Pawel Solarz

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ON SOME PROPERTIES OF ORIENTATION-PRESERVING SURJECTIONS ON THE CIRCLE

Paweł Solarz

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ABSTRACT. Some properties of orientation-preserving surjections with non-empty set of periodic points are studied. In particular, orientation-preserving homeomorphisms of the whole circle S^1 are considered.

Let S^1 denote the unit circle in the complex plane and let $u, w, z \in S^1$, then there exist unique $t_1, t_2 \in \langle 0, 1)$ such that $we^{2\pi i t_1} = z$, $we^{2\pi i t_2} = u$. Define

$$u \prec w \prec z \quad \text{if and only if} \quad 0 < t_1 < t_2$$

(see [2]). Some properties of this relation can be found in [3] and [4]. For any distinct elements $u, z \in S^1$ put $\overrightarrow{(u,z)} := \{w \in S^1 : u \prec w \prec z\}$, $\overrightarrow{\langle u,z\rangle} := \overrightarrow{(u,z)} \cup \{u, z\}$ and $\overrightarrow{\langle u,z)} := \overrightarrow{(u,z)} \cup \{u\}$. Moreover, if $u = z$ set $\overrightarrow{(u,z)} := S^1 \setminus \{u\}$. These sets are called *arcs.*

Let $B \subset S^1$ be a set which has at least three elements. We say that a function $F \colon B \longrightarrow S^1$ *preserves the orientation* if for any $u, w, z \in B$ such that $u \prec w \prec z$ we have $F(u) \prec F(w) \prec F(z)$. It can be easily proved that any orientation-preserving function is an injection and F^{-1} and $F \circ G$ preserve the orientation if F and G are orientation-preserving maps (see [3]). For any function $f \colon X \longrightarrow X$, a point $x \in X$ is called a *periodic point* of f if $f^k(x) = x$ for some $k \in \mathbb{N} := \{1, 2, \ldots\}$. By $\operatorname{Per} f$ we denote the set of all periodic points of f. Finally, a set of the form $\{x, f(x), \ldots, f^{n-1}(x)\}$, where $x \in X$, $f^n(x) = x$ and $f^i(x) \neq f^j(x)$ for $i, j \in \mathbb{N} \cup \{0\}$, $i \neq j$, is called a *cycle of order* $n \in \mathbb{N}$ and the number of its elements is called a *period* of x.

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Throughout the paper the set $\{0, \dots, n-1\}$, where $n \in \mathbb{N}$, is denoted by $\mathbb{Z}_n$.

Llibre in [9] studied how a continuous map of the circle having periodic points acts on a cycle. He proved that if f is a continuous map of a circle and $P = \{p_1, \dots, p_n\}$, where $n > 1$, is a cycle such that $P \cap \overrightarrow{(p_k, p_{k+1})} = \emptyset$ for $k = 1, \dots, n-1$ and $P \cap \overrightarrow{(p_n, p_1)} = \emptyset$, then $f(p_k) = p_{\tau^t(k)}$, where $\tau(k) - k + 1$ for $k = 1, \dots, n-1$, $\tau(n) = 1$ and $1 < t < n$ is relatively prime to n.

Let $F\colon B \longrightarrow B$, where $B \subset S^1$ be an orientation-preserving surjection. In this paper we prove the similar result for any non-empty and finite set which is an invariant set of F. We also generalize the known fact that every two periodic points of an orientation-preserving homeomorphism have the same period (see for example [6, p. 16]). Finally, we consider orientation-preserving surjections of the whole circle.

We start with the following observation.

Lemma 1. *Let A, B be closed subsets of S^1 such that* $\operatorname{card} A \geq 3$ *and* $\operatorname{card} B \geq 3$. *If $F\colon B \longrightarrow A$ preserves the orientation and maps B onto A, then F is a homeomorphism.*

Proof. Notice that it is sufficient to show that F is continuous. If there existed $z \in B$ a cluster point of B such that F were discontinuous at z, we would get the existence of a sequence $(z_n)_{n\in\mathbb{N}}$ of distinct elements of $B \setminus \{z\}$ such that $\lim_{n\to\infty} z_n = z$ and

$$\lim_{n\to\infty} F(z_n) =: u_0 \neq F(z). \tag{1}$$

Clearly, $u_0 \in A$. Put $z_0 := F^{-1}(u_0)$, then by (1) $z \neq z_0$. Without loss of generality we may assume that there exists a subsequence $(z_{n_m})_{m\in\mathbb{N}}$ of $(z_n)_{n\in\mathbb{N}}$ such that

$$F(z_{n_m}) \in \overrightarrow{(F(z_0), F(z))}, \qquad m \in \mathbb{N}.$$

Let z^* be one of the elements of $(z_{n_m})_{m\in\mathbb{N}}$. Define

$$p := \max\left\{n_m :\ F(z_{n_m}) \in \overrightarrow{\langle F(z^*), F(z))},\ m \in \mathbb{N}\right\},$$

then $F(z_{n_m}) \in \overrightarrow{(F(z_0), F(z^*))}$ for every $n_m > p$. Whence $z_{n_m} \in \overrightarrow{(z_0, z^*)}$ for $n_m > p$. On the other hand, $z \in \overrightarrow{(z^*, z_0)}$. But $\lim_{n\to\infty} z_{n_m} - z$, so we have a contradiction. □

