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# On the Lindelöf property of spaces of continuous functions over a Tychonoff space and its subspaces

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*Abstract.* We study relations between the Lindelöf property in the spaces of continuous functions with the topology of pointwise convergence over a Tychonoff space and over its subspaces. We prove, in particular, the following: a) if $C_p(X)$ is Lindelöf, $Y = X \cup \{p\}$, and the point $p$ has countable character in $Y$, then $C_p(Y)$ is Lindelöf; b) if $Y$ is a cozero subspace of a Tychonoff space $X$, then $l(C_p(Y)^\omega) \leq l(C_p(X)^\omega)$ and $\mathrm{ext}(C_p(Y)^\omega) \leq \mathrm{ext}(C_p(X)^\omega)$.

*Keywords:* pointwise convergence, Lindelöf property

*Classification:* 54C35, 54D20

All spaces below are assumed to be Tychonoff (that is, completely regular Hausdorff). We use terminology and notation as in [Eng].

Given two spaces $X$ and $Z$, we denote by $C_p(X, Z)$ the space of all continuous functions from $X$ to $Z$ equipped with the topology of pointwise convergence (that is, the topology of the subspace of the space $Z^X$ of all functions from $X$ to $Z$ endowed with the Tychonoff product topology). The space $C_p(X, \mathbb{R})$ is denoted as $C_p(X)$.

If $p\colon X \to Y$ is a continuous mapping, the *dual mapping* $p^*\colon C_p(Y, Z) \to C_p(X, Z)$ is defined by the rule: $p^*(f) = f \circ p$ for all $f \in C_p(Y)$. The dual mapping is always continuous, is a homeomorphic embedding if $p$ is onto, and is a closed embedding if $p$ is quotient; see [Arh2].

A space $X$ is a $\mathcal{K}_{\sigma\delta}$*-space* if it is an $F_{\sigma\delta}$-set in $\beta X$; $\mathcal{K}$*-analytic* spaces are continuous images of $\mathcal{K}_{\sigma\delta}$-spaces.

In [Buz] Buzyakova raised some questions about the behavior of the Lindelöf property of the spaces $C_p(X)$ and $C_p(X, Y)$ for some simple spaces $Y$ under "slight changes" of the spaces $X$ and $Y$. In this article we give complete or partial answers to a few of these questions.

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## 1. Adding a point of countable character

**Proposition 1.1.** *Let $X$ be a non-pseudocompact space. Then $C_p(X) \times \omega^\omega$ is homeomorphic to a closed subspace of $C_p(X)$.*

PROOF: Since $X$ is not pseudocompact, there is a discrete family $\{U_n : n \in \omega\}$ of non-empty open sets in $X$. Choose a point $x_n$ in each $U_n$; then the set $D = \{x_n : n \in \omega\}$ is closed and discrete in $X$. For every $n \in \omega$ choose a continuous function from $\phi_n\colon X \to [0,1]$ so that $\phi_n(x_n) = 1$ and $\phi_n(X \setminus U_n) = \{0\}$. For every $f \in \mathbb{R}^D$ put

$$h(f)(x) = \sum_{n=1}^{\infty} f(x_n)\phi_n(x).$$

Note that, by the discreteness of the family $\{U_n : n \in \omega\}$, in a neighborhood of every $x \in X$ at most one term in the sum in the definition of $h(f)$ is distinct from zero; clearly, $h(f)(x_n) = f(x_n)$. It follows that $h$ is a linear extension operator from $C_p(D) = \mathbb{R}^D \to C_p(X)$. Since the value of $h(f)$ at a point $x \in X$ is completely and continuously determined by the value of $f$ at at most one point of $D$ (the one such that $x \in \bar{U}_n$, if there is any), $h$ is continuous.

By Proposition 2.1 in [Ar1], the space $C_p(X)$ is homeomorphic to $C \times C_p(D) = C \times \mathbb{R}^\omega$ where $C$ is the subset of $C_p(X)$ consisting of all functions equal to 0 on $D$. Thus, we have homeomorphisms $C_p(X) = C \times \mathbb{R}^\omega = C \times \mathbb{R}^\omega \times \mathbb{R}^\omega = C_p(X) \times \mathbb{R}^\omega$. Since $\omega^\omega$ is homeomorphic to a closed subspace of $\mathbb{R}^\omega$, we get the statement of the proposition. □

**Corollary 1.2.** *If $X$ is a non-pseudocompact space, $C_p(X)$ is Lindelöf, and $Y$ is a $\mathcal{K}$-analytic space, then $C_p(X) \times Y$ is Lindelöf.*

