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Dedicated to Belo Riečan, mathematicus et musicus

A FORMAL ANALYSIS OF MUSICAL SCORES

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ABSTRACT. Musical scores are very complicated examples of symbolic languages. Is it possible (and interesting) to represent a musical score as a peculiar example of a *formal language*? We give a positive answer to this question by introducing the notion of *formal representation of a musical score*.

1. Musical scores and formal languages

Musical scores are very complicated examples of symbolic languages. It is interesting to analyze how information is coded by musical scores in comparison with the standard formal languages that are used for scientific theories. The most important differences seem to be the following:

- Formal scientific languages are basically *linear* and *compositional*: *words* and *well-formed expressions* are represented as *strings* consisting of symbols that belong to a well determined alphabet. The relevant syntactical relations can be adequately represented by a Turing Machine, which refers to a one-dimensional tape.

- Scores, instead, are two-dimensional syntactical objects, which have at the same time a horizontal and a vertical component. Any attempt to linearize a score would lead to totally counter-intuitive results. From a semantic point of view, the characteristic two-dimensionality of musical notation seems to be significantly connected with the deep parallel structures that have an essential role in our perception and intellectual elaboration of musical experiences.

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We will investigate the following question: is it possible (and interesting) to represent a musical score as a peculiar example of a *formal language*? In a sense, are scores *formalizable*? We will positively answer this question by introducing the notion of *formal representation of a musical score*.

Let us first discuss the role of *proper names* in science and in music. As is well known, in the framework of standard formal languages, *proper names* represent "privileged" expressions that *denote* objects belonging to an appropriate *universe of discourse*. Do musical languages use proper names in a similar way? Let us refer to the tempered scale system.[1] We can think of the 84 $(= 7 \cdot 12)$ keys — from the lowest c to the highest b — of a well tuned piano (whose a_4 has the fundamental frequency of approximately 440 Hertz).[2] A natural name-system for the sounds corresponding to these keys is the following set (called the *basic note-name system*):

$$BNames^{Notes} = \left\{\mathbf{c_i}, \mathbf{c_i^\sharp}, \mathbf{d_i}, \mathbf{d_i^\sharp}, \mathbf{e_i}, \mathbf{f_i}, \mathbf{f_i^\sharp}, \mathbf{g_i}, \mathbf{g_i^\sharp}, \mathbf{a_i}, \mathbf{a_i^\sharp}, \mathbf{b_i} : \ 1 \leq i \leq 7\right\},$$

where $\mathbf{c_i}$ corresponds to the c of the the ith octave, $\mathbf{c_i^\sharp}$ corresponds the $c^\sharp$ of the the ith octave, and so on. We will use ν, ν_1, ν_2, ... as variables ranging over $BNames^{Notes}$, while $\nu^{(i)}$, $\nu_1^{(i)}$, $\nu_2^{(i)}$, ... will range over the set $BNames_i^{Notes} = \left\{\mathbf{c_i}, \mathbf{c_i^\sharp}, \mathbf{d_i}, \mathbf{d_i^\sharp}, \mathbf{e_i}, \mathbf{f_i}, \mathbf{f_i^\sharp}, \mathbf{g_i}, \mathbf{g_i^\sharp}, \mathbf{a_i}, \mathbf{a_i^\sharp}, \mathbf{b_i}\right\}$ consisting of all the basic note-names for the elements of the ith octave. We will also write $\nu^{(i_j)}$ for the name of the jth note of the ith octave.

A linear (lexicographic) order can be naturally defined on the set of all basic note-names:

$$\nu^{i_j} \preceq \nu^{h_k} \quad \text{iff} \quad \left([i < h] \text{ or } [i = h \text{ and } j \leq k]\right).$$

The *intended meaning* of a basic note-name is, of course, a sound-pitch (which can be thought of as a particular frequency). As a consequence, we have:

$\nu^{i_j} \prec \nu^{h_k}$ iff the pitch denoted by ν^{i_j} is lower than the pitch denoted by ν_{h_k}.

