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SOME REMARKS ON FUNCTIONS WITH VALUES IN PROBABILISTIC NORMED SPACES

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ABSTRACT. In this paper we consider an enlargement of the notion of the probabilistic normed space. For this new class of probabilistic normed spaces we give some topological properties. By using properties of the probabilistic norm we prove some differential and integral properties of functions with values into probabilistic normed spaces. As special cases, results for deterministic and random functions can be obtained.

1. Introduction

In [16] A. N. Sherstnev endowed a set having an algebraic structure of linear space with a probabilistic norm. He used the K. Menger's idea from [12], where the probabilistic concept of distance was proposed. The number $d(p,q)$, the distance between two points p, q, was replaced by a probabilistic distribution function $F_{p,q}$. These ideas led to a large development of probabilistic analysis. Applications to systems having hysteresis, mixture processes, the measuring error were also given. For an extensive view of this subject we refer [3] [4], [7]–[8] and [15].

In [1] C. Alsina, B. Schweizer and A. Sklar gave a new definition of probabilistic normed spaces which is based on a characterization of normed spaces by means of a betweenness relation and includes the definition of A. N. Sherstnev as a special case. Another results in relating with these spaces were obtained in [5], [9]. In the second section of this paper we introduce a new class of probabilistic normed spaces which also includes the probabilistic

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normed spaces defined by A. N. S h e r s t n e v as a special case. We have generalized the axiom which give a connection between the distribution functions of a vector and its product by a real number.

The third section is devoted to the study of functions with values into such a probabilistic normed space. By using the properties of the probabilistic norm we analyze some differential and integral properties of such functions.

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$ and $I = [0, 1]$, the closed unit interval. A mapping $F\colon \mathbb{R} \to I$ is called a *distribution function* if it is non-decreasing, left continuous with $\inf F = 0$ and $\sup F = 1$. D^+ denotes the set of all distribution functions for which $F(0) = 0$. Let F, G be in D^+, then we write $F \leq G$ if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. If $a \in \mathbb{R}_+$, then H_a will be the element of D^+ defined by $H_a(t) = 0$ if $t < a$ and $H_a(t) = 1$ if $t > a$. It is obvious that $H_0 \geq F$, for all $F \in D^+$. The set D^+ will be endowed with the natural topology defined by the modified Lévy metric d_L ([15]).

A *t-norm* T is a two place function $T\colon I \times I \to I$ which is associative, commutative, non decreasing in each place and such that $T(a, 1) = a$, for all $a \in [0, 1]$.

A *triangle function* τ is a binary operation on D^+ which is commutative, associative, non decreasing in each place and for which H_0 is the identity, that is, $\tau(F, H_0) = F$ for every $F \in D^+$. T-norms and triangle functions have been very important in writing the appropriate probabilistic triangle inequality.

2. On probabilistic normed spaces

Let φ be a function defined on the real field $\mathbb{R}$ into itself, with the following properties:

(a) $\varphi(-t) = \varphi(t)$ for every $t \in \mathbb{R}$;

(b) $\varphi(1) = 1$;

(c) φ is strictly increasing and continuous on $[0, \infty)$,
$\varphi(0) = 0$ and $\lim\limits_{\alpha \to \infty} \varphi(\alpha) = \infty$.

Examples of such functions are: $\varphi(\alpha) = |\alpha|$; $\varphi(\alpha) = |\alpha|^p$, $p \in (0, \infty)$; $\rho(\alpha) = \frac{2\alpha^{2n}}{|\alpha|+1}$, $n \in \mathbb{N}^+$.

Definition 1. Let L be a linear space, τ a triangle function and let $\mathcal{F}$ be a mapping from L into D^+. If the following conditions are satisfied:

(1) $F_x = H_0$, if and only if $x = \theta$;

(2) $F_{\alpha x}(t) = F_x(\frac{t}{\varphi(\alpha)})$ for every $t > 0$, $\alpha \in \mathbb{R}$ and $x \in L$;

(3) $F_{x+y} \geq \tau(F_x, F_y)$, whenever $x, y \in L$;

then $\mathcal{F}$ is called a *probabilistic* φ*-norm* on L and the triple $(L, \mathcal{F}, \tau)$ is called a *probabilistic* φ*-normed space (of Sherstnev type)*. The pair $(L, \mathcal{F})$ is said to be *probabilistic* φ*-seminormed space* if the mapping $\mathcal{F}\colon L \to D^+$ satisfies the conditions (1) and (2). We have made the conventions: $F_x(\frac{t}{0}) = 1$, for $t > 0$, $F_x(\frac{0}{0}) = 0$ and $\mathcal{F}(x)$ is denoted by F_x.