PROOF: Every $\mathcal{K}$-analytic space is an image of $\omega^\omega$ under a compact-valued upper semicontinuous mapping (see e.g. [RJ]). Hence, by Proposition 1.1, $C_p(X) \times \mathcal{K}$ is an image under a compact-valued upper semicontinuous mapping of a closed subspace of $C_p(X)$. The statement of the corollary now follows from the well-known fact that compact-valued upper semicontinuous mappings do not raise the Lindelöf number. □

**Corollary 1.3.** *Let $X$ be a non-pseudocompact space such that $C_p(X)$ is Lindelöf, and $P$ an $F_{\sigma\delta}$-subspace of $C_p(X)$. Then $P$ is Lindelöf.*

PROOF: Let $P = \bigcap_{n\in\omega} \bigcup_{m\in\omega} F_{nm}$ where each $F_{nm}$ is a closed set in $C_p(X)$. Then $P$ is the image under the projection onto $C_p(X)$ of the closed subset

$$B = \{(f,\phi) : \forall\, n \in \omega\; f \in F_{n\phi(n)}\}$$

of $C_p(X) \times \omega^\omega$. □

The next theorem provides a positive answer to Question 3.1 in [Buz].

**Theorem 1.4.** *Let $Y = X \cup \{p\}$ and assume that the point $p$ has countable character in $Y$. If $C_p(X)$ is Lindelöf, then $C_p(Y)$ is Lindelöf.*

Proof: If $p$ is an isolated point in $Y$, then $C_p(Y) = C_p(X) \times \mathbb{R}$, and $C_p(Y)$ is Lindelöf. So assume that $p$ is not isolated. Then $X$ is not pseudocompact, by the well-known fact that a pseudocompact space is $G_\delta$-dense in any its extension.

Let $C_0 = \{f \in C_p(Y) : f(p) = 0\}$. Then $C_p(Y)$ is homeomorphic to $C_0 \times \mathbb{R}$ (by virtue of the homeomorphism $f \mapsto (f - f(p), f(p))$ for every $f \in C_p(Y)$). Therefore, it suffices to show that $C_0$ is Lindelöf. The restriction mapping $r\colon C_p(Y) \to C_p(X)$ embeds $C_0$ homeomorphically into $C_p(X)$, so we need to show that the subspace $C = r(C_0)$ of $C_p(X)$ is Lindelöf. Clearly, $C = \{f \in C_p(X) : \lim_{x\to p} f(x) = 0\}$.

Let $\{V_n : n \in \omega\}$ be a countable open base for $p$ in $Y$, and let $U_n = V_n \cap X$, $n \in \omega$. Then

$$C = \{f \in C_p(X) : \forall n \in \omega\, \exists m \in \omega\, \forall x \in U_m \quad |f(x)| \le 1/(n+1)\}.$$

Thus,

$$C = \bigcap_{n\in\omega} \bigcup_{m\in\omega} \bigcap_{x\in U_m} \{f \in C_p(X) : |f(x)| \le 1/(n+1)\}$$

is an $F_{\sigma\delta}$-set in $C_p(X)$, hence is Lindelöf by Corollary 1.3. □

Theorem 1.4 may be slightly generalized:

**Theorem 1.5.** *Let $Y = X \cup K$ where $K$ is a metrizable compact space, $X$ is dense in $Y$, $K \cap X = \emptyset$, and $\chi(K, Y) \le \omega$. If $C_p(X)$ is Lindelöf, then $C_p(Y)$ is Lindelöf.*

Proof: Since $K$ is compact metrizable, there is a continuous linear extension operator $h\colon C_p(K) \to C_p(Y)$ [Ar1], so by Proposition 2.1 in [Ar1], $C_p(Y)$ is homeomorphic to $C_0 \times C_p(K)$ where $C_0$ is the set of all functions in $C_p(Y)$ whose restrictions to $K$ are zero.

Let $Z = Y/K$ be the quotient space, $q\colon Y \to Z$ the natural mapping, and $\{p\} = q(K)$. Since $K$ is compact, $q$ is a perfect mapping, the space $Z$ is Tychonoff, and since the character of $K$ in $Y$ is countable, the character of $p$ in $Z$ is countable. Furthermore, $X = q^{-1}(q(Z \setminus \{p\}))$, so $q|X$ is a perfect bijection from $X$ to $Z \setminus \{p\}$. Thus, $Z \setminus \{p\}$ is homeomorphic to $X$. By Theorem 1.4, $C_p(Z)$ is Lindelöf.