Consider the sequence of all basic note-names in the *natural order*:

$$\mathbf{c_1}, \mathbf{c_1^\sharp}, \ldots, \mathbf{c_i}, \mathbf{c_i^\sharp}, \ldots, \mathbf{a_7^\sharp}, \mathbf{b_7}.$$

We also write:

$$\nu^{[1]}, \nu^{[2]}, \ldots, \nu^{[n]}, \ldots \nu^{[7 \cdot 12]},$$

[1]We refer to the tempered system for the sake of simplicity. The formal analysis we are proposing can be obviously generalized also to other systems.

[2]For arithmetical simplicity we refer to exactly seven octaves. But, of course, we might consider all the keys of a modern piano and possibly also some other lower and higher sounds.

where $\nu^{[1]} = \mathbf{c_1}, \ldots, \nu^{[7\cdot 12]} = \mathbf{b_7}$ and $\nu^{[n]}$ represents the nth basic note-name ($1 \leq n \leq 7 \cdot 12$). Of course, we have: $\nu^{(i_j)} = \nu^{[12(i-1)+j]}$, where $1 \leq i \leq 7$ and $1 \leq j \leq 12$.

Apparently, basic note-names are an imperfect way to represent notes, since they do not take into consideration the note-values. We will now introduce the notion of *valued basic note-name*. Let $\frac{x}{y}$ be a meter-indication (say, $\frac{3}{4}$). Any rational number $\frac{b}{ay}$ such that $\frac{b}{ay} \leq \frac{x}{y}$ is called a *submeter* of $\frac{x}{y}$. We will also write: $\frac{b}{ay}\left[\frac{x}{y}\right]$. For instance, if $\frac{x}{y} = \frac{3}{4}$, the expression $\frac{1}{2\cdot 4}\left[\frac{3}{4}\right] = \frac{1}{8}\left[\frac{3}{4}\right]$ represents a submeter of $\frac{3}{4}$ (read as: $\frac{1}{8}$ of $\frac{3}{4}$). By *valued basic note-name* we mean an expression having the following form: ${}^{\frac{b}{ay}[\frac{x}{y}]}\nu$, where ν is a basic note-name, while $\frac{b}{ay}\left[\frac{x}{y}\right]$ is a submeter of a given meter $\frac{x}{y}$.

The translation of a valued note-name into the standard staff-notation is the expected one. For instance, the name ${}^{\frac{1}{4}[\frac{2}{4}]}\mathbf{c_4}$ corresponds to the following standard notation:

We will use v as a variable ranging over the set of all possible note-values $\frac{b}{ay}\left[\frac{x}{y}\right]$. Hence ${}^{v}\nu$ will represent a generic valued note-name.

Valued basic note-names do not give any indication about the *emission* of the corresponding sounds. It is important to distinguish names denoting *onset-sounds* from the names that denote a *tied sound* (a *legato*, where the sound is prolonged without any new emission). As is well known, in the framework of the standard staff notation, such a distinction is realized by means of the *legato*-symbol.

EXAMPLE 1.1.

1. Two onset-notes

2. An onset-note followed by a tied note

In our formalization, we will use the following convention: we will write $\searrow\nu$ to indicate an onset-note, while $\frown\nu$ will represent a tied note. In the following, the symbol $\underline{\nu}$ will represent either $\searrow\nu$ or $\frown\nu$. We will call any $\underline{\nu}$ a *signed* note-name; while any ${}^{v}\underline{\nu}$ will be called a *complete note-name* (or, simply, a *note-name*). Given a note-name ${}^{v}\underline{\nu}$, we will say that v represents the value of ${}^{v}\underline{\nu}$, while ν represents the basic note-name corresponding to ${}^{v}\underline{\nu}$.