For any map $f\colon X \longrightarrow X$ such that $\operatorname{Per} f \neq \emptyset$ and any $x \in \operatorname{Per} f$ let $n_f(x)$ denote the period of x and

$$n_f := \min\{n_f(x) :\ x \in \operatorname{Per} f\}.$$

Theorem 1. *Let $B \subset S^1$ be such that* $\operatorname{card} B \geq 3$ *and let $F\colon B \longrightarrow B$ be an orientation-preserving surjection such that* $\operatorname{Per} F \neq \emptyset$. *If $z_0, z_1 \in \operatorname{Per} F$, then $n_F(z_0) = n_F(z_1)$.*

Proof. It is well known (see [8]) that if a function $f\colon X \longrightarrow X$ is a bijection, then every cycle of order $n \in \mathbb{N}$ is an equivalence class of the following relation on X:

$$x \sim_f y \iff \exists m, n \in \mathbb{N} \cup \{0\}: \ f^n(x) = f^m(y).$$

Therefore, it is sufficient to consider the case

$$\left\{z_0, F(z_0), \dots, F^{n_F(z_0)-1}(z_0)\right\} \cap \left\{z_1, F(z_1), \dots, F^{n_F(z_1)-1}(z_1)\right\} = \emptyset. \tag{2}$$

To obtain a contradiction suppose that $n_F(z_1) < n_F(z_0)$. Define

$$a_i \in \left\{z_0, F(z_0), \dots, F^{n_F(z_0)-1}(z_0)\right\} \qquad \text{for} \quad i \in \mathbb{Z}_{n_F(z_0)}$$

in the following manner:

$$a_0 := z_0 \qquad \text{and} \qquad \operatorname{Arg} \frac{a_i}{a_0} < \operatorname{Arg} \frac{a_{i+1}}{a_0}, \quad i \in \{0, \dots, n_F(z_0) - 2\}.$$

For the convenience put also $a_{n_F(z_0)} := a_0$. By (2), $z_1 \in \overrightarrow{(a_i, a_{i+1})}$ for some $i \in \mathbb{Z}_{n_F(z_0)}$. Since F preserves the orientation we have

$$z_1 = F^{n_F(z_1)}(z_1) \in \overrightarrow{\left(F^{n_F(z_1)}(a_i), F^{n_F(z_1)}(a_{i+1})\right)}.$$

Thus

$$\overrightarrow{(a_i, a_{i+1})} \cap \overrightarrow{\left(F^{n_F(z_1)}(a_i), F^{n_F(z_1)}(a_{i+1})\right)} \neq \emptyset. \tag{3}$$

As $n_F(z_1) < n_F(z_0)$ we have $F^{n_F(z_1)}(a_i) \neq a_i$. Consequently,

$$\overrightarrow{(a_i, a_{i+1})} \subset \overrightarrow{\left(F^{n_F(z_1)}(a_i), F^{n_F(z_1)}(a_{i+1})\right)}.$$

From the fact that $a_i \in \overrightarrow{\left(F^{n_F(z_1)}(a_i), F^{n_F(z_1)}(a_{i+1})\right)}$ we have

$$F^{n_F(z_0)-n_F(z_1)}(a_i) \in \overrightarrow{\left(F^{n_F(z_0)}(a_i), F^{n_F(z_0)}(a_{i+1})\right)} = \overrightarrow{(a_i, a_{i+1})},$$

but $F^{n_F(z_0)-n_F(z_1)}(a_i) = a_j$ for an $j \in \mathbb{Z}_{n_F(z_0)}$, and we have a contradiction. □

Corollary 1. *If $F\colon B \longrightarrow B$, where $B \subset S^1$ is such that* $\operatorname{card} B \geq 3$, *is an orientation-preserving surjection such that* $\operatorname{Per} F \neq \emptyset$, *then*

$$\operatorname{Per} F = \left\{z \in B: \ F^{n_F}(z) = z\right\}.$$

Now let $F\colon B \longrightarrow B$, where $B \subset S^1$ is such that $\operatorname{card} B \geq 3$, be an orientation-preserving surjection with all points periodic and having a fixed point. Then the above theorem yields $F = \mathrm{id}_B$. This is a generalization of the result obtained by W. Jarczyk for an orientation-preserving homeomorphism of the whole circle (see [7, Theorem 1]).

The following remark is easy to check

Remark 1. Let $A \subset X$ be a non-empty finite set and let $f\colon X \longrightarrow X$ be a map such that $f(A) = A$, then $A \subset \operatorname{Per} f$.