The dual mapping $q^*\colon C_p(Z) \to C_p(Y)$ is a closed embedding and $C_0$ is contained in $q^*(C_p(Z))$. Since $C_0$ is closed in $C_p(Y)$, it is homeomorphic to a closed subspace of $C_p(Z)$. By the density of $X$ in $Y$, $X$ is not pseudocompact (except the trivial case $K = \emptyset$). The space $C_p(K)$ is $\mathcal{K}$-analytic (in fact, a $\mathcal{K}_{\sigma\delta}$-space, see [Arh2]), so by Corollary 1.2, $C_p(Z) \times C_p(K)$ is Lindelöf. Since $C_0$ is homeomorphic to a closed set in $C_p(Z)$, $C_0 \times C_p(K)$ is Lindelöf, and $C_p(Y)$ is Lindelöf.

□

Theorem 1.5 does not hold if we only require that $K$ be an Eberlein compact space. Indeed, if $Y$ is the one-point compactification of a Mrówka space, then it is the union of a countable discrete subspace $X$ and the compact space $K$ homeomorphic to the one-point compactification of a discrete space, which is an Eberlein compact space; $K$ has countable character in $Y$, because its complement is countable (so it is a $G_\delta$-set) and $Y$ is compact. That $C_p(Y)$ for such $Y$ cannot be Lindelöf was proved in [Pol].

On the other hand, a statement similar to Theorem 1.5 holds, with a similar proof, if we require the existence of an extension operator.

**Theorem 1.6.** *Let $Y = X \cup K$ where $K$ is an Eberlein compact space, $X$ is dense in $Y$, $K \cap X = \emptyset$, and $\chi(K, Y) \leq \omega$. If $C_p(X)$ is Lindelöf, and there is a continuous extension operator $h\colon C_p(K) \to C_p(Y)$, then $C_p(Y)$ is Lindelöf.*

## 2. Spaces of functions on cozero sets

In [Buz], Buzyakova proved that *If $X$ is zero-dimensional compact, $C_p(X)$ is Lindelöf, and $p$ is a point of countable character in $X$, then $C_p(X \setminus \{p\})$ is Lindelöf*, and asks if the same holds for every compact space, or for any space $X$.

In this section we prove some statements in this direction, which generalize the theorem of Buzyakova.

**Theorem 2.1.** *Let $X$ be a space such that $C_p(X)^\omega$ is Lindelöf, and $Y$ a cozero set in $X$. Then $C_p(Y)^\omega$ is Lindelöf.*

Proof: Let $h\colon X \to [0,1]$ be a continuous function such that $Y = h^{-1}((0,1])$.

For each $n \in \omega$ put $F_n = h^{-1}([1/(n+1), 1])$ and $F = X \setminus Y$. Clearly, $F$ and $F_n$, $n \in \omega$, are zero sets, $F_n \subset \operatorname{Int} F_{n+1}$, and $Y = \bigcup\{F_n : n \in \omega\}$.

Put

$$P = \{G \in C_p(X)^\omega : G(n)|F_n = G(m)|F_n \text{ for all } m, n \in \omega,\ m \geq n\}.$$

Then

$$P = \bigcap_{n\in\omega} \bigcap_{m\geq n} \bigcap_{x\in F_n} \{G \in C_p(X)^\omega : G(m)(x) = G(n)(x)\},$$

so $P$ is closed in $C_p(X)^\omega$, and $P^\omega$ is Lindelöf.

Define $T\colon P \to \mathbb{R}^Y$ by the rule:

$$T(G)(x) = G(n)(x) \text{ if } x \in F_n.$$

Obviously, $T$ is well-defined. Let $G \in P$ and $x \in Y$. Then $x \in F_n$ for some $n$, and $x \in \operatorname{Int} F_{n+1}$. Since $T(G)|F_{n+1} = G(n+1)|F_{n+1}$, $T(G)$ coincides with the continuous function $G(n+1)$ in the neighborhood $F_{n+1}$ of $x$, and therefore is continuous at $x$. Thus, $G$ is continuous on $Y$, and we have proved $T(P) \subset C_p(Y)$.