If (1) (2) are satisfied and the probabilistic triangle inequality (3) is formulated under a t-norm T:

(4) $F_{x+y}(t_1 + t_2) \geq T(F_x(t_1), F_y(t_2))$ for all $x, y \in L$ and $t_1, t_2 \in \mathbb{R}_+$,

then $(L, \mathcal{F}, T)$ is called a *Menger* φ*-normed space.*

Proposition 1. *If T is a left continuous t-norm and τ_T is the triangle function defined by* $\tau_T(F, G)(t) = \sup_{t_1+t_2<t} T(F(t_1), G(t_2))$, $t > 0$, *then* $(L, \mathcal{F}, \tau_T)$ *is a probabilistic* φ*-normed space if, and only if,* $(L, \mathcal{F}, T)$ *is a Menger* φ*-normed space.*

If we define $\mathcal{F}^m(x, y) = F_{x-y}$, then a probabilistic φ-normed space $(L, \mathcal{F}, \tau)$ becomes a probabilistic metric space $(L, \mathcal{F}^m, \tau)$ under the same triangle function τ. In what follows we will consider probabilistic φ-normed spaces under a continuous triangle function $\tau \geqslant \tau_{T_m}$, where $T_m(a, b) = \text{Max}\{a + b - 1, 0\}$. This condition ensures the existence of a linear topology on L.

By a φ*-normed space* we mean a pair $(L, \|\cdot\|)$, where L is a linear space, $\|\cdot\|$ is a real valued mapping defined on L such that the following conditions are satisfied:

(5) $\|x\| \geqslant 0$ for all $x \in L$, $\|x\| = 0$ if and only if $x = \theta$;

(6) $\|\alpha \cdot x\| = \varphi(\alpha)\|x\|$, whenever $x \in L$, $\alpha \in \mathbb{R}$ and φ is a function with the above properties;

(7) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in L$.

Remark 1. For $\varphi(\alpha) = |\alpha|^p$, $0 < p < 1$, one obtains a p-normed space ([11], [2]), for $\varphi(\alpha) = |\alpha|$ one obtains an ordinary normed space.

Example 1. Let $(L, \|\cdot\|)$ be a p-normed space. It is easy to check that it can be, in a natural way, made a probabilistic φ-normed space $(L, \mathcal{F}, T)$, by setting $F_x(t) = H_0(t - \|x\|)$ for every $x \in L$, $t \in \mathbb{R}_+$, $\varphi(\alpha) = |\alpha|^p$ and $T = \text{Min}$. Moreover, we have $F_{\alpha x}(t) = H_{\varphi(\alpha)\|x|)}(t)$ and $(L, \mathcal{F}, T)$, is a probabilistic normed space if, and only if, $p = 1$. This example shows us that probabilistic φ-normed spaces include, in a natural way, φ-normed spaces (p-normed spaces). This fact is not possible in the case of probabilistic normed spaces.

Example 2. We will show that, by starting from a φ-normed space, for particular dilatation or contraction functions φ and for different distribution functions G, different probabilistic φ-normed spaces can be obtained. Let $G \in D^+$ be different from H_0, let $(L, \|\cdot\|)$ be a p-normed space. We define $\mathcal{F}\colon L \to D^+$ by $F_\theta = H_0$ and if $x \neq \theta$ by

$$F_x(t) = G\Big(\frac{t}{\|x\|}\Big) \qquad (t \in \mathbb{R}_+).$$

The triple $(L, \mathcal{F}, \tau_T)$ becomes a probabilistic φ-normed space under the t-norm $T = \text{Min}$ and $\varphi(\alpha) = |\alpha|^p$, $p \in (0, \infty)$. This is called a *simple probabilistic φ-normed space generated by the distribution function G and the p-normed space* $(L, \|\cdot\|)$. This example shows us that probabilistic φ-normed spaces have a large statistical disposal. So, different processes of measurement for vectorial amounts can be set in a statistical framework by using an appropriate probabilistic φ-normed space.

