+p+1}}}{\gamma} H_p & 0 & \ldots & 0 & 0 & 0 & & 0
\end{array}\right].
\tag{37}
$$

Assumption 4. The constants $\tilde{\tau}_1 > 0, \ldots, \tilde{\tau}_{\tilde{k}} > 0$ are assumed to be chosen such that

(i) $\check{E}_1 > 0$.

(ii) $\check{E}_2 > 0$.

(iii) $\check{D}_{11}\check{D}_{11}' < I$.

Theorem 1. Let $\beta_1 > 0, \ldots, \beta_p > 0$ be given constants and suppose that the uncertain system (18), (15), (3), (19) satisfies Assumptions 1 and 3 and that there exist constants $\tau_1 > 0, \ldots, \tau_{\tilde{k}} > 0$ such that Assumption 2 is satisfied and the Riccati equation (26) has a solution $X > 0$. Furthermore, suppose there exist constants $\tilde{\tau}_1 > 0, \ldots, \tilde{\tau}_{\tilde{k}} > 0$ such that Assumption 4 is satisfied and the Riccati equations (33) and (34) have solutions $\check{X} > 0$ and $\check{Y} > 0$ such that the spectral radius of their product satisfies $\rho(\check{X}\check{Y}) < 1$. Then the uncertain system (18), (15), (3), (19) is absolutely stabilizable with disturbance attenuation γ via a stable linear controller of the form (7) where

$$\begin{aligned} A_c &= \check{A}_c - B_c\check{D}_{22}C_c \\ \check{A}_c &= \check{A} + \check{B}_2C_c - B_c\check{C}_2 + (\check{B}_1 - B_c\check{D}_{21})\check{B}_1'\check{X} \\ B_c &= (I - \check{Y}\check{X})^{-1}(\check{Y}\check{C}_2' + \check{B}_1\check{D}_{21}')\check{E}_2^{-1} \\ C_c &= -\check{E}_1^{-1}(\check{B}_2'\check{X} + \check{D}_{12}'\check{C}_1). \end{aligned} \tag{38}$$

P r o o f. It follows via a similar argument to the proof of Theorem 4.1 of [7] that the uncertain system (29), (19), (31) is absolutely stabilizable with disturbance attenuation γ via a controller of the form (7) if and only if there exist constants $\tilde{\tau}_1 > 0, \ldots, \tilde{\tau}_{\tilde{k}} > 0$ such that the controller (7) solves the H^∞ control problem defined by the system

$$\begin{aligned} \dot{x}(t) &= \tilde{A}x(t) + \check{B}_1\check{w}(t) + \tilde{B}_2u(t); \\ \check{z}(t) &= \check{C}_1x(t) + \check{D}_{11}\check{w}(t) + \check{D}_{12}u(t); \\ y(t) &= C_2x(t) + \check{D}_{21}\check{w}(t) \end{aligned} \tag{39}$$

and the H^∞ norm bound condition

$$\check{J} \triangleq \sup_{\check{w}(\cdot)\in \boldsymbol{L}_2[0,\infty),x(0)=0,x_c(0)=0} \frac{\|\check{z}(\cdot)\|_2^2}{\|\check{w}(\cdot)\|_2^2} < 1. \tag{40}$$

Here,

$$\check{w}(\cdot) = \begin{bmatrix} \gamma w(\cdot) \\ \sqrt{\tilde{\tau}_1}\xi_1(\cdot) \\ \vdots \\ \sqrt{\tilde{\tau}_{k+1}}\xi_{k+1}(\cdot) \\ \sqrt{\tilde{\tau}_{k+2}}\tilde{\xi}_1(\cdot) \\ \vdots \\ \sqrt{\tilde{\tau}_{k+p+1}}\tilde{\xi}_p(\cdot) \end{bmatrix}; \quad \check{z}(\cdot) = \begin{bmatrix} z(\cdot) \\ \sqrt{\tilde{\tau}_1}\zeta_1(\cdot) \\ \vdots \\ \sqrt{\tilde{\tau}_k + 1}\zeta_{k+1}(\cdot) \\ \sqrt{\tilde{\tau}_{k+2}}\tilde{\zeta}_1(\cdot) \\ \vdots \\ \sqrt{\tilde{\tau}_{k+p+1}}\tilde{\zeta}_p(\cdot) \end{bmatrix}$$