What about chords? Consider the following example of a chord (written according to the standard notation):

A *molecular* term that denotes such a chord can be naturally written as follows (according to a matrix-like notation):

$$\begin{pmatrix} \searrow {}^{\frac{2}{4}[\frac{4}{4}]}\mathbf{c}_5 \\ \searrow {}^{\frac{2}{4}[\frac{4}{4}]}\mathbf{g}_4 \\ \searrow {}^{\frac{2}{4}[\frac{4}{4}]}\mathbf{e}_4 \\ \searrow {}^{\frac{2}{4}[\frac{4}{4}]}\mathbf{c}_4 \end{pmatrix}.$$

Generally, by *chord-name* we mean an expression having the following form:

$$\begin{pmatrix} {}^{\frac{b}{ay}[\frac{x}{y}]}\underline{\nu}_n \\ \vdots \\ {}^{\frac{b}{ay}[\frac{x}{y}]}\underline{\nu}_1 \end{pmatrix},$$

where $\nu_1 \preceq \nu_2 \preceq \cdots \preceq \nu_n$ (from the lowest to the highest note). Needless to say, note-names are limit-cases of chord-names (representing *improper* chord-names). Proper and improper chord-names will be also called *note-terms*. We will indicate by $Terms^{Notes}$ the set of all note-terms, and we will use ${}^{\frac{b}{ay}[\frac{x}{y}]}\underline{\tau}$ (or ${}^{v}\underline{\tau}$) as a variable ranging over this set. We assume that the set of all note-terms contains all pause-symbols, indicated by ${}^{v}\pi$ (where v represents the value of ${}^{v}\pi$). Unlike valued note-names, any pause-symbol is clearly determined by its value.

Consider now a standard score $\mathbf{S}$ (for instance the score of a string Trio), which consists of a sequence of measures:

$$(\mathbf{M}^{[\mathbf{1}]}, \dots, \mathbf{M}^{[\mathbf{s}]}).$$

We call *length* of $\mathbf{S}$ the number s of measures of $\mathbf{S}$. A *formal representation* of $\mathbf{S}$ can be described as an abstract syntactical object $\mathbf{FS}$ that expresses in a canonical way the relevant information contained in $\mathbf{S}$. As happens in the case of $\mathbf{S}$, also $\mathbf{FS}$ consists of a sequence of s *formal measures* (where s is the length of $\mathbf{S}$):

$$(\mathbf{FM}^{[\mathbf{1}]}, \dots, \mathbf{FM}^{[\mathbf{s}]}).$$

Any $\mathbf{FM}^{[\mathbf{r}]}$ is a *formal representation* of the corresponding standard measure $\mathbf{M}^{[\mathbf{r}]}$ of $\mathbf{S}$. The relation that holds between $\mathbf{FM}^{[\mathbf{r}]}$ and $\mathbf{M}^{[\mathbf{r}]}$ can be determined as follows.

Let $\mathrm{INS} = \{\mathrm{Ins}_1, \ldots, \mathrm{Ins}_m\}$ be the set of *instruments* (or *voices*) which $\mathbf{S}$ refers to (say, INS = {*Violin*, *Viola*, *Cello*}).

Any $\mathbf{FM}^{[\mathbf{r}]}$ is represented as a matrix-like configuration, where the number of rows corresponds to the number of instruments. By simplicity, we assume that this number remains constant through the whole formal score $\mathbf{FS}$. Whenever a given instrument *tacet*, we suppose that some appropriate pause-symbols are written for the instrument in question. The formal representation $\mathbf{FM}^{[\mathbf{r}]}$ of $\mathbf{M}^{[\mathbf{r}]}$ has the following form:

$$\begin{pmatrix} \mathrm{Ins}_1 : A_{11}^{[r]} \ldots A_{1n}^{[r]} \\ \mathrm{Ins}_2 : A_{21}^{[r]} \ldots A_{2n}^{[r]} \\ \vdots \quad\; \vdots \quad\quad \vdots \\ \mathrm{Ins}_m : A_{m1}^{[r]} \ldots A_{mn}^{[r]} \end{pmatrix}$$

where any $A_{ij}^{[r]}$, called a *score-atom* of $\mathbf{FM}^{[\mathbf{r}]}$, represents a piece of information for the ith instrument at the jth position of $\mathbf{FM}^{[\mathbf{r}]}$.