Example 3. Now, we consider an example of probabilistic φ-normed space having, as a base space, a set of random variables with values in a p normed space, $p \in (0, 1]$.

Let $(X, \|\cdot\|)$ be a p-normed space. We suppose that $(\Omega, \mathcal{K}, P)$ is a complete probability measure space and $(X, \mathcal{B})$ is the measurable space, where $\mathcal{B}$ is the σ-algebra of Borel subsets of the p-normed space $(X, \|\cdot\|)$. We denote by L a linear subspace of random variables defined on $(\Omega, \mathcal{K}, P)$ with values in $(X, \mathcal{B})$ and we will identify the random variables which are equal with the probability one. For all $x \in L$, $t \in \mathbb{R}$, and $t > 0$ we define

$$F_x(t) = P(\{\omega \in \Omega : \ \|x(\omega)\| < t\}).$$

The triple $(L, \mathcal{F}, T_m)$, where $\mathcal{F}_x(t) = F_x(t)$, is a probabilistic φ normed space with $\varphi(\alpha) = |\alpha|^p$. We verify only the conditions (2) and (3) of the Definition 1, the condition (1) is obviously satisfied. $F_{\alpha x}(t) = P\big(\{\omega \in \Omega : \ \|\alpha x(\omega)\| < t\}\big) = P\big(\{\omega \in \Omega : \ |\alpha|^p \|x(\omega)\| < t\}\big) = P\big(\{\omega \in \Omega : \ \|x(\omega)\| < \frac{t}{|\alpha|^p}\}\big) = F_x\big(\frac{t}{|\alpha|^p}\big)$. For $x, y \in L$, and $t_1, t_2 \in \mathbb{R}_+ - \{0\}$ we define the sets:

$$\begin{aligned} A &= \{\omega \in \Omega : \ \|x(\omega)\| < t_1\}, \\ B &= \{\omega \in \Omega : \ \|y(\omega)\| < t_2\}, \\ C &= \{\omega \in \Omega : \ \|[x(\omega) + y(\omega)]\| < t_1 + t_2\}. \end{aligned}$$

The triangle inequality of a p-normed space implies that $A \cap B \subset C$. By the properties of the probability measure P we have

$$P(C) \geq P(A \cap B) \geq P(A) + P(B) - P(A \cap B) \geq P(A) + P(B) - 1.$$

Taking into account that $P(A) = F_x(t_1)$, $P(B) = F_y(t_1)$ and $P(C) = F_{x+y}(t_1 + t_2)$, it follows that the inequality (4) is satisfied for $T = T_m$. By the Proposition 1 the condition (4) is equivalent with (3).

The following theorem give a topological structure of a probabilistic φ-normed space.

THEOREM 1. *Let $(L, \mathcal{F}, T)$ be a Menger φ-normed space under a continuous t-norm T such that $T \geq T_m$, then:*

(a) $\mathcal{V} = \{V(\varepsilon, \lambda) : \ \varepsilon > 0, \ \lambda \in (0,1)\}$, $V(\varepsilon, \lambda) = \{x \in L : \ F_x(\varepsilon) > 1 - \lambda\}$ *is a complete system of neighbourhoods of null vector for a linear topology on L generated by the φ-norm $\mathcal{F}$.*

(b) *The family of subsets of L: $\mathcal{U} = \{U(\varepsilon, \lambda) : \ \varepsilon > 0, \ \lambda \in (0,1)\}$,*

$$U(\varepsilon, \lambda) = \{(x, y) \in L \times L : \ F_{x-y}(\varepsilon) > 1 - \lambda\}$$

is a complete system of neighbourhoods for a uniformity on L.

Proof. We will prove only the point (a), the proof of (b) is similar to that of (a) and we will omit it.

Let $V(\varepsilon_k, \lambda_k)$, $k = 1, 2$ be in $\mathcal{V}$, consider $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ and $\lambda = \min\{\lambda_1, \lambda_2\}$, then $V(\varepsilon, \lambda) \subset V(\varepsilon_1, \lambda_1) \cap V(\varepsilon_2, \lambda_2)$.