and the matrix coefficients $\breve{B}_1,\ \breve{C}_1,\ \breve{D}_{11},\ \breve{D}_{12},\ \breve{D}_{21}$ are defined by (35), (36), (37). Furthermore, it follows from standard loop shifting arguments in H^∞ control theory (e. g., see Sections 4.5.1 and 5.5.1 in [1] and Section 17.2 in [15]) that the H^∞ control problem (39), (40) has a solution if and only if the Riccati equations (33) and (34) have solutions $\check{X} > 0$ and $\check{Y} > 0$ and such that the spectral radius of their product satisfies $\rho(\check{X}\check{Y}) < 1$. Furthermore in this case, a controller of the form (7) which solves the H^∞ control problem (39), (40) is defined by the equations (38).

We can now conclude that if the conditions of the theorem are satisfied, then the controller (7), (38) is absolutely stabilizing with disturbance attenuation γ for the uncertain system (29), (19), (31). Then, using the arguments given above, it follows that the controller (7), (38) is stable and is absolutely stabilizing with disturbance attenuation γ for the uncertain system (18), (15), (3), (19). □

We now recall the construction of the uncertain system (18), (15), (3), (19) was such that if a stable controller of the form (7) with transfer function matrix (8) is absolutely stabilizing with disturbance attenuation γ and if this controller is such that the transfer function matrices defined by (13) satisfy the bounds (19), then the corresponding stable decentralized controller defined by (12) is absolutely stabilizing with disturbance attenuation γ. This leads to the following theorem which is the main result of this paper.

Theorem 2. Let $\beta_1 > 0, \ldots, \beta_p > 0$ be given constants and suppose that the uncertain system (1), (3), satisfies Assumptions 1 and 3 and that there exist constants $\tau_1 > 0, \ldots, \tau_{\tilde{k}} > 0$ such that Assumption 2 is satisfied and the Riccati equation (26) has a solution $X > 0$. Furthermore, suppose there exist constants $\tilde{\tau}_1 > 0, \ldots, \tilde{\tau}_{\tilde{k}} > 0$ such that Assumption 4 is satisfied and the Riccati equations (33) and (34) have solutions $\check{X} > 0$ and $\check{Y} > 0$ such that the spectral radius of their product satisfies $\rho(\check{X}\check{Y}) < 1$. Also, suppose that the stable linear controller defined by (7), (38) with transfer function matrix (8) is such that the transfer function matrices defined by (13) satisfy the bounds (19). Then the corresponding stable decentralized controller defined by (12) is absolutely stabilizing with disturbance attenuation γ for the uncertain system (1), (3).

Proof. If the conditions of the theorem are satisfied then it follows from Theorem 1 that the uncertain system (18), (15), (3), (19) is absolutely stabilizable with disturbance attenuation γ via the stable linear controller of the form (7), (38). Furthermore, if the controller transfer function matrix $H(s)$ (8) is such that the transfer function matrices defined by (13) satisfy the bounds (19), then it follows that the corresponding uncertainty inputs defined in (14) satisfy the IQCs (19). Furthermore, as noted above in the construction of the uncertain system (18), (15), (3), (19), the closed loop system obtained by applying the decentralized controller $u(s) = \tilde{H}(s)y(s)$ defined by (12) to the uncertain system (1), (3) is identical to the closed loop obtained by applying the controller (7) to the uncertain system (18), (15), (3), (19) when uncertainty inputs defined in (14) are applied. Hence, it follows that the decentralized controller defined by (7), (38), (8), (12) is absolutely stabilizing with disturbance attenuation γ for the uncertain system (1), (3). This completes the proof of the theorem. □