Since every two different cycles of a bijection are disjoint sets and since, by Corollary 1, in the case of circle maps they have the same number of elements, we have:

Corollary 2. *If $F\colon B \longrightarrow B$, where $B \subset S^1$ and $\operatorname{card} B \geq 3$, is an orientation-preserving surjection such that $F(A) = A$ for some non-empty and finite $A \subset B$, then n_F divides $\operatorname{card} A$.*

Now for any set A satisfying the assumptions of Corollary 2 put

$$k_F(A) := \frac{\operatorname{card} A}{n_F}. \tag{4}$$

Before we write the next lemma notice that $\gcd(0, k) = k$ for every $k \in \mathbb{N}$. In particular $\gcd(0, 1) = 1$.

Lemma 2. *Suppose that $B \subset S^1$ is such that $\operatorname{card} B \geq 3$, $F\colon B \longrightarrow B$ is an orientation-preserving surjection and $A \subset B$ is a non-empty finite set such that $F(A) = A$. Let $a_0 \in A$ be an arbitrary element and if $\operatorname{card} A =: N_A > 1$ let $a_1, \ldots, a_{N_A-1} \in A$ satisfy the following condition:*

$$\operatorname{Arg} \frac{a_i}{a_0} < \operatorname{Arg} \frac{a_{i+1}}{a_0}, \qquad i \in \{0, \ldots, N_A - 2\}. \tag{5}$$

There exists a unique $q = q(F) \in \mathbb{Z}_{n_F}$ such that $\gcd(q, n_F) = 1$ and

$$F(a_i) = a_{(i+k_F(A)q) \pmod{N_A}}, \qquad i \in \mathbb{Z}_{N_A}. \tag{6}$$

P r o o f. It is clear that if $n_F = 1$, then a_i for $i \in \mathbb{Z}_{k_F(A)}$ are fixed points of F, so $F(a_i) = a_i$ for every $i \in \mathbb{Z}_{k_F(A)}$. In this case $q = q(F) = 0$ is the only number which has the desired properties.

Let $n_F \geq 2$. Fix $i \in \mathbb{Z}_{N_A}$. Therefore, there exist a $p \in \mathbb{Z}_{n_F}$ and an $r \in \mathbb{Z}_{k_F(A)}$ such that

$$i = k_F(A)p + r. \tag{7}$$

We show that

$$\{a_i, F(a_i), \ldots, F^{n_F-1}(a_i)\} = \{a_r, a_{r+k_F(A)}, \ldots, a_{r+(n_F-1)k_F(A)}\}. \tag{8}$$

Of course, if $k_F(A) = 1$, then $N_A = n_F$, $r = 0$ and (8) holds. Let $k_F(A) > 1$ and $b_k \in \{a_i, F(a_i), \ldots, F^{n_F-1}(a_i)\}$ for $k \in \mathbb{Z}_{n_F}$ be such that

$$b_0 = b_{n_F} := a_i \qquad \text{and} \qquad \operatorname{Arg} \frac{b_k}{b_0} < \operatorname{Arg} \frac{b_{k+1}}{b_0}, \quad k \in \{0, \ldots, n_F - 2\}.$$

Notice that

$$\operatorname{card} \left(\bigcup_{k=0}^{n_F-1} \overrightarrow{(b_k, b_{k+1})} \cap A \right) = (k_F(A) - 1)n_F. \tag{9}$$

Suppose that for some $k \in \mathbb{Z}_{n_F}$

$$\operatorname{card} \left(\overrightarrow{(b_k, b_{k+1})} \cap A \right) < k_F(A) - 1,$$

then for every $l \in \mathbb{Z}_{n_F}$ we have

$$\operatorname{card} \left(F^l \left(\overrightarrow{(b_k, b_{k+1})} \cap A \right) \right) < k_F(A) - 1.$$

From this and the fact that

$$\bigcup_{l=0}^{n_F-1} F^l \left(\overrightarrow{(b_k, b_{k+1})} \cap A \right) = A \setminus \{b_0, b_1, \ldots, b_{n_F-1}\}$$

we have a contradiction. Hence for every $k \in \mathbb{Z}_{n_F}$ we obtain

$$\operatorname{card} \left(\overrightarrow{(b_k, b_{k+1})} \cap A \right) \geq k_F(A) - 1.$$

From this and (9) it follows that

$$\operatorname{card} \left(\overrightarrow{(b_k, b_{k+1})} \cap A \right) = k_F(A) - 1, \qquad k \in \mathbb{Z}_{n_F}. \tag{10}$$

Now fix $k \in \mathbb{Z}_{n_F}$. Let $j \in \mathbb{Z}_{N_A}$ be such that $b_k = a_j$. From (10) and the definition of a_i we get

$$\overrightarrow{(b_k, b_{k+1})} \cap A = \left\{ a_{(j+1) \pmod{N_A}}, \ldots, a_{(j+k_F(A)-1) \pmod{N_A}} \right\}.$$

Hence

$$b_{k+1} = a_{(j+k_F(A)) \pmod{N_A}}.$$

This and the fact that $b_0 = a_i$ give

$$b_k = a_{(i+kk_F(A)) \pmod{N_A}} \qquad \text{for all} \quad k \in \mathbb{Z}_{n_F}. \tag{11}$$