Let $\alpha \in \mathbb{R}$ such that $0 \leq |\alpha| \leq 1$ and $x \in \alpha V(\varepsilon, \lambda)$, then $x = \alpha y$, where $y \in V(\varepsilon, \lambda)$ and we have

$$F_x(\varepsilon) = F_{\alpha y}(\varepsilon) = F_y\left(\tfrac{\varepsilon}{\varphi(\alpha)}\right) \geq F_y(\varepsilon) > 1 - \lambda.$$

This shows us that $x \in V(\varepsilon, \lambda)$, hence $\alpha V(\varepsilon, \lambda) \subset V(\varepsilon, \lambda)$.

Let us show that, for every $V \subset \mathcal{V}$ and $x \in L$ there exists $\alpha \in \mathbb{R}$, $\alpha \neq 0$, such that $\alpha x \in V$. If $V \in \mathcal{V}$, then there exist $\varepsilon > 0$, $\lambda \in (0,1)$ such that $V = V(\varepsilon, \lambda)$. Let x be arbitrarily fixed in L and $\alpha \in \mathbb{R}$, $\alpha \neq 0$, then $F_{\alpha x}(\varepsilon) = F_x\left(\frac{\varepsilon}{\varphi(\alpha)}\right)$. Since, $\lim\limits_{\alpha \to 0} F_x\left(\frac{\varepsilon}{\varphi(\alpha)}\right) = 1$ it follows that, there exists $\alpha \in \mathbb{R}$ such that $F_{\alpha x}(\epsilon) = F_x\left(\frac{\varepsilon}{\varphi(\alpha)}\right) > 1 - \lambda$, hence $\alpha x \in V$.

Let us prove that, for every $V \in \mathcal{V}$ there exists $V_0 \in \mathcal{V}$ such that $V_0 + V_0 \subset V$. If $V = V(\varepsilon, \lambda)$ and $x \in V(\varepsilon, \lambda)$, then there exists $\eta > 0$ such that $F_x(\varepsilon) > 1 - \eta > 1 - \lambda$. If $V_0 = V(\frac{\varepsilon}{2}, \frac{\eta}{2})$ and $x, y \in V_0$, by the triangle inequality (4) we have

$$F_{x+y}(\varepsilon) \geq T\left(F_x\left(\frac{\varepsilon}{2}\right), F_y\left(\frac{\varepsilon}{2}\right)\right) \geq T_m\left(1 - \frac{\eta}{2}, 1 - \frac{\eta}{2}\right) > 1 - \eta > 1 - \lambda.$$

The above inequalities show us that $V_0 + V_0 \subset V$.

Now, we show that $V \in \mathcal{V}$ and $\alpha \in \mathbb{R}$, $\alpha \neq 0$, imply $\alpha V \in \mathcal{V}$. Let us remark that $\alpha V = \alpha V(\varepsilon, \lambda) = \left\{\alpha x : \ F_x(\varepsilon) > 1 - \lambda\right\}$ and $F_x(\varepsilon) > 1 - \lambda \iff F_x\left(\frac{\varphi(\alpha)\varepsilon}{\varphi(\alpha)}\right) = F_{\alpha x, a}(\varphi(\alpha)\varepsilon) > 1 - \lambda$. This shows us that $\alpha V = V(\varphi(\alpha)\varepsilon, \lambda, A)$, hence $\alpha V \in \mathcal{V}$.

The above statements show us that $\mathcal{V}$ is a base for a system of neighborhoods of the null vector in the linear space L. It is easy to see that the uniformity generated by $\mathcal{U}$ and the topology generated by $\mathcal{V}$ are compatible. $\square$

PROPOSITION 2. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in L and let $(L, \mathcal{F}, T)$ be a Menger φ-normed space under a continuous t-norm T, then:*

(a) *$\{x_n\}$ converges to x in the topology generated by the probabilistic φ-norm $\mathcal{F}$ on L if, and only if, $F_{x_n-x}(t)$, converges to $H_0(t)$ for every $t > 0$;*

(b) *$\{x_n\}$ is a Cauchy sequence in the uniformity generated by the probabilistic φ-norm $\mathcal{F}$ on L if, and only if, $F_{x_n-x_m}(t)$ converges to $H_0(t)$ for all $t > 0$.*

3. Functions with values in probabilistic normed spaces

Let $(\Omega, \mathcal{K}, P)$ be a complete probability measure space, i.e., Ω is a nonempty set, $\mathcal{K}$ is a σ-algebra on Ω and P is a complete probability measure on $\mathcal{K}$. Let $(X, \mathcal{B})$ be a measurable space, where $(X, \|\cdot\|)$ is a separable Banach space and $\mathcal{B}$ is the σ-algebra of the Borel subsets of $(X, \|\cdot\|)$.

