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Basic Pseudorings*

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Abstract

The concept of a basic pseudoring is introduced. It is shown that every orthomodular lattice can be converted into a basic pseudoring by using of the term operation called Sasaki projection. It is given a mutual relationship between basic algebras and basic pseudorings. There are characterized basic pseudorings which can be converted into othomodular lattices.

Key words: Basic algebra, basic pseudoring, orthomodular lattice.

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It is well-known that every Boolean algebra can be converted into a Boolean ring by using of the symmetrical difference, see e.g. [2]. Also conversely, every Boolean ring can be converted into a Boolean algebra. For orthomodular lattices (instead of Boolean algebras) a similar construction giving a ring-like structure called Boolean quasiring was settled in [6], [7] and generalized for bounded lattices with an antitone involution in [8] and [9]. The natural question is for which algebras used in non-classical logics a similar conversion into a ring-like structure is possible. Of course, Boolean algebras serve as axiomatization of the classical propositional logic and orthomodular lattices play a similar role in the logic of quantum mechanics, see e.g. [1], [7], [8], [9].

In this study we are concentrated in an algebraic counterpart of many-valued logics. This is usually considered to be an MV-algebra for many-valued Łukasiewicz logic. However, it was generalized for more wide class as the concept of basic algebra, see e.g. [3], [4] as sources.

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Let us note that a certain ring-like structures corresponding to MV-algebras were investigated by the first author and H. Länger in [5] and analogously, it was done for pseudo MV-algebras by Y. Shang in [10]. We will involve a similar approach which, however, can be used both for MV-algebras and orthomodular lattices.

The concept of basic algebra was introduced in [3] as a common generalization of an MV-algebra and an orthomodular lattice. Recall that a *basic algebra* (see e.g. [3], [4]) is an algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the following identities

(BA1) $x \oplus 0 = x$;

(BA2) $\neg\neg x = x$ (double negation);

(BA3) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$ (Łukasiewicz axiom);

(BA4) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = 1$ (where $1 := \neg 0$).

Let us note that every basic algebra satisfies also the identities $1 \oplus x = 1 = x \oplus 1$, $0 \oplus x = x$, $x \oplus \neg x = \neg x \oplus x = 1$ (see e.g. [3]). In every basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$, the partial order can be defined by $x \leq y$ if and only if $\neg x \oplus y = 1$. The ordered set $(A; \leq)$ is a bounded lattice where $x \vee y = \neg(\neg x \oplus y) \oplus y$, $x \wedge y = \neg(\neg x \vee \neg y)$ and $1 = \neg 0$. Moreover, it satisfies $y \leq x \oplus y$ and the mapping $x \mapsto \neg x$ is antitone for every $x, y \in A$.

A basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ is called *commutative* if it satisfies the identity $x \oplus y = y \oplus x$.

The concept of symmetrical difference can be introduced for basic algebras in a way similar to that of [6] for orthomodular lattices, however, an operation $\oplus$ is considered instead of $\vee$ in orthomodular lattice because $\oplus$ expresses the logical connective disjunction in the corresponding logic.

Searching for an appropriate ring-like structure, we choose the following one from a number of possible ways.

Definition 1 By a *basic pseudoring* we mean an algebra $\mathcal{R} = (R; +, \cdot, 0, 1)$ of type $(2, 2, 0, 0)$ satisfying the identities

(R1) $1 + 0 = 1$;

(R2) $x \cdot 1 = x$;

(R3) $1 + (1 + x) = x$;

(R4) $(1 + x \cdot (1 + y)) \cdot (1 + y) = (1 + y \cdot (1 + x)) \cdot (1 + x)$;

(R5) $1+(1+(1+(1+((1+x)\cdot(1+y)))\cdot(1+y))\cdot(1+z))\cdot((1+x)\cdot(1+z)) = 1$.

One can immediately mention that this concept differs from the concept of a Boolean quasiring or a generalized Boolean quasiring as defined in [7], [8], [9]. From this point it can be of interest that this ring-like structure can be also reached from every orthomodular structure. Of course, this conversion differs due to the fact that instead of a symmetrical difference (see [6]) the *Sasaki* *operation* (alias *Sasaki projection*, see [1]) is used. Let us recall that by a Sasaki operation of an orthomodular lattice is meant a term operation

$$(x \vee y') \wedge y.$$

We are ready to state our first result.