Applying (7) to (11) we get

$$b_k = a_{(r+(p+k)k_F(A)) \pmod{N_A}}, \qquad k \in \mathbb{Z}_{n_F}. \tag{12}$$

Let $k := n_F - p$, then $0 < \bar{k} \leq n_F$. Since $k_F(A)n_F = N_A$ we get

$$b_{\bar{k}} = b_{n_F - p} = a_{(r+N_A) \pmod{N_A}} = a_r.$$

Thus by (12), when $p > 0$ we obtain

$$b_{k+l} = a_{(r+lk_F(A)) \pmod{N_A}} = a_{r+lk_F(A)} \qquad \text{for} \quad l \in \{0, \ldots, n_F - 1 - k\} = \mathbb{Z}_p.$$

On the other hand, inequalities $r \leq k_F(A) - 1$ and $l - k \leq -1$ for $l \in \mathbb{Z}_k$ imply

$$r + (l + n_F - \bar{k})k_F(A) \leq k_F(A) - 1 + (n_F - 1)k_F(A) - N_A - 1.$$

Hence

$$b_l = a_{(r+(l+n_F-k)k_F(A)) \pmod{N_A}} = a_{(r+(l+n_F-k)k_F(A))}, \qquad l \in \mathbb{Z}_k.$$

Finally,

$$\{b_0, \ldots, b_{n_F-1}\} = \{a_r, a_{r+k_F(A)}, \ldots, a_{r+(n_F-1)k_F(A)}\},$$

which proves (8).

By (8) and since $n_F \geq 2$ we obtain

$$F\left(a_{k_F(A)p+r}\right) = a_{k_F(A)l+r} \tag{13}$$

for some $l \in \mathbb{Z}_{n_F}$, $l \neq p$.

Now consider two cases:

(i) $l - p > 0$. Clearly, $l - p < n_F$. Put $q := l - p$, thus by (13)

$$F(a_i) = F(a_{k_F(A)p+r}) = a_{k_F(A)(p+q)+r} = a_{i+k_F(A)q} = a_{(i+k_F(A)q) \pmod{N_A}},$$

since $i + k_F(A)q < N_A$.

(ii) $l - p < 0$. Then $0 < l - p + n_F < n_F$ and setting $q := l - p + n_F$ we get

$$\begin{aligned} F(a_i) &= F(a_{k_F(A)p+r}) = a_{k_F(A)l-k_F(A)p+k_F(A)p+r} \\ &= a_{(i+k_F(A)(l-p)+N_A) \pmod{N_A}} = a_{(i+k_F(A)q) \pmod{N_A}}, \end{aligned}$$

since $i + k_F(A)(l - p) < N_A$.

If there existed another $q_1 \in \mathbb{Z}_{n_F}$, $q_1 \neq q$, satisfying (6), we would have q_1 $q + dn_F$ for some $d \in \mathbb{Z} \setminus \{0\}$, which is impossible.

Our next goal is to show that q defined above is one for all a_i, $i \in \mathbb{Z}_{N_A}$. For this purpose assume that for some $j \in \mathbb{Z}_{N_A}$, $j \neq i$, there exists a $q_1 \in \{1, \ldots, n_F - 1\}$ such that

$$F(a_j) = a_{(j+k_F(A)q_1) \pmod{N_A}}. \tag{14}$$

There is no loss of generality in assuming that $i < j$. On the other hand, since

$$F(a_i) = a_{(i+k_F(A)q) \pmod{N_A}}$$

and F is an orientation-preserving map we get

$$F(a_{i+1}) = a_{((i+k_F(A)q) \pmod{N_A}+1) \pmod{N_A}} = a_{(i+1+k_F(A)q) \pmod{N_A}}.$$

Repeating this argument $j - i$ times we get

$$F(a_j) = a_{(j+k_F(A)q) \pmod{N_A}}.$$

This and (14) lead to $q = q_1$.

It remains to prove that $\gcd(q, n_F) = 1$. Suppose, contrary to our claim, that $\gcd(q, n_F) = d > 1$. Hence there exist $p_1, p_2 \in \mathbb{N}$ such that $q = p_1 d$ and $n_F = p_2 d$. This and (6) yield

$$F^{p_2}(a_0) = a_{k_F(A)qp_2 \pmod{N_A}} = a_{N_A p_1 \pmod{N_A}} = a_0,$$

a contradiction. □

Now we turn to the case $B = S^1$. In view of Lemma 1 every orientation-preserving function mapping S^1 onto S^1 is a homeomorphism. Therefore, we recall the basic definitions and notations for homeomorphisms of the circle.