The number m of *rows* is determined by the number of instruments. In order to determine the number n of *columns*, we first define the *atomic value* $\alpha^{\mathbf{M}}$ of $\mathbf{M}$. Suppose that $\mathit{Meter}^{\mathbf{M}} = \frac{x}{y}$ (say, $\frac{x}{y} = \frac{3}{4}$). Suppose that the minimum value of all notes/pauses occurring in $\mathbf{M}$ is greater than $\frac{1}{y}$. Then,

$$\alpha^{\mathbf{M}} := \frac{1}{ay}, \qquad \text{where} \quad a = 1 .$$

Otherwise,

$$\alpha^{\mathbf{M}} := \frac{1}{ay},$$

where $\frac{1}{ay}$ is the minimum value of the notes/pauses occurring in $\mathbf{M}$.

EXAMPLE.

$$\frac{x}{y} = \frac{3}{4}; \qquad \alpha^{\mathbf{M}} = \frac{1}{ay} = \frac{1}{2 \cdot 4} = \frac{1}{8} .$$

On this basis, the number n of columns is determined as follows:

$$n = xa, \qquad \text{where} \quad a = \frac{1}{\alpha^{\mathbf{M}} y} .$$

EXAMPLE.

$$\frac{x}{y} = \frac{3}{4}; \quad \alpha^{\mathbf{M}} = \frac{1}{2 \cdot 4}; \qquad a = 2; \;\; n = xa = 3 \cdot 2 = 6 .$$

The number n of columns is also called the *length* of the measure $\mathbf{M}^{[\mathbf{r}]}$ and of its formal representation $\mathbf{FM}^{[\mathbf{r}]}$.

For any row i (corresponding to the ith instrument), the sequence

$$A_{i1}^{[r]}, \dots, A_{in}^{[r]}$$

represents the rth formal measure for the instrument in question (say, for the violin). We assume that the canonical form of $A_{ij}^{[r]}$ is a sequence consisting of three basic kinds of information. More precisely,

$$A_{ij}^{[r]} = \left(\mathrm{Inf}_{ij}^{[r]},\ \mathit{Notes}_{ij}^{[r]},\ \mathit{Perform}_{ij}^{[r]}\right),$$

where:

- $\mathrm{Inf}_{ij}^{[r]}$ sums up some general pieces of information (which appear either at the beginning of a movement of $\mathbf{S}$ or at the beginning of the measure $\mathbf{M}^{[r]}$ of $\mathbf{S}$). We suppose that $\mathrm{Inf}_{ij}^{[r]}$ consists at least of the following items:
  - A *Tempo-indication* (*Allegro, Adagio, ...*);
  - A *Meter-indication* $\left(\frac{3}{4}, \frac{4}{4}, \frac{6}{8}, \dots\right)$;
  - A *Metronomic-indication.*

  Hence, $\mathrm{Inf}_{ij}^{[r]}$ has the following general form:

  $$\mathrm{Inf}_{ij}^{[r]} = (\mathit{Tempo},\ \mathit{Meter},\ \mathit{Metronome}, \dots).$$

- $\mathit{Notes}_{ij}^{[r]}$ represents a note-term for the ith instrument at the jth position. In order to determine $\mathit{Notes}_{ij}^{[r]}$, we first transform the measure $\mathbf{M}^{[\mathbf{r}]}$ into a (musically) equivalent measure $\left(\mathbf{M}^{[\mathbf{r}]}\right)'$ that only contains notes or pauses whose value is the atomic value $\alpha^{\mathbf{M}}$.