Theorem 1 *Let $\mathcal{L} = (L; \vee, \wedge, ', 0, 1)$ be an orthomodular lattice. Define*

$$x \cdot y = (x \vee y') \wedge y \quad \textit{and} \quad x + y = ((x' \cdot y)' \cdot (x \cdot y')')'.$$

Then $\mathcal{R}(L) = (R; +, \cdot, 0, 1)$ is a basic pseudoring satisfying the conditions
(a) $x \cdot x = x$
(b) $x \cdot (1+y) = 0 \Rightarrow 1+(1+((1+(1+y)\cdot(1+x))\cdot(1+x))\cdot(1+x))\cdot(1+x) = y$.

Proof It is an immediate reflexion that

$$x \cdot x = (x \vee x') \wedge x = 1 \wedge x = x$$

proving (a).

Further, $1 \cdot x = (1 \vee x') \wedge x = x$ and $0 \cdot x = (0 \vee x') \wedge x = 0$. Hence,

$$1 + x = ((1' \cdot x)' \cdot (1 \cdot x')')' = (0' \cdot x'')' = (1 \cdot x)' = x'.$$

This yields $1 + 0 = 0' = 1$ proving (R1). Evidently,

$$x \cdot 1 = (x \vee 1') \wedge 1 = x$$

proving (R2) and $1 + (1 + x) = x'' = x$ proving (R3). For (R4) we compute

$$\begin{aligned}(1 + x \cdot (1 + y)) \cdot (1 + y) &= (x \cdot y')' \cdot y' = ((x \vee y) \wedge y')' \cdot y' \\ &= (((x \vee y) \wedge y')' \vee y) \wedge y' = ((x \vee y)' \vee y) \wedge y' = (x \vee y)'\end{aligned}$$

due to the orthomodular law since $(x \vee y)' \leq y'$. By symmetry we obtain (R4).

Since

$$\begin{aligned}1 + (1 + ((1 + x) \cdot (1 + y))) \cdot (1 + y) &= 1 + ((x' \vee y) \wedge y')' \cdot y' \\ &= (((x \wedge y') \vee y) \wedge y')' = ((x' \vee y) \wedge y') \vee y = x' \vee y\end{aligned}$$

by the orthomodular law, for (R5) we have

$$\begin{aligned}&1 + (1 + (1 + (1 + ((1 + x) \cdot (1 + y))) \cdot (1 + y)) \cdot (1 + z)) \cdot ((1 + x) \cdot (1 + z)) \\ &\quad = 1 + (1 + (x' \vee y) \cdot (1 + z)) \cdot ((1 + x) \cdot (1 + z)) \\ &\quad = 1 + (1 + (x' \vee y) \cdot z') \cdot ((x' \vee z) \wedge z') \\ &\quad = ((((x' \vee y) \cdot z')') \cdot ((x' \vee z) \wedge z'))' \\ &\quad = (((((x' \vee y) \vee z)' \vee z) \vee ((x' \vee z) \wedge z')') \wedge ((x' \vee z) \wedge z'))' \\ &\quad = ((((x' \vee y) \vee z) \wedge z') \wedge ((x' \vee z) \wedge z')) \vee ((x' \vee z) \wedge z')' \\ &\quad = ((x' \vee z) \wedge z') \vee ((x' \vee z) \wedge z')' = 1.\end{aligned}$$

It remains to prove (b). Assume $x \cdot (1+y) = 0$. Then $0 = x \cdot y' = (x \vee y) \wedge y'$ thus $x \vee y = y$ whence $x \leq y$. Thus

$$\begin{aligned}&1 + (1 + ((1 + (1 + y) \cdot (1 + x)) \cdot (1 + x)) \cdot (1 + x)) \cdot (1 + x)\\ &= 1 + (1 + (y \wedge x') \cdot (1 + x)) \cdot (1 + x) = (y \wedge x') \vee x = y\end{aligned}$$

by the orthomodular law. □

Now, we are going to describe a mutual relationship between basic pseudorings and basic algebras.