Let $F\colon S^1 \longrightarrow S^1$ be an orientation-preserving homeomorphism, then there exists a homeomorphism $f\colon \mathbb{R} \longrightarrow \mathbb{R}$, unique up to translation by an integer, such that $F\left(\mathrm{e}^{2\pi \mathrm{i} x}\right) = \mathrm{e}^{2\pi \mathrm{i} f(x)}$ and $f(x+1) = f(x)+1$ for all $x \in \mathbb{R}$. The function f is called a *lift* of F (see [5]). Moreover, the number $\alpha(F) \in \langle 0, 1)$ defined as

$$\alpha(F) := \lim_{n\to\infty} \frac{f^n(x)}{n} \pmod 1, \qquad x \in \mathbb{R},$$

always exists and does not depend on x and f. This number is called the *rotation number* of F and is rational if and only if F has a periodic point (see for example [1], [5]).

Notice that if A is a cycle of order n_F of F, Lemma 2 gives the following:

Corollary 3. *Let $F\colon S^1 \longrightarrow S^1$ be an orientation-preserving surjection such that* $\operatorname{Per} F \neq \emptyset$. *If $z \in \operatorname{Per} F$ and $b_k \in \left\{z, F(z), \dots, F^{n_F-1}(z)\right\}$ for $k \in \mathbb{Z}_{n_F}$ are such that*

$$b_0 := z$$

$$\text{and if} \quad n_F \geq 2 \qquad \operatorname{Arg}\frac{b_k}{b_0} < \operatorname{Arg}\frac{b_{k+1}}{b_0}, \qquad k \in \{0, \dots, n_F - 2\}, \tag{15}$$

then

$$F(b_k) = b_{(k+q) \pmod{n_F}}, \qquad k \in \mathbb{Z}_{n_F}, \tag{16}$$

where $q = q(F)$.

The next lemma is a consequence of Corollary 3 and of the definition of the rotation number.

Lemma 3. *If $F\colon S^1 \longrightarrow S^1$ is an orientation-preserving surjection with* $\operatorname{Per} F \neq \emptyset$. *Then $\alpha(F) = \frac{q}{n_F}$, where $q = q(F)$.*

Proof. Fix $z \in \operatorname{Per} F$ and define $b_k \in \left\{z, F(z), \dots, F^{n_F-1}(z)\right\}$ for $k \in \mathbb{Z}_{n_F}$ by (15). Obviously, if $n_F = 1$, then $q = 0$ and $b_0 = z$ is a fixed point of F. Hence $\alpha(F) = 0$. Suppose that $n_F > 1$, thus since $\gcd(0, k) = k$ for $k \in \mathbb{N}$, we have $q \geq 1$. Let $x_0 \in \langle 0, 1)$ be such that $\mathrm{e}^{2\pi \mathrm{i} x_0} = b_0$. There exist $x_1, \dots, x_{n_F-1} \in (x_0, x_0 + 1)$ such that

$$x_0 < x_1 < \dots < x_{n_F-1} < x_0+1 \quad \text{and} \quad \mathrm{e}^{2\pi \mathrm{i} x_k} = b_k, \quad k \in \{1, \dots, n_F-1\}. \tag{17}$$

Put

$$x_k := x_{k-n_F} + 1, \qquad k \in \mathbb{N}, \quad k \geq n_F, \tag{18}$$

and

$$x_k := x_{k+n_F} - 1, \qquad k \in \mathbb{Z} \setminus (\mathbb{N} \cup \{0\}). \tag{19}$$

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a strictly increasing lift of F. By (16) and (17) we get

$$\mathrm{e}^{2\pi \mathrm{i} f(x_0)} = F\left(\mathrm{e}^{2\pi \mathrm{i} x_0}\right) = F(b_0) = b_{q \pmod{n_F}} = b_q = \mathrm{e}^{2\pi \mathrm{i} x_q}.$$

Hence $f(x_0) = x_q + l$ for some integer l. Put $f := f - l$, then

$$f(x_0) = x_q.$$

Fix $k \in \{1, \dots, n_F - 1\}$ and observe that since f is a strictly increasing lift of F and $x_k \in (x_0, x_0 + 1)$ we obtain

$$x_q = f(x_0) < f(x_k) < f(x_0 + 1) = f(x_0) + 1 = x_q + 1. \tag{20}$$

On the other hand, (17) and (16) lead to

$$\mathrm{e}^{2\pi \mathrm{i} f(x_k)} = F(b_k) = b_{(k+q) \pmod{n_F}} = \mathrm{e}^{2\pi \mathrm{i} x_{(k+q) \pmod{n_F}}}.$$

Hence we get

$$f(x_k) = x_{(k+q) \pmod{n_F}} + d$$

for some integer d. Notice that $(k + q) \pmod{n_F} = (k + q - mn_F)$ for an $m \in \{0, 1\}$ as $k + q < 2n_F - 1$. Therefore, by (19)

$$f(x_k) = x_{k+q} + d - m.$$

Inserting the above equality to (20) we obtain

$$x_q < x_{k+q} + d - m < x_q + 1.$$

Since $0 < k < n_F$ it follows that $0 < x_{k+q} - x_q < 1$, hence $-1 < d - m < 1$, but $d - m \in \mathbb{Z}$, so $d - m = 0$. Finally,

$$f(x_k) = x_{k+q}, \qquad k \in \mathbb{Z}_{n_F}. \tag{21}$$

Now let $k \in \mathbb{Z} \setminus \mathbb{Z}_{n_F}$, then $k = pn_F + r$ for some $p \in \mathbb{Z}$ and $r \in \mathbb{Z}_{n_F}$. Using this notation (18), (19) and (21) we get

$$f(x_k) = f(x_{pn_F+r}) = f(x_r + p) = f(x_r) + p = x_{r+q} + p = x_{r+pn_F+q} = x_{k+q}.$$