A mapping $x\colon \Omega \to X$ is said to be a random variable with values in X if $x^{-1}(B) \in \mathcal{K}$ for all $B \in \mathcal{B}$ ([2], [14]). Let $\mathcal{X}$ be the set of all random variables (equal in probability) and let $\mathcal{F}$ be the probabilistic norm on $\mathcal{X}$ defined by

$$F_x(t) = P(\{\omega \in \Omega : \ \|x(\omega)\| < t\}).$$

It is known that $(\mathcal{X}, \mathcal{F}, \tau_{T_m})$ is a complete probabilistic normed space of Sherstnev type. Furthermore, the (ε, λ)-topology on $\mathcal{X}$ induced by the probabilistic norm $\mathcal{F}$ is equivalent to the topology of the convergence in probability on $\mathcal{X}$.

A mapping f is said to be a *random function* defined on the subset A of real line with values in a separable Banach space X if, for every $t \in A$ the mapping $f(t, \cdot)\colon \Omega \to X$ is a X-valued random variable. Two X-valued random functions f and g are said to be equivalent if $f(t, \omega)$ and $g(t, \omega)$ are equal almost surely for every $t \in A$.

Random functions have had a special importance in the probability theory as well as in its applications. Regarding time series as random functions their predictability have increased and random functions have given important new tools in solving economics and engineering problems. Now, let f be a X-valued random function defined on $A \subset \mathbb{R}$, then one can define the mapping $\widetilde{f}$ on A with values in the random normed space $(\mathcal{X}, \mathcal{F}, T_m)$ by $A \ni t \mapsto \widetilde{f}(t)$, where $[\widetilde{f}(t)](\omega) = f(t, \omega)$. Conversely, for every function $\widetilde{f}\colon A \to (\mathcal{X}, \mathcal{F}, T_m)$ one can define the X-valued random function on A by $f(t, \omega) = [\widetilde{f}(t)](\omega)$, for every $t \in A$ and $\omega \in \Omega$. Furthermore the correspondence $f \mapsto \widetilde{f}$ is one to one and onto. By this correspondence results obtained for functions with values in a probabilistic φ-normed space can be applied to the study of random functions with values in a separable Banach space.

These considerations have determined us to approach the study of functions with values into a probabilistic φ-normed space.

PROPOSITION 3. *Let f be a function and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions defined on a non-empty subset A of real line with values in a Menger φ-normed spaces $(L,\mathcal{F},T)$. Then we have:*

(a) *The function f is continuous in $t_0 \in A$, if and only if, for every $\varepsilon > 0$ and $\lambda \in (0,1)$ there is $\delta(\varepsilon,\lambda) > 0$ such that for all $t \in A$ with $|t - t_0| < \delta(\varepsilon,\lambda)$*

$$F_{f(t)-f(t_0)}(\varepsilon) > 1 - \lambda.$$

(b) *The sequence $\{f_n\}_{n\in\mathbb{N}}$ converges on the set A to the function f if and only if for every $\varepsilon > 0$, $\lambda \in (0,1)$ and $t \in A$ there is an integer $N(\varepsilon,\lambda,t)$ such that, for all $n > N(\varepsilon,\lambda,t)$ we have*

$$F_{f_n(t)-f(t)}(\varepsilon) > 1 - \lambda.$$

The above statements are valid because the family $\{V_x(\varepsilon,\lambda) : V_x(\varepsilon,\lambda) = \{y \in L : F_{x-y}(\varepsilon) > 1-\lambda\},\ \varepsilon > 0,\ \lambda \in (0,1)\}$ is a complete system of neighbourhoods for the point x in the topology generated by the Menger φ-norm $\mathcal{F}$ of $(L,\mathcal{F},T)$.