4. ILLUSTRATIVE EXAMPLE

To illustrate the results of this paper with a numerical example, we consider the problem of controlling a pair of vehicles. This example is modified from the example considered in [12]. As in [12], the dynamics of the ith vehicle can be described as

$$\begin{aligned}
\dot{d}_i &= v_{i-1} - v_i; \\
\dot{v}_i &= a_i + w_{1i}; \\
\dot{a}_i &= h_i(v_i, a_i) + u_i; \\
z_i &= \begin{bmatrix} d_i \\ \varepsilon_1 u_i \end{bmatrix}; \\
\zeta_i &= \begin{bmatrix} v_i \\ a_i \end{bmatrix}; \\
y_i &= \begin{bmatrix} d_i + \varepsilon_2 w_{2i} \\ v_i + \varepsilon_3 w_{3i} \end{bmatrix}
\end{aligned}$$

for $i = 1, 2$. Here d_2 represents the distance between the two vehicles and d_1 represents the position of the first vehicle. Also, v_i is the velocity of the ith vehicle, a_i is the acceleration of the ith vehicle, u_i is the control input for the ith vehicle and y_i is the measured output for the ith vehicle. Furthermore, the w_{1i}, w_{2i}, w_{3i} represent disturbance inputs and the z_i are controlled outputs in the H^∞ control problem to be considered. The parameters ε_1, ε_2, ε_3 are treated as design parameters in H^∞ control problem. Moreover, the functions $h_i(v_i, a_i)$ represent uncertain nonlinearities which will be bounded as shown below. Note that in [12], the case of multiple vehicles is considered whereas in our example we only consider two vehicles. However, [12] considers the state feedback case whereas we consider the output feedback case.

In order to obtain an uncertain system of the form (1), we let

$$x_i = \begin{bmatrix} d_i \\ v_i \\ a_i \end{bmatrix}; \quad w_i = \begin{bmatrix} w_{1i} \\ w_{2i} \\ w_{3i} \end{bmatrix}$$

and consider the state equations

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} A_0 & 0 \\ A_1 & A_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_{10} & 0 \\ 0 & B_{10} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} D_0 & 0 \\ 0 & D_0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \\
&\quad + \begin{bmatrix} B_{20} & 0 \\ 0 & B_{20} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}; \\
\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} &= \begin{bmatrix} C_{10} & 0 \\ 0 & C_{10} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} D_{120} & 0 \\ 0 & D_{120} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}; \\
\begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} &= \begin{bmatrix} K_0 & 0 \\ 0 & K_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \\
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \begin{bmatrix} C_{20} & 0 \\ 0 & C_{20} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} D_{210} & 0 \\ 0 & D_{210} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}
\end{aligned}$$

where

$$
\begin{aligned}
A_0 &= \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}; \quad A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad B_{10} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \\
D_0 &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \quad B_{20} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \quad C_{10} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad D_{120} = \begin{bmatrix} 0 \\ \varepsilon_1 \end{bmatrix}; \\
K_0 &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad C_{20} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}; \quad D_{210} = \begin{bmatrix} 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}.
\end{aligned}
$$

The uncertain nonlinearity is such that $h(\xi) = [h_1(v_1, a_1)\ h_2(v_2, a_2)]'$ satisfies the bound