EXAMPLE 1.2. Consider the following measure $\mathbf{M}$:

Clearly, $\mathbf{M}$ is musically equivalent to the following measure $\mathbf{M}'$:

where:

$$\frac{x}{y} = \frac{2}{4}; \quad \alpha^{\mathrm{M}} = \frac{1}{ay} = \frac{1}{2 \cdot 4} = \frac{1}{8}; \qquad n = xa = 2 \cdot 2 = 4.$$

We say that a measure $\mathbf{M}$ is in *normal form* when all the notes/pauses occurring in $\mathbf{M}$ have the same value, which corresponds to the atomic value for $\mathbf{M}$. Clearly, any measure can be normalized.

A normalized measure $\mathbf{M}$ whose length is n contains n symbolic notations for notes, chords or pauses, having all the same value $\alpha^{\mathbf{M}}$. Each notation depends on an initial information (concerning the *Clef*, the *Accidentals*, ...), that is normally written at the beginning of $\mathbf{M}$. This uniquely determines a note-term whose value $\alpha^{\mathbf{M}}$ (corresponding to the atomic value of the measure) is constant. On this basis, $Notes_{ij}^{[r]}$ can be identified with the note-term ${}^{\alpha^{\mathbf{M}}[\frac{x}{y}]}\underline{T}$ corresponding to the note-indication that appears at the jth position for the ith instrument in the normalized measure $(\mathbf{M}^{[\mathbf{r}]})'$.

- $Perform_{ij}^{[r]}$ represents the *performance-prescriptions* that concern $Notes_{ij}^{[r]}$. Besides the note-terms, a (normalized) measure $\mathbf{M}$ generally contains dynamic indications (like *p*, *pp*, *f*, *mf*, *dolce*, *espressivo*, *crescendo*, ...), and other indications that possibly concern the sound-emission (like *staccato*, *legato*, *pizzicato*, *arco*, ...). We call such indications *performance-prescriptions*. Generally such prescriptions may concern the whole measure and possibly also a sequence of measures. Of course, one can always distribute a performance-prescription over single note-terms. We say that a normalized measure $\mathbf{M}$ is *completely normalized* when all the performance-prescriptions occurring in $\mathbf{M}$ refer to single note-terms. On this basis, $Perform_{ij}^{[r]}$ is identified with the performance-prescription that concerns the note-indication appearing at the jth position for the ith instrument in the completely normalized conventional measure corresponding to $\mathbf{M}^{[\mathbf{r}]}$.

On this basis, the *formal representation* $\mathbf{FS}$ of the score $\mathbf{S}$ is identified with the sequence

$$\left(\mathbf{FM}^{[\mathbf{1}]}, \ldots \mathbf{FM}^{[\mathbf{s}]}\right),$$

where any $\mathbf{FM}^{[\mathbf{r}]}$ $(1 \leq r \leq s)$ is a formal representation of the corresponding measure $\mathbf{M}^{[\mathbf{r}]}$ of $\mathbf{S}$.

How to describe the language of a formal score? As is well known, the language of a (formalized) scientific theory is based on a set of primitive symbols, that is also called the *alphabet* of the *object-language* of the theory in question. Any finite sequence of symbols of the alphabet represents a *word* of the language. At the same time, the *syntactical rules* of the language determine which words are *well-formed expressions*, that may receive a *meaning* in the framework of a semantic interpretation. Important examples of well-formed expressions are represented by *individual terms* (which denote objects) and *sentences* (which may be either *true* or *false*).

Is it reasonable to describe the language of a formal score in a similar way? How to determine the alphabet and the syntactical rules that give rise to well-formed score-expressions? The alphabet can be naturally identified with the set of all basic symbolic notations that may occur in the score. This set should contain at least the following syntactical objects:

- the basic note-names and the pause-symbols;
- the names of the meters and of the possible note-values;
- all the symbolic notations that may occur in the parts of a score-atom that we have conventionally indicated by $\mathrm{Inf}_{ij}^{[r]}$ and by $Perform_{ij}^{[r]}$ (which include dynamic and performance-indications);
- the proper names of the instruments (*violin*, *viola* and so on).