DEFINITION 2. A sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions defined on a set $A \subset \mathbb{R}$ with values in a Menger φ-normed space $(L,\mathcal{F},T)$ is called a *Cauchy sequence* if for every $\varepsilon > 0$ and $\lambda \in (0,1)$ there is an integer $N(\varepsilon,\lambda) > 0$ such that $F_{f_n(t)-f_m(t)}(\varepsilon) > 1-\lambda$ for all $t \in A$ and $n, m \geq N(\varepsilon,\lambda)$.

THEOREM 2. *A sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions defined on the set $A \subset \mathbb{R}$ with values in a complete Menger φ-normed space $(L,\mathcal{F},T)$ is uniformly convergent on A if and only if $\{f_n\}_{n\in\mathbb{N}}$ is a Chauchy sequence on A.*

In what follows we will use the probabilistic φ-norm to introduce the fundamental concepts of differential and integral calculus for functions with values in a probabilistic φ-normed space.

Some particular results show us that these concepts assure a natural frame for the study of random functions.

DEFINITION 3. Let f be a function defined on a set $A \subset \mathbb{R}$ with values in a probabilistic φ-normed space $(L,\mathcal{F},T)$ and let $t_0 \in I \subset A$, where I is an open interval. The function f is said to *be differentiable in the point* t_0 if there exists an element $x_0 \in L$ such that, for every $\varepsilon > 0$ and $\lambda \in (0,1)$ there exists $\delta(\varepsilon,\lambda) > 0$ such that

$$F_{\frac{f(t)-f(t_0)}{t-t_0}-x_0}(\varepsilon) > 1 - \lambda$$

for all $t \in I$, $t \neq t_0$, with $|t - t_0| < \delta(\varepsilon,\lambda)$. The element $x_0 \in L$ is called the *derivative* of f in the point t_0 and it is denoted by $f'(t_0)$. If the function f is differentiable in each point $t \in A$, then the function f is said to be *differentiable on the set A*.

Proposition 4. *If the function $f\colon A \to (L, \mathcal{F}, T)$ is differentiable in $t_0 \in A$, then the derivative $f'(t_0)$ is unique.*

Proof. Let us consider $\varepsilon > 0$, $\lambda \in (0,1)$ and x_0, $y_0 \in L$ such that $f'(t_0) = x_0$ and $f'(t_0) = y_0$. By Definition 3 it results that there exists $\delta(\varepsilon, \lambda) > 0$ and $\eta \in (0,1)$ such that

$$F_{\frac{f(t)-f(t_0)}{t-t_0}-x_0}\left(\frac{\varepsilon}{2}\right) > 1 - \frac{\eta}{2} > 1 - \frac{\lambda}{2}$$

and

$$F_{\frac{f(t)-f(t_0)}{t-t_0}-y_0}\left(\frac{\varepsilon}{2}\right) > 1 - \frac{\eta}{2} > 1 - \frac{\lambda}{2}$$

for every $t \in A$, $t \neq t_0$, and $|t - t_0| < \delta(\varepsilon, \lambda)$. Then we have :

$$\begin{aligned} F_{x_0-y_0}(\varepsilon) &\geq T\left(F_{\frac{f(t)-f(t_0)}{t-t_0}-x_0}\left(\frac{\varepsilon}{2}\right), F_{\frac{f(t)-f(t_0)}{t-t_0}-y_0}\left(\frac{\varepsilon}{2}\right)\right) \\ &\geq T_m\left(1 - \frac{\eta}{2}, 1 - \frac{\eta}{2}\right) \geq 1 - \eta > 1 - \lambda. \end{aligned}$$

If $\lambda \to 0$, it results that $F_{x_0,y_0}(\varepsilon) = 1$ for every $\varepsilon > 0$. Hence $F_{x_0-y_0} = H_0$ and $x_0 = y_0$. □

Proposition 5. *If the function $f\colon A \to (L, \mathcal{F}, T)$ is differentiable in $t_0 \in A$, then it is continuous in the point t_0.*

Now, let f be a function defined on a interval $[a,b]$ and let Δ be a division of the interval $[a,b]$ given by $a = t_0 < t_1 < \cdots < t_n = b$. Let us denote, by $\nu(\Delta) = \max_{0 \leq i \leq n-1} (t_{i+1} - t_i)$, the norm of the division Δ and let $u = (u_i)_{0 \leq i \leq n-1}$, $u_i \in [t_i, t_{i+1}]$.