$$h(\zeta)'h(\zeta) \leq \alpha^2 \zeta'\zeta$$

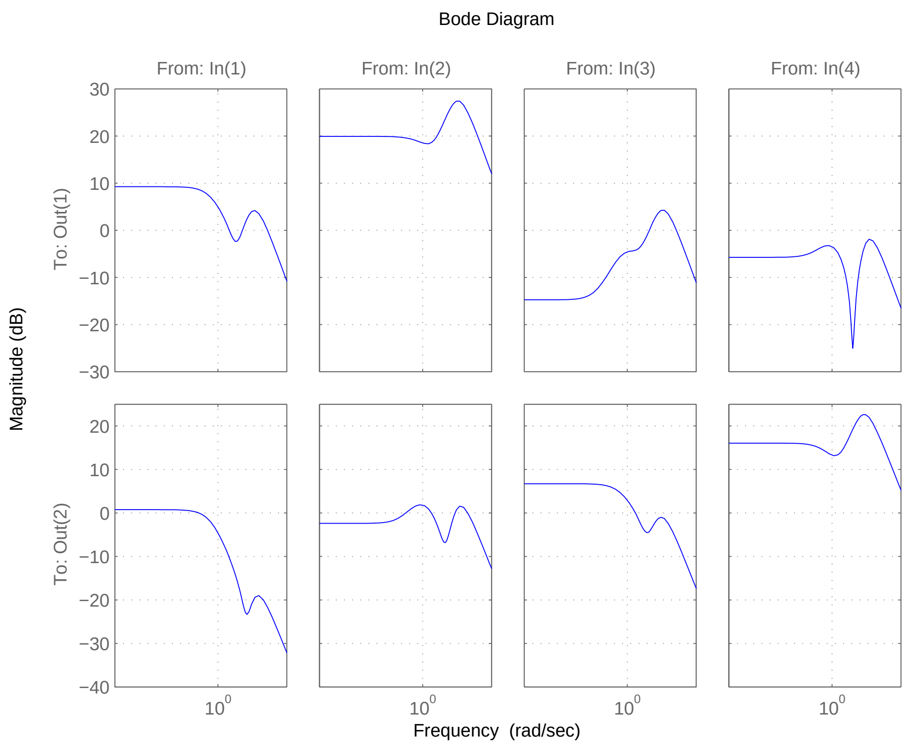

Fig. 3. Magnitude bode plots of the non-decentralized controller (8). The decentralized controller (12) corresponds to the block diagonal part of this controller. In taking the block diagonal part, we group inputs 1 and 2 as well as grouping inputs 3 and 4.

where $\alpha > 0$ is a given constant. Letting $\xi = h(\zeta)$ and integrating, we obtain an IQC of the form (3). Thus, we have defined an uncertain system of the form (1), (3). We then apply our approach to this uncertain system to obtain a decentralized output feedback controller. Indeed, choosing the parameter values $\tau_1 = \tau_2 = \tau_3 = \tau_4 = 1$, $\tilde{\tau}_1 = \tilde{\tau}_2 = 4$, $\tilde{\tau}_3 = 0.5$, $\tilde{\tau}_4 = 3$, $\tilde{\tau}_5 = 2$, $\beta_1 = 3.5$, $\beta_2 = 2.1$, $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0.2$, $\gamma = 5$, $\alpha = 0.1$, it was found that the Riccati equations (26), (33), (34), had suitable stabilizing solutions such that the conditions (19) are satisfied. Then, according to Theorem 2, the corresponding stable decentralized controller defined by (12) is absolutely stabilizing with disturbance attenuation γ for the uncertain system (1), (3). Magnitude bode plots of the non-decentralized controller (8) are shown in Figure 3. The decentralized controller (12) corresponds to the block diagonal part of this controller.

5. CONCLUSIONS

In this paper we have presented a new approach to robust H^∞ control via a stable decentralized output feedback controller. The key idea of our approach is to treat the off-diagonal blocks in the controller transfer function matrix as uncertainties so that they can be neglected to yield a decentralized output feedback controller. One advantage of this approach is that it enables the coupling between the subsystems to be exploited by the controller.

In order for this approach to work, it is necessary that the controller be a stable output feedback controller. This has been achieved by adding additional uncertainties to the system to force the controller to be stable.

A number of possible areas for future research are motivated by the results of this paper. These include an an investigation of the nonlinear constrained optimization problem which needs to be solved in order to find values of the parameters scaling parameters which are used in our solution. Also, it would be of interest to see if our solution to the robust decentralized control problem could be reformulated in terms of LMIs.

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