The syntactical rules should determine at least the following:

- how to construct *valued note-names*, *complete note-names* and *chord-names*;
- how to construct *well-formed formal measures* (according to the abstract indications sketched above).

In a sense, the role of formal measures and of sequences of formal measures in a formal score might be compared with the role played by *sentences* in scientific formal languages. As is well known, sentences can be combined by means of *logical connectives* (like *not*, *and*, *or*) giving rise to new sentences. In a similar way, formal measures can be combined, giving rise to syntactical structures that are usually called *musical phrases*. We will see, however, that some basic syntactical relations seem to be much more complicated in music than in science.

2. Intervals, phrases, themes

A fundamental role in the structure of musical compositions is played by *intervals*. We will try and analyze this notion in a syntactical framework.

Consider again the set of all basic note-names in the *natural order*:

$$BNames^{Notes} = \left\{\nu^{[1]}, \nu^{[2]}, \ldots, \nu^{[n]}, \ldots, \nu^{[7\cdot 12]}\right\}.$$

One can define on this set a natural *tone-distance* function. As is well known (in the framework of the tempered system), the interval between the pitches denoted by $\nu^{(i_j)}$ and $\nu^{(i_{j+1})}$ (with $1 \leq j \leq 11$) is one semi-tone. Hence, the interval between the pitches denoted by two names $\nu^{(i_j)}$ and $\nu^{(h_k)}$ will correspond to a given number n of semi-tones.

DEFINITION 2.1. (*The tone-distance between two basic note-names*)
Let $\nu^{[m]}$ and $\nu^{[n]}$ be the mth and the nth basic note-name, respectively. The *tone-distance* $d(\nu^{[m]}, \nu^{[n]})$ is defined as follows:

$$d(\nu^{[m]}, \nu^{[n]}) = |n - m|$$

(where $|n - m|$ is the absolute value of $n - m$).

Hence the values assumed by the tone-distance function d belong to the set $\{0, 1, \ldots, 7{\cdot}12{-}1\}$. As an example, consider the tone-distance between $\mathbf{c_3}$ and $\mathbf{d_3^\sharp}$, which corresponds to a *minor third interval*. As expected, by applying our definition, we obtain:

$$d(\mathbf{c_3}, \mathbf{d_3^\sharp}) = d(\nu^{[25]}, \nu^{[28]}) = 3\,.$$

On can easily show that d is a "good" distance-relation (satisfying the standard mathematical conditions). Consequently, the pair $(BNames^{Notes}, d)$ turns out to be a standard *metric space*.

On this basis, any number that is a value of the tone-distance function applied to two basic note-names represents a (*syntactical*) *interval*. Suppose that $d(\nu^{[m]}, \nu^{[n]}) = c$. When $m < n$, we also briefly write: $\nu^{[n]} = \nu^{[m]} + c$. When $m > n$, we write: $\nu^{[n]} = \nu^{[m]} - c$. In such cases, we will say that $+c$ represents an *ascending interval*, while $-c$ represents a *descending interval*.[3] Of course, the tone-distance function can be applied to (complete) note-names as well.

We will now discuss the notion of *musical phrase* (which has a very important role in all interpretations of scores). From a syntactical point of view, musical phrases can be described as special *linguistic objects* that represent fragments of a formal score. Our formal definition will take into account the following characteristic properties of concrete musical phrases (briefly *phrases*):

- A phrase generally consists of a (small) number of measures.
- A phrase does not necessarily begin at the beginning of a measure and does not necessarily end at the end of a measure.
- Phrases are generally *transversal* with respect to the instruments of the score: an instrument may begin a given phrase, while other instruments will continue it.
- A phrase does not generally concern *all* the score-atoms contained in the measures where parts of the phrase in question appear. Hence, phrases can be formally represented as pieces of score with "holes", which will be also called *empty score-atoms* (and indicated by the symbol $\clubsuit$).[4]

[3] The correspondence between these intervals and the standard interval-terminology is straightforward. For instance, the ascending (descending) octave-interval is represented by the number $+12$ (-12); the ascending (descending) fifth-interval is represented by the number $+7$ (-7), and so on.