Now, we define:

$$\sigma_\triangle(f,u) = \sum_{i=0}^{n-1} (t_{i+1} - t_i) f(u_i).$$

Definition 4. A function $f\colon [a,b] \to (L, \mathcal{F}, T)$ is said to be *Riemann integrable* on $[a,b]$ if there exist $x_0 \in L$ such that for every $\varepsilon > 0$ and $\lambda \in (0,1)$ there exists $\delta(\varepsilon, \lambda) > 0$ such that, if $\nu(\Delta) < \delta(\varepsilon, \lambda)$, then we have

$$F_{\sigma_\triangle(f,u)-x_0}(\varepsilon) > 1 - \lambda$$

for every $u = (u_i)_{0 \leq i \leq n-1}$. The element $x_0 \in L$ is called the *Riemann integral* of the function f on the interval $[a,b]$ and it is denoted by $x_0 = \int_a^b f(t)\,\mathrm{d}t$.

Proposition 6. *If the function $f\colon [a,b] \to (L,\mathcal{F},T)$ is integrable on $[a,b]$, then $\int_a^b f(t)\,\mathrm{d}t$ is unique.*

The proof is similar to that of Proposition 4 and we omitted it. One can similarly prove that the other known properties of integrals are valid.

The following theorem assures us that a large class of functions defined on a interval $[a,b] \subset \mathbb{R}$, with values in a Menger φ-normed space $(L,\mathcal{F},T)$ are integrable.

We say that a continuous t-norm T is of *Hadzić type* if the family $\{T^n\}_{n\in\mathbb{N}}$, where $T^1(t) = t$, $T^2(t) = T(t,t)$ and $T^{n+1}(t) = T(T^n(t),t)$, is equicontinuous at $t = 1$.

Theorem 3. *Let $(L,\mathcal{F},T)$ be a complete Menger φ-normed space under a continuous t-norm T of Hadzić type. If f is a continuous function defined on $[a,b]$ with values in $(L,\mathcal{F},T)$, then f is Riemann integrable on $[a,b]$.*

Proof. Let $\Delta' : \ a = t_0' < t_1' < \cdots < t_n' = b$ and $\Delta'' : \ a = t_0'' < t_1'' < \ldots \cdots < t_n'' = b$ be two divisions of $[a,b]$. We will show that, for every $\varepsilon > 0$ and $\lambda \in (0,1)$ there exists $\delta(\varepsilon,\lambda) > 0$ such that, if $\max\{\nu(\Delta'),\nu(\Delta'')\} \le \delta(\varepsilon,\lambda)$, then

$$F_{\sigma_{\Delta'}(f,u)-\sigma_{\Delta''}(f,u)}(\varepsilon) > 1-\lambda.$$

If the t-norm T is continuous and of Hadzić type, then for every $\lambda \in (0,1)$ there exists $\eta \in (0,1)$ such that $T^n(1-\eta) > 1-\lambda$ for all $n \ge 1$. Since f is continuous on $[a,b]$, it results that, for every $\varepsilon > 0$ and $\eta > 0$ previously considered, there exists $\delta_1(\varepsilon,\lambda) > 0$ and $\eta_1 \in (0,1)$ such that

$$F_{f(t')-f(t'')}\left(\frac{\varepsilon}{b-a}\right) > 1-\eta_1 > 1-\eta$$

for every $t',t'' \in [a,b]$ with $|t'-t''| < \delta_1(\varepsilon,\lambda)$. Let us consider $\delta(\varepsilon,\lambda) = \frac{1}{4}\delta_1(\varepsilon,\lambda)$ and the division Δ of $[a,b]$ given by the union of the divisions Δ' and Δ''. We assume that $\Delta : \ a_0 = t_1 < t_2 < \cdots < t_p = b$, $u_i' \in [t_i',t_{i+1}']$ for $0 \le i \le n-1$ and $u_i'' \in [t_j'',t_{j+1}'']$ for $0 \le j \le m-i$. Let us denote $u_k = u_i'$ for $[t_k,t_{k+1}] \subset [t_i',t_{i+1}']$ and $\tilde{u}_k = u_j''$ for $[t_k,t_{k+1}] \subset [t_j'',t_{j+1}'']$ for any $0 \le i \le n-1$, $0 \le j \le m-1$ and $0 \le k \le p-1$.