Theorem 2 *Let $\mathcal{R} = (R; +, \cdot, 0, 1)$ be a basic pseudoring. Define*

$$x \oplus y = 1 + (1 + x) \cdot (1 + y) \quad \textit{and} \quad \neg x = 1 + x.$$

Then $\mathcal{A}(R) = (R; \oplus, \neg, 0)$ is a basic algebra.

Proof We will check the axioms of a basic algebra.

(BA1): $x \oplus 0 = 1 + (1 + x) \cdot (1 + 0) = 1 + (1 + x) \cdot 1 = 1 + (1 + x) = x$;

(BA2): $\neg\neg x = 1 + (1 + x) = x$;

(BA3): $\neg(\neg x \oplus y) \oplus y =$
$= 1 + (1 + (1 + (\neg x \oplus y))) \cdot (1 + y) = 1 + (\neg x \oplus y) \cdot (1 + y)$
$= 1 + (1 + (1 + (1 + x)) \cdot (1 + y)) \cdot (1 + y) = 1 + (1 + x \cdot (1 + y)) \cdot (1 + y)$
$= 1 + (1 + y \cdot (1 + x)) \cdot (1 + x) = 1 + (\neg y \oplus x) \cdot (1 + x)$
$= \neg(\neg y \oplus x) \oplus x$;

(BA4): $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) =$
$= \neg(\neg((1 + (1 + ((1 + x) \cdot (1 + y))) \cdot (1 + y)) \oplus z) \oplus (1 + (1 + x) \cdot (1 + z))$
$= (1 + (1 + ((1 + x) \cdot (1 + y))) \cdot (1 + y)) \cdot (1 + z) \oplus (1 + (1 + x) \cdot (1 + z))$
$= 1 + (1 + (1 + (1 + ((1 + x) \cdot (1 + y))) \cdot (1 + y)) \cdot (1 + z)) \cdot ((1 + x) \cdot (1 + z))$
$= 1.$ □

We can prove the converse.

Theorem 3 *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra. Define*

$$x + y = \neg(x \oplus \neg y) \oplus \neg(\neg x \oplus y) \quad \textit{and} \quad x \cdot y = \neg(\neg x \oplus \neg y) \quad \textit{and} \quad 1 = \neg 0.$$

Then $\mathcal{R}(A) = (A; +, \cdot, 0, 1)$ is a basic pseudoring satisfying the correspondence identity

$$1 + (1 + (1 + x) \cdot y) \cdot (1 + x \cdot (1 + y)) = x + y. \tag{CI}$$

Proof First we mention that

$$1 + x = \neg(1 \oplus \neg x) \oplus \neg(0 \oplus x) = \neg 1 \oplus \neg x = 0 \oplus \neg x = \neg x.$$

Now we check the axioms of a basic pseudoring.

(R1): $1 + 0 = \neg(1 \oplus \neg 0) \oplus \neg(\neg 1 \oplus 0) = \neg 1 \oplus \neg 0 = 1$;

(R2): $x \cdot 1 = \neg(\neg x \oplus \neg 1) = \neg(\neg x \oplus 0) = \neg\neg x = x$;

(R3): $1+(1+x) = \neg\neg x = x$;

(R4): $(1 + x\cdot(1+y))\cdot(1+y)$
$= (\neg(x\cdot\neg y))\cdot\neg y = (\neg x\oplus y)\cdot\neg y = \neg(\neg(\neg x\oplus y)\oplus y) = \neg(\neg(\neg y\oplus x)\oplus x)$
$= (1+y\cdot(1+x))\cdot(1+x)$;

(R5): $1+(1+(1+(1+((1+x)\cdot(1+y)))\cdot(1+y))\cdot(1+z))\cdot((1+x)\cdot(1+z))$
$= 1+(1+(1+(x\oplus y)\cdot\neg y)\cdot\neg z)\cdot(\neg x\cdot\neg z)$
$= 1+(1+(\neg((x\oplus y)\cdot\neg y))\cdot\neg z)\cdot\neg(x\oplus z)$
$= 1+(1+(\neg(x\oplus y)\oplus y)\cdot\neg z)\cdot\neg(x\oplus z)$
$= 1+(\neg(\neg(x\oplus y)\oplus y)\oplus z)\cdot\neg(x\oplus z)$
$= 1+\neg(\neg(\neg(\neg(x\oplus y)\oplus y)\oplus z)\oplus(x\oplus z))$
$= \neg(\neg(\neg(x\oplus y)\oplus y)\oplus z)\oplus(x\oplus z)$
$= 1.$