Let us verify the inverse inclusion. Let $f \in C_p(Y)$. Fix a continuous function $\theta\colon [0,1] \to [0,1]$ so that $\theta(1) = 1$ and $\theta([0,1/2]) = \{0\}$. For every $n \in \omega$ fix a continuous function $h_n\colon X \to [0,1]$ so that $h_n(F_n) \subset \{1\}$ and $h_n(F) \subset \{0\}$, and let $s_n(x) = \theta \circ h_n$. Then $s_n\colon X \to [0,1]$ is continuous, $s_n(F_n) \subset \{1\}$, and $s_n$ is zero in a neighborhood of $F$. It follows that the function $g_n\colon X \to \mathbb{R}$ defined by the rule

$$g_n(x) = \begin{cases} f(x)s_n(x) & \text{if } x \in Y, \\ 0 & \text{if } x \in F \end{cases}$$

is continuous on $X$, and coincides with $f$ on $F_n$. Thus, $f = T(G)$ where $G(n) = g_n$ for all $n \in \omega$. This finishes the proof that $T(P) = C_p(Y)$.

Finally, let us verify that $T$ is continuous. For an open set $W$ in $\mathbb{R}$ and $x \in Y$ denote $O(x,W) = \{f \in C_p(Y) : f(x) \in W\}$. The sets $O(x,W)$ form an open subbase for the topology of $C_p(Y)$, so it suffices to verify that their preimages under $T$ are open in $P$. So fix $x$ and $W$; find an $m \in \omega$ so that $x \in F_m$. Then $x \in F_n$ for all $n \geq m$, so $G(n)(x) = G(m)(x)$ for all $G \in P$ and $n \geq m$. We have therefore

$$\begin{aligned} T^{-1}(O(x,W)) &= \{G \in P : G(m)(x) \in W\} \\ &= P \cap \{H \in C_p(X)^\omega : H(m)(x) \in W\}, \end{aligned}$$

an open set in $P$.

Thus, $C_p(Y)$ is a continuous image of the set $P$, whence $C_p(Y)^\omega$ is Lindelöf. □

The condition "$C_p(X)^\omega$ is Lindelöf" appears much stronger than "$C_p(X)$ is Lindelöf"; however, as far as the author knows by the moment, whether the two conditions are equivalent is an open problem, both for compact spaces $X$ and in the general case. In some particular cases, however, it is known that the two conditions are equivalent. Thus, R. Pol showed in [Pol] that if $X$ is zero-dimensional compact and $C_p(X)$ is Lindelöf, then $C_p(X)^\omega$ is Lindelöf. We can slightly improve this statement.

**Theorem 2.2.** *Let $X$ be a $\sigma$-compact zero-dimensional space. If $C_p(X)$ is Lindelöf, then $C_p(X)^\omega$ is Lindelöf.*

Proof: Since the Cantor cube $2^\omega$ is homeomorphic to a closed subspace of $\mathbb{R}$, the space $C_p(X, 2^\omega)$ is homeomorphic to a closed subspace of $C_p(X)$, and therefore is Lindelöf. We have $C_p(X, 2^\omega) = C_p(X, 2)^\omega$, so $C_p(X, 2)^\omega$ is Lindelöf. Since $X$ is zero-dimensional, $C_p(X, 2)$ separates points and closed sets of $X$. It follows that the diagonal product $\Phi = \Delta C_p(X, 2)\colon X \to \mathbb{R}^{C_p(X,2)}$ is an embedding; obviously, $\Phi(X) \subset C_p(C_p(X, 2))$. Thus, $X$ is homeomorphic to a $\sigma$-compact subspace of $C_p(Y)$ where $Y = C_p(X, 2)$. Then $X \times \omega$ is $\sigma$-compact and homeomorphic to a subspace of $C_p(Y^+) = C_p(Y) \times \mathbb{R}$, where $Y^+$ is the space obtained by adding an isolated point to $Y$. The space $(Y^+)^\omega$ is Lindelöf: $Y^+$ is a continuous image of $Y \times 2$, so $(Y^+)^\omega$ is a continuous image of $Y^\omega \times 2^\omega$. By Corollary 2.8 in [Oku], $C_p(X)^\omega = C_p(X \times \omega)$ is Lindelöf. □

**Corollary 2.3.** *Let $X$ be a zero-dimensional $\sigma$-compact space such that $C_p(X)$ is Lindelöf. Then for every cozero set $Y$ in $X$, $C_p(Y)$ is Lindelöf.*

This corollary can also be deduced directly from Theorem 2.2 and Corollary 2.8 in [Oku], using the observation that a cozero set in a $\sigma$-compact space is $\sigma$-compact.