[4] Needless to observe, empty score-atoms (which represent "holes") should not be confused with pause-symbols.

Accordingly, a phrase will be identified with a special sequence of score-columns, that may contain empty score-atoms. We will require the following natural condition: all the columns that constitute a phrase shall contain at least one non-empty score atom.

In order to define the notion of phrase, we will first introduce the concepts of *score-segment*, of *subsegment* and of *full subsegment* of a score-segment. Consider a formal score **FS** consisting of the following sequence of s *formal measures*:

$$\left(\mathbf{FM}^{[1]}, \ldots, \mathbf{FM}^{[s]}\right).$$

Definition 2.2. (*Score segment*)
A *score segment* of **FS** is a subsequence

$$\mathrm{Seg} = \left(\mathbf{FM}^{[r]}, \mathbf{FM}^{[r+1]}, \ldots, \mathbf{FM}^{[r+u]}\right)$$

of the sequence of the formal measures that constitute **FS** (where $1 \leq r \leq s$ and $u \leq s - r$).

Apparently, a score-segment always begins at the beginning of a measure and ends at the end of a measure.

Definition 2.3. (*Subsegment*)
A *subsegment* of a score-segment

$$\mathrm{Seg} = \left(\mathbf{FM}^{[r]}, \mathbf{FM}^{[r+1]}, \ldots, \mathbf{FM}^{[r+u]}\right)$$

is a sequence

$$\mathrm{Seg}' = \left(\mathbf{FM}'^{[r]}, \mathbf{FM}'^{[r+1]}, \ldots, \mathbf{FM}'^{[r+u]}\right),$$

that is obtained by substituting in Seg some score-atoms with the empty score-atom $\clubsuit$.

In Section 1 we have seen that any formal measure $\mathbf{FM}^{[r]}$ can be represented as a sequence of columns:

$$\begin{pmatrix} A_{11}^{[r]} \ldots A_{1n}^{[r]} \\ A_{21}^{[r]} \ldots A_{2n}^{[r]} \\ \vdots \quad\quad \vdots \\ A_{m1}^{[r]} \ldots A_{mn}^{[r]} \end{pmatrix}$$

where each $A_{ij}^{[r]}$ is a *score-atom*, while each row $A_{i1}^{[r]} \ldots A_{in}^{[r]}$ is associated to the ith instrument of the score. For the sake of simplicity, we can briefly write:

$$\mathbf{FM}^{[r]} = \left(\chi_1^{[r]}, \ldots, \chi_n^{[r]}\right),$$

where $\chi_1^{[r]}$ represents the first column of $\mathbf{FM}^{[r]}$, $\chi_2^{[r]}$ represents the second column of $\mathbf{FM}^{[r]}$, and so on.

In a similar way, also the segments of a formal score can be naturally represented as sequences of columns. Consider the following segment of $\mathbf{FS}$:

$$\mathrm{Seg} = \left(\mathbf{FM^{[r]}}, \mathbf{FM^{[r+1]}}, \ldots, \mathbf{FM^{[r+u]}}\right),$$

where $\mathbf{FM^{[r]}} = \left(\chi_1^{[r]}, \ldots, \chi_{n_r}^{[r]}\right)$, $\mathbf{FM^{[r+1]}} = \left(\chi_1^{[r+1]}, \ldots, \chi_{n_{r+1}}^{[r+1]}\right)$, and so on.