Thus we have proved

$$f(x_k) = x_{k+q}, \qquad k \in \mathbb{Z}.$$

From this and (18) we have

$$f^{n_F}(x_0) = x_{qn_F} = x_0 + q,$$

Consequently,

$$f^{jn_F}(x_0) = x_{qjn_F} = x_0 + jq, \qquad j \in \mathbb{N},$$

which in view of the definition of the rotation number gives $\alpha(F) = \frac{q}{n_F}$. □

From now on suppose that $F\colon S^1 \longrightarrow S^1$ is an orientation-preserving surjection such that $\emptyset \neq \operatorname{Per} F \neq S^1$. Since $\operatorname{Per} F = \{z \in S^1 : F^{n_F}(z) = z\}$ it is a closed subset of S^1 and it follows that $S^1 \setminus \operatorname{Per} F$ is a sum of non-empty, pairwise disjoint open arcs. Denote this family by $\mathscr{B}_F$. Therefore,

$$S^1 \setminus \operatorname{Per} F = \bigcup_{I \in \mathscr{B}_F} I.$$

Lemma 4. *Let $F\colon S^1 \longrightarrow S^1$ be an orientation-preserving surjection such that $\emptyset \neq \operatorname{Per} F \neq S^1$ and let $I \in \mathscr{B}_F$. Then either*

$$\bigcup_{i \in \mathbb{Z}} \overrightarrow{\Big\langle F^{in_F}(z), F^{(i+1)n_F}(z)\Big)} = I, \qquad z \in I \tag{22}$$

or

$$\bigcup_{i \in \mathbb{Z}} \overrightarrow{\Big\langle F^{(i+1)n_F}(z), F^{in_F}(z)\Big)} = I, \qquad z \in I. \tag{23}$$

Moreover, $\overrightarrow{(z, F^{n_F}(z))} \subset I$ for every $z \in I$ or $\overrightarrow{(F^{n_F}(z), z)} \subset I$ for every $z \in I$.

Proof. Fix $I \in \mathscr{B}_F$ and $z \in I$. Then $F^{n_F}(z) \in F^{n_F}(I) = I$ and $F^{n_F}(z) \neq z$. Suppose that

$$\overrightarrow{(z, F^{n_F}(z))} \subset I. \tag{24}$$

Hence

$$\overrightarrow{\Big\langle F^{ln_F}(z), F^{(l+1)n_F}(z)\Big)} \subset I \qquad \text{for} \quad l \in \mathbb{Z} \tag{25}$$

and in consequence

$$\bigcup_{l \in \mathbb{Z}} \overrightarrow{\Big\langle F^{ln_F}(z), F^{(l+1)n_F}(z)\Big)} \subset I.$$

To show the opposite inclusion suppose that $I := \overrightarrow{(a, b)}$, where $a, b \in \operatorname{Per} F$ and notice that (24) yields

$$\lim_{n \to \infty} F^{nn_F}(z) = b \tag{26}$$

and

$$\lim_{n \to \infty} F^{-nn_F}(z) = a. \tag{27}$$

Now fix $v \in I$. From (25), (26) and (27) it follows that there exists a $k \in \mathbb{Z}$ such that

$$v \in \overrightarrow{\left\langle F^{kn_F}(z), F^{(k+1)n_F}(z)\right)}.$$

Consequently,

$$I \subset \bigcup_{l \in \mathbb{Z}} \overrightarrow{\left\langle F^{ln_F}(z), F^{(l+1)n_F}(z)\right)}$$

and (22) is proved. Applying similar arguments to the case $\overrightarrow{(F^{n_F}(z), z)} \subset I$ we get (23).

To prove the second assertion suppose that $\overrightarrow{(z, F^{n_F}(z))} \subset I$. Now let $u \in I$. Notice that if $u = F^{n_F l}(z)$ for some $l \in \mathbb{Z}$ the assertion follows from (25). Otherwise, by (22) we get

$$\bigcup_{l \in \mathbb{Z}} \overrightarrow{(F^{n_F l}(z), F^{n_F(l+1)}(z))} = I \setminus \{F^{n_F l}(z) : \ l \in \mathbb{Z}\}.$$