Hence, $\mathcal{R}(A) = (A;+,\cdot,0,1)$ is a basic pseudoring. It remains to prove (CI). For this, we compute

$$\begin{aligned} &1+(1+(1+x)\cdot y)\cdot(1+x\cdot(1+y)) \\ &= \neg(\neg(\neg x\cdot y)\cdot\neg(x\cdot\neg y)) = \neg(x\oplus\neg y)\oplus\neg(\neg x\oplus y) = x+y \end{aligned}$$ □

In what follows we show that this relationship is in fact a one-to-one correspondence if $\mathcal{R}$ satisfies the correspondence identity.

Theorem 4 *(a) Let $\mathcal{A} = (A;\oplus,\neg,0)$ be a basic algebra and $\mathcal{R}(A)$ the induced basic pseudoring and $\mathcal{A}(\mathcal{R}(A))$ the induced basic algebra. Then $\mathcal{A}(\mathcal{R}(A)) = \mathcal{A}$.*

(b) Let $\mathcal{R} = (R;+,\cdot,0,1)$ be a basic pseudoring satisfying the correspondence identity (CI), let $\mathcal{A}(R)$ be the induced basic algebra and $\mathcal{R}(\mathcal{A}(R))$ the induced basic pseudoring. Then $\mathcal{R}(\mathcal{A}(R)) = \mathcal{R}$.

Proof Denote by $\widehat{\oplus}$ and $\widehat{\neg}$ the binary and the unary operation of $\mathcal{A}(\mathcal{R}(A))$. Then clearly,

$$\widehat{\neg}x = 1+x = \neg(1\oplus\neg x)\oplus\neg(\neg 1\oplus x) = 0\oplus\neg x = \neg x$$

and

$$x\widehat{\oplus}y = 1+(1+x)\cdot(1+y) = \neg(\neg x\cdot\neg y) = \neg(\neg(x\oplus y)) = x\oplus y$$

thus $\mathcal{A}(\mathcal{R}(A)) = \mathcal{A}$.

Denote by $\widehat{+}$ and $\widehat{\cdot}$ the binary operations of $\mathcal{R}(\mathcal{A}(R))$. Then, due to (CI) we compute

$$\begin{aligned} x\widehat{+}y &= \neg(x\oplus\neg y)\oplus\neg(\neg x\oplus y) = (1+x)\cdot y\oplus x\cdot(1+y) \\ &= 1+(1+(1+x)\cdot y)\cdot(1+x\cdot(1+y)) = x+y \end{aligned}$$

and

$$\begin{aligned} &x\,\widehat{\cdot}\,y = \neg(\neg x\oplus\neg y) = 1+((1+x)\oplus(1+y)) \\ &= 1+(1+(1+(1+x))\cdot(1+(1+y))) = 1+(1+x\cdot y) = x\cdot y \end{aligned}$$

thus also $\mathcal{R}(\mathcal{A}(R)) = \mathcal{R}$. □

Several interesting properties of basic pseudorings are described by the following theorem and its corollary.

Theorem 5 *Let $\mathcal{R} = (R; +, \cdot, 0, 1)$ be a basic pseudoring and $a, b \in R$. Then*

$$a + b = 0 \quad \textit{if and only if} \quad a = b.$$

Proof Let $\mathcal{R} = (R; +, \cdot, 0, 1)$ be a basic pseudoring and $\mathcal{A}(R) = (R; \oplus, \neg, 0)$ the induced basic algebra. In $\mathcal{A}(R)$ we have $c \leq d$ if and only if $\neg c \oplus d = 1$. Since $x \leq x$ and $\neg x \leq \neg x$, we get $\neg x \oplus x = 1$ and $x \oplus \neg x = \neg\neg x \oplus \neg x = 1$ whence

$$x + x = \neg(x \oplus \neg x) \oplus \neg(\neg x \oplus x) = \neg 1 \oplus \neg 1 = 0 \oplus 0 = 0.$$

Assume now that $c, d \in R$ and $c \oplus d = 0$. Since $d \leq c \oplus d = 0$, we conclude $d = 0$ and hence $c = c \oplus 0 = c \oplus d = 0$, i.e.