Then we have

$$
\begin{aligned}
&F_{\sigma_{\Delta'}(t,u')-\sigma_{d''}(f,u'')}(\varepsilon)\\
&=F_{\sum\limits_{i=0}^{n-1}(t'_{i+1}-t'_i)f(u'_i)-\sum\limits_{j=0}^{m-1}(t'_{j+1}-t'_j)f(u''_j)}(\varepsilon)\\
&=F_{\sum\limits_{k=1}^{p-1}(t_{k+1}-t_k)(f(u_k)-f(\tilde{u}_k))}\left[\frac{\varepsilon}{b-a}\cdot\sum_{k=0}^{p-1}(t_{k+1}-t_k)\right]\\
&\geq \underbrace{T(T(\dots T(}_{(p-1)times} F_{f(u_0)-f(\tilde{u}_0)}\big(\tfrac{\varepsilon}{b-a}\big), F_{f(u_1)-f(\tilde{u}_1)}\big(\tfrac{\varepsilon}{b-a}\big),\dots\\
&\quad\dots, F_{f(u_{p-1})-f(\tilde{u}_{p-1})}\big(\tfrac{\varepsilon}{b-a}\big))\dots) \geq T^{p-1}(1-\eta_1) > T^{p-1}(1-\eta) > 1-\lambda
\end{aligned}
$$

for every two divisions Δ', Δ'' of the interval $[a,b]$ with $\max\{\nu(d'),\nu(d'')\} \leq \delta(\varepsilon,\lambda)$ and for every choice of points $u_i \in [t'_i, t'_{i+1}]$ and $u''_j \in [t''_j, t''_{j+1}]$, where $0 \leq i \leq n-1$ and $0 \leq j \leq m-1$.

Now, let $\{\Delta_n\}_{n\geq 1}$ be a sequence of divisions of the interval $[a,b]$ such that $\lim\limits_{n\to\infty} \nu(\Delta_n) = 0$. Then, for any $\delta_1 > 0$ there exists $n_0 \in \mathbb{N}$ such that, for $n', n'' \geq n_0$, $\max\{\nu(\Delta_{n'}), \nu(\Delta''_n)\} < \delta$. If we choose this δ_1 such that $\delta_1 < \delta(\varepsilon,\lambda)$, then we have

$$F_{\sigma_{\Delta_{n'}}(f,u'_n)-\sigma_{\Delta_{n''}}(f,u''_n)}(\varepsilon) > 1-\lambda.$$

This show us that the sequence $\{\sigma_{\Delta_n}(f,u)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the RN-space $(L,\mathcal{F},T)$. This being complete, it results that there exists $x_0 = \lim\limits_{n\to\infty} \sigma_{\Delta_n}(f,u) = \int\limits_a^b f(t)\,dt$. This completes the proof of the theorem. □

Remark 2. We know that every continuous function defined on $[0,1]$ with values in a complete metric linear space (L,d) is Riemann integrable if and only if (L,d) is a locally convex topological linear space ([14]). We remark that, if the t-norm T is not of Hadzić type, then there exists a Menger φ-normed spaces $(L,\mathcal{F},T)$ which, endowed with the (ε,λ)-topology generated by the Menger φ-norm $\mathcal{F}$, is not locally convex. These shows us that the above theorem offers the largest class of Menger φ-normed spaces $(L,\mathcal{F},T)$ which have the property: every continuous function $f\colon [a,b] \to (L,\mathcal{F},T)$ is Riemann integrable.

Remark 3. The condition as the t-norm T to be of Hadzić type is also a necessary condition of the Theorem 3 ([8], [14]).

Remark 4. If $(L, \mathcal{F}, T)$ is a random normed space under a t-norm T which is not of Hadzić type, there are continuous functions which are not integrable. So, an open problem is to find which classes of functions are integrable in which classes of probabilistic normed spaces, especially, for T_m and product t-norm T_p, which are not of Hadzić type.

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