Thus it follows that there exists a $j \in \mathbb{Z}$ such that

$$u \in \overrightarrow{(F^{n_F j}(z), F^{n_F(j+1)}(z))}. \tag{28}$$

Hence

$$F^{n_F}(u) \in \overrightarrow{(F^{n_F(j+1)}(z), F^{n_F(j+2)}(z))}.$$

This and (28) lead to

$$\overrightarrow{(u, F^{n_F(j+1)}(z))} \subset \overrightarrow{(F^{n_F j}(z), F^{n_F(j+1)}(z))} \subset I$$

and

$$\overrightarrow{(F^{n_F(j+1)}(z), F^{n_F}(u))} \subset \overrightarrow{(F^{n_F(j+1)}(z), F^{n_F(j+2)}(z))} \subset I.$$

Finally, since $F^{n_F(j+1)}(z) \in I$ we get $\overrightarrow{(u, F^{n_F}(u))} \subset I$. □

Lemma 5. *Let $F \colon S^1 \longrightarrow S^1$ be an orientation-preserving surjection such that $\emptyset \neq \operatorname{Per} F \neq S^1$ and let $I \in \mathscr{B}_F$. If $\overrightarrow{(z, F^{n_F}(z))} \subset I$ (respectively, $\overrightarrow{(F^{n_F}(z), z)} \subset I$) for a $z \in I$, then $\overrightarrow{(z_1, F^{n_F}(z_1))} \subset F(I)$ (respectively, $\overrightarrow{(F^{n_F}(z_1), z_1)} \subset F(I)$ for all $z_1 \in F(I)$.*

P r o o f. For the proof suppose that for some $z \in I$, F fulfils the condition

$$\overrightarrow{(z, F^{n_F}(z))} \subset I.$$

Fix $z_1 \in F(I)$. Since F is a surjection it follows that there exists a $z_0 \in I$ such that $F(z_0) = z_1$. As F preserves the orientation and since Lemma 4 yields $\overrightarrow{(z_0, F^{n_F}(z_0))} \subset I$ we get

$$\overrightarrow{(z_1, F^{n_F}(z_1))} = F\left(\overrightarrow{(z_0, F^{n_F}(z_0))}\right) \subset F(I),$$

which ends the proof. □

We finish with some properties of orientation-preserving surjections with a finite and non-empty set of periodic points. Therefore, from now on we impose on F the following general condition:

(H_1) $F\colon S^1 \longrightarrow S^1$ is an orientation-preserving surjection such that

$$0 < N_F := \operatorname{card} \operatorname{Per} F < \infty.$$

Notice that if a function F satisfies (H_1), then

$$k_F := k_F(\operatorname{Per} F) = \frac{N_F}{n_F}$$

is a number of cycles of $F|_{\operatorname{Per} F}$ and n_F is a number of elements in each such a cycle. In this case, for the convenience, we enumerate the arcs of the family $\mathscr{B}_F$, i.e. for a fixed $z \in \operatorname{Per} F$ define $a_i \in \operatorname{Per} F$ for $i \in \mathbb{Z}_{N_F}$ in the following way:

$$a_0 := z$$
$$\text{and if} \quad N_F > 1 \quad \text{let} \qquad \operatorname{Arg} \frac{a_i}{a_0} < \operatorname{Arg} \frac{a_{i+1}}{a_0}, \qquad i \in \{0, \ldots, N_F - 2\}. \tag{29}$$

Set moreover $a_{N_F} := a_0$ and define

$$I_i := \overrightarrow{(a_i, a_{i+1})} \qquad \text{for} \quad i \in \mathbb{Z}_{N_F}. \tag{30}$$

Notice that if F fulfils (H_1), then

$$S^1 \setminus \operatorname{Per} F = \bigcup_{i=0}^{N_F - 1} I_i.$$

Now for a given homeomorphism $F\colon S^1 \longrightarrow S^1$ satisfying (H_1) we may define two types of arcs of the family $\mathscr{B}_F$.

Definition 1. Let $F\colon S^1 \longrightarrow S^1$ satisfy (H_1). Put

$$Z^+(F) := \left\{ i \in \mathbb{Z}_{N_F} : \ \overrightarrow{(z, F^{n_F}(z))} \subset I_i \text{ for all } z \in I_i \right\}$$

and

$$Z^-(F) := \left\{ i \in \mathbb{Z}_{N_F} : \ \overrightarrow{(F^{n_F}(z), z)} \subset I_i \ \text{ for all } z \in I_i \right\},$$

where I_i for $i \in \mathbb{Z}_{N_F}$ is the family defined by (29) and (30).

From Lemma 4 it follows that $Z^+(F) \cup Z^-(F) = \mathbb{Z}_{N_F}$.