$$c \oplus d = 0 \Rightarrow c = d = 0. \qquad (**)$$

Suppose $a, b \in R$ and $a + b = 0$. Then

$$\neg(a \oplus \neg b) \oplus \neg(\neg a \oplus b) = 0$$

and, by $(**)$, $\neg(a \oplus \neg b) = 0 = \neg(\neg a \oplus b)$, i.e. $a \oplus \neg b = 1$ and $\neg a \oplus b = 1$ thus $\neg a \leq \neg b$ and $a \leq b$. However, the first inequality yields $b \leq a$ thus $a = b$. □

Corollary 1 *(a) Every basic pseudoring satisfies the identity $x + x = 0$.*

(b) If a pseudoring $\mathcal{R}$ satisfies the identity $x \cdot y = y \cdot x$ then $\mathcal{A}(R)$ is a commutative basic algebra.

(c) If a basic algebra $\mathcal{A}$ is commutative then $\mathcal{R}(A)$ satisfies the identities $x \cdot y = y \cdot x$ and $x + y = y + x$.

In what follows, we are going to show that not only every basic algebra induces a basic pseudoring and vice versa as shown by Theorems 2 and 3 but also Theorem 1 can be inverted, i.e. every orthomodular lattice induces a basic pseudoring satisfying the conditions (a), (b) but also every such basic pseudoring induces an orthomodular latttice.

Now, we are ready to prove the following

Theorem 6 *Let $\mathcal{R} = (R; +, \cdot, 0, 1)$ be a basic pseudoring satisfying the identities (a) and (b) of Theorem 1. Define a binary relation $\leq$ on R as follows*

$$x \leq y \quad \textit{if and only if} \quad x \cdot (1 + y) = 0.$$

Then $\leq$ is an order on R and $(R; \leq)$ is an orthomodular lattice where

$$x \vee y = 1 + (1 + x \cdot (1 + y)) \cdot (1 + y) \quad \textit{and} \quad x' = 1 + x.$$

Proof Let $\mathcal{R} = (R; +, \cdot, 0, 1)$ be a basic pseudoring satisfying (a) and (b). Consider the induced basic algebra $\mathcal{A}(R) = (R; \oplus, \neg, 0)$. Then clearly

$$x \cdot (1 + y) = 0 \quad \text{iff} \quad \neg x \oplus y = 1 \quad \text{iff} \quad x \leq y$$

thus $\leq$ is an order on R and $(R; \leq)$ is the lattice induced by the basic algebra $\mathcal{A}(R)$ where $x \vee y = \neg(\neg x \oplus y) \oplus y = 1 + (1 + x \cdot (1 + y)) \cdot (1 + y)$ and $\neg x = 1 + x$ (as already shown by Theorem 2). Hence, for $x \wedge y = (x' \vee y')'$ we have that $(R; \vee, \wedge, ', 0, 1)$ is a bounded lattice with an antitone involution (i.e. $x'' = x$ and $x \leq y \Rightarrow y' \leq x'$).

Further, by (a) we have $x = x \cdot x = \neg(\neg x \oplus \neg x)$, i.e. $\neg x = \neg x \oplus \neg x$ and, due to the double negation law in $\mathcal{A}(R)$, also $x \oplus x = x$ for each $x \in R$. Thus $\neg x \vee x = \neg(x \oplus x) \oplus x = \neg x \oplus x = 1$ and, due to De Morgan law, also $x \wedge \neg x = \neg(\neg x \vee x) = \neg 1 = 0$ thus $x' = \neg x$ is a complement of x, i.e. $(R; \vee, \wedge, ', 0, 1)$ is an ortholattice.

Finally,

$$\begin{aligned} &1 + (1 + ((1 + (1 + y) \cdot (1 + x)) \cdot (1 + x)) \cdot (1 + x)) \cdot (1 + x) \\ &\quad = 1 + (1 + (y \wedge x') \cdot (1 + x)) \cdot (1 + x) = (y \wedge x') \vee x, \end{aligned}$$

thus $x \leq y \Rightarrow x \cdot (1 + y) = 0$ and, by (b) and the previous computation, $x \vee (x' \wedge y) = y$, which is the orthomodular law. Hence, $(R; \vee, \wedge, ', 0, 1)$ is an orthomodular lattice. $\square$

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