**Corollary 2.4.** *Let $X$ be a zero-dimensional $\sigma$-compact space such that $C_p(X)$ is Lindelöf. Then for every compact $G_\delta$-set $K$ in $X$, $C_p(X \setminus K)$ is Lindelöf.*

The proof of Theorem 2.1 actually gives the following statement:

**Theorem 2.5.** *Let $Y$ be a cozero set in $X$. Then $C_p(Y)$ is a continuous image of a closed subset of $C_p(X)^\omega$.*

We now can deduce various other corollaries, related to classes of spaces invariant with respect to countable powers, closed subspaces, and continuous images.

**Corollary 2.6.** *If $C_p(X)$ is a Lindelöf $\Sigma$-space, and $Y$ a cozero set in $X$, then $C_p(Y)$ is a Lindelöf $\Sigma$-space.*

**Corollary 2.7.** *If $C_p(X)$ is a $\mathcal{K}$-analytic space, and $Y$ a cozero set in $X$, then $C_p(Y)$ is a $\mathcal{K}$-analytic space.*

**Corollary 2.8.** *If $Y$ is a cozero subspace of $X$, then $l(C_p(Y)^\omega) \leq l(C_p(X)^\omega)$ and $\operatorname{ext}(C_p(Y)^\omega) \leq \operatorname{ext}(C_p(X)^\omega)$.*

**Corollary 2.9.** *If $C_p(X)$ is a $L\Sigma(\leq \omega)$-space, and $Y$ a cozero set in $X$, then $C_p(Y)$ is an $L\Sigma(\leq \omega)$-space.*

(See [KOS] for definition and basic properties of $L\Sigma(\leq \omega)$-spaces.)

And, generally,

**Corollary 2.10.** *Let $\mathcal{P}$ be a class of spaces invariant with respect to countable powers, closed subspaces and continuous images. If $C_p(X) \in \mathcal{P}$, and $Y$ is a cozero set in $X$, then $C_p(Y) \in \mathcal{P}$.*

A similar argument applies to the spaces $C_p(X, I)$ where $I = [0, 1]$:

**Theorem 2.11.** *Let $Y$ be a cozero set in $X$. Then $C_p(Y, I)$ is a continuous image of a closed subset of $C_p(X, I)^\omega$.*

## 3. Some open problems

**Question 3.1.** Let $X$ be a pseudocompact space such that $C_p(X)$ is Lindelöf. Must the product $C_p(X) \times \omega^\omega$ be Lindelöf?

**Question 3.2.** Let $Y = X \cup K$ where $K$ is a metrizable compact space, $X$ is dense in $Y$, $\chi(K, Y) \leq \omega$, and $C_p(X)$ is Lindelöf. Must $C_p(Y)$ be Lindelöf?

The question here is if we can omit the condition "$X \cap K = \emptyset$" in Theorem 1.5. If we assume that $C_p(X)^\omega$ is Lindelöf and that $X \setminus K$ is dense in $Y$, the answer is "yes" by Theorems 1.5 and 2.1.

**Question 3.3.** Let $X$ be a space such that $C_p(X)^\omega$ is Lindelöf, and $Y$ an open $F_\sigma$-subspace of $X$. Must $C_p(Y)$ be Lindelöf?

Note that for normal spaces $X$ an affirmative answer to this question follows from Theorem 2.1.

## References

[Ar1] Arhangel'skii A.V., *On linear topological and topological classification of spaces $C_p(X)$*, Zb. Rad. **3** (1989), 3–12.

[Arh2] Arhangel'skii A.V., *Topological Function Spaces*, Kluwer Acad. Publ., Dordrecht, 1992.

[Buz] Buzyakova R.Z.,, *How sensitive is $C_p(X, Y)$ to changes in $X$ and/or $Y$?*, Comment. Math. Univ. Carolin. **49** (2008), no. 4, 657–665.

[Eng] Engelking R., *General Topology*, Sigma Series in Pure Mathematics, 6, Helderman, Berlin, 1989.

[KOS] Kubiš W., Okunev O., Szeptycki P.J., *On some classes of Lindelöf $\Sigma$-spaces*, Topology Appl. **153** (2006), 2574–2590.

[Oku] Okunev O., *On Lindelöf $\Sigma$-spaces of continuous functions in the pointwise topology*, Topology Appl. **49** (1993), 149–166.

[Pol] Pol R., *A theorem on the weak topology of $C(X)$ for compact scattered $X$*, Fund. Math. **106** (1980), no. 2, 135–140.

[RJ] Rogers A., Jayne E. (Eds.), *Analytic Sets*, Academic Press, London, 1980.

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