Example. Let $\bar{f}\colon \langle 0,1) \longrightarrow \langle 0,1)$ be defined as follows

$$\bar{f}(x) = \begin{cases} -x^2 + \frac{3}{2}x, & x \in \left\langle 0, \frac{1}{2}\right), \\ 2x^2 - 2x + 1, & x \in \left\langle \frac{1}{2}, 1\right). \end{cases}$$

For every $x \in \mathbb{R}$ put $f(x) := \bar{f}(x - E(x)) + E(x)$, where $E(x)$ denotes the integer part of x. Then $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is a strictly increasing homeomorphism such that $f(x+1) = f(x) + 1$ for $x \in \mathbb{R}$. Moreover, for every $x \in \left(0, \frac{1}{2}\right)$ we have

$$f(x) > x \qquad \text{and} \qquad f(x) \in \left(0, \tfrac{1}{2}\right)$$

and for every $x \in \left(\frac{1}{2}, 1\right)$ we have

$$f(x) < x \qquad \text{and} \qquad f(x) \in \left(\tfrac{1}{2}, 1\right).$$

Therefore, $(x, f(x)) \subset \left(0, \frac{1}{2}\right)$ for $x \in \left(0, \frac{1}{2}\right)$ and $(f(x), x) \subset \left(\frac{1}{2}, 1\right)$ for $x \in \left(\frac{1}{2}, 1\right)$. Let $F\colon S^1 \longrightarrow S^1$ be a homeomorphism defined by

$$F\left(\mathrm{e}^{2\pi \mathrm{i} x}\right) := \mathrm{e}^{2\pi \mathrm{i} f(x)}, \qquad x \in \mathbb{R}.$$

Then $n_F = 1$, $N_F = 2$ and $\operatorname{Per} F = \{-1, 1\}$. Put $a_0 := 1$ and $a_1 := -1$, then $I_0 = \overrightarrow{(a_0, a_1)}$ and $I_1 = \overrightarrow{(a_1, a_0)}$. Fix $z \in I_0$. There exists a unique $x \in \left(0, \frac{1}{2}\right)$ such that $z = \mathrm{e}^{2\pi \mathrm{i} x}$. Notice that

$$\overrightarrow{(z, F(z))} = \left\{ \mathrm{e}^{2\pi \mathrm{i} t} : \ t \in (x, f(x)) \right\} \subset \left\{ \mathrm{e}^{2\pi \mathrm{i} t} : \ t \in \left(0, \tfrac{1}{2}\right) \right\} = I_0.$$

Thus $0 \in Z^+(F)$. Similarly we get that $1 \in Z^-(F)$. Hence $Z^+(F) = \{0\}$ and $Z^-(F) = \{1\}$.

From Lemma 2 and the fact that $F(I) \in \mathscr{B}_F$ for any $I \in \mathscr{B}_F$ we obtain:

Theorem 2. *Suppose that F fulfils (H_1), then*

$$F(I_i) = I_{(i + k_F q) \pmod{N_F}}, \qquad i \in \mathbb{Z}_{N_F}, \tag{31}$$

where $q = q(F)$ and I_i for $i \in \mathbb{Z}_{N_F}$ are defined by (29) and (30).

As a consequence of Theorem 2 and Lemma 5 we get:

COROLLARY 4. *Let F satisfy* (H_1) *and let* $i \in \mathbb{Z}_{N_F}$. *Then* $i \in Z^+(F)$ *iff* $(i + k_F q) \pmod{N_F} \in Z^+(F)$.

Notice that Theorem 2 lets us classify the orientation-preserving homeomorphisms with non-empty and finite set of periodic points in the following way:

DEFINITION 2. Let $n \in \mathbb{N}$ and $q \in \mathbb{Z}_n$ be such that $\gcd(q, n) = 1$. By $\mathscr{P}_{q,n}$ denote the set of all maps $F\colon S^1 \longrightarrow S^1$ satisfying (H_1) and such that $q(F) = q$ and $n_F = n$.

By Lemma 2 we get:

Remark 2. If F satisfies (H_1), then there exists a unique pair (q, n) such that $n \in \mathbb{N}$, $q \in \mathbb{Z}_n$, $\gcd(q, n) = 1$ and $F \in \mathscr{P}_{q,n}$.

We finish with some characterization of the family $\mathscr{P}_{q,n}$.

THEOREM 3. *Let* $n \in \mathbb{N}$ *and* $q \in \mathbb{Z}_n$ *satisfy* $\gcd(q, n) = 1$. *Then* $F \in \mathscr{P}_{q,n}$ *if and only if F satisfies* (H_1) *and* $\alpha(F) = \frac{q}{n}$.

P r o o f. Let us observe that the necessary condition follows from Definition 2 and Lemma 3. To prove the sufficient condition assume that F satisfies (H_1), $\alpha(F) = \frac{q}{n}$ and $F \notin \mathscr{P}_{q,n}$. By Remark 2 there exists a unique pair (q', n') such that $n' \in \mathbb{N}$, $q' \in \mathbb{Z}_{n'}$, $\gcd(q', n') = 1$, $(q, n) \neq (q', n')$ and $F \in \mathscr{P}_{q',n'}$. Using the first part of the theorem we obtain $\alpha(F) = \frac{q'}{n'}$. Therefore, $\frac{q}{n} = \frac{q'}{n'}$ and consequently, since $\gcd(q, n) = \gcd(q', n') = 1$ we get $q = q'$ and $n = n'$, which contradicts our assumption. □

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Institute of Mathematics
Pedagogical University
ul. Podchorążych 2
PL 30-084 Kraków
POLAND
E-mail: psolarz@ap.krakow.pl