We can put:

$$\mathrm{Seg} = \left(\chi_1^{[r]}, \ldots, \chi_{n_r}^{[r]}, \chi_1^{[r+1]}, \ldots, \chi_{n_{r+1}}^{[r+1]}, \ldots, \chi_1^{[r+u]}, \ldots, \chi_{n_{r+u}}^{[r+u]}\right).$$

Following a more usual way of representing musical measures, we will also write:

$$\mathrm{Seg} = \left|\chi_1^{[r]} \ldots, \chi_{n_r}^{[r]}\right|\chi_1^{[r+1]}, \ldots, \chi_{n_{r+1}}^{[r+1]}\right| \ldots \left|\chi_1^{[r+u]}, \ldots, \chi_{n_{r+u}}^{[r+u]}\right|,$$

where $|$ corresponds to the standard measure-separation symbol.

Consider now a subsegment of a segment Seg:

$$\mathrm{Seg}' = \left(\mathbf{FM'^{[r]}}, \mathbf{FM'^{[r+1]}}, \ldots, \mathbf{FM'^{[r+u]}}\right).$$

Of course, also Seg' will correspond to an appropriate sequence of columns:

$$\mathrm{Seg}' = \left(\chi_1'^{[r]}, \ldots, \chi_{n_r}'^{[r]}, \chi_1'^{[r+1]}, \ldots, \chi_{n_{r+1}}'^{[r+1]}, \ldots \chi_1'^{[r+u]}, \ldots, \chi_{n_{r+u}}'^{[r+u]}\right),$$

which can be also written as follows:

$$\mathrm{Seg}' = \left|\chi_1'^{[r]}, \ldots, \chi_{n_r}'^{[r]}\right|\chi_1'^{[r+1]}, \ldots, \chi_{n_{r+1}}'^{[r+1]}\right| \ldots \left|\chi_1'^{[r+u]}, \ldots, \chi_{n_{r+u}}'^{[r+u]}\right|.$$

The columns of Seg' that only contain empty atoms are called *empty columns*. By *beginning* of Seg' we mean the first column of Seg' that contains a non-empty atom; while the last column of Seg' that contains a non-empty atom is called the *end* of Seg'.

Definition 2.4. (*Full subsegment of a score-segment*)
A *full subsegment* of a score-segment is a subsegment where all columns between the beginning and the end are non-empty.

Definition 2.5. (*Phrase*)
A *phrase* of a formal score $\mathbf{FS}$ is a full subsegment Phr of a score-segment of $\mathbf{FS}$.

Definition 2.6. (*Monodic phrase*)
A *monodic phrase* is a phrase Phr that satisfies the following condition: each column of Phr contains exactly one non-empty atom, whose note-term is either a note-name or a pause-symbol.

In other words, monodic phrases do not contain any chord-name. Notice that, according to our definition, the *melody* expressed by a monodic phrase may be realized by different instruments.

EXAMPLE 2.1. The first measures of the violins and of the violas of Beethoven's 9th symphony give rise to a monodic phrase that satisfies our formal definition. Here, we suppose that the corresponding measures of all other instruments consist of "holes" of the score.

Needless to stress, our definition of musical phrase is quite "liberal", since it only assumes some minimal syntactical conditions. The choice of the *significant* phrases arises at the semantic level and represents the basic step for any *interpretation* and *performance* of a score.

Let us now turn to the notion of *theme*. As is well known, according to the usual terminology, a *monodic theme* represents a musical structure that is not bound to a particular choice of notes or tonalities. Generally, themes appear in different forms in the framework of a musical composition: they are exemplified by different note-sets in different tonalities, and give rise to a number of variations. We need only think of the *sonata-form*, which is essentially based on *developments* and *variations* of two fundamental themes. As a consequence, themes cannot be identified with fragments of formal scores, where notes are always indicated by proper names. In a syntactical framework, a *theme* can be regarded as an abstract structure underlying a number of different musical phrases that turn out to satisfy a special *isomorphism-relation*.

Let us first introduce the notion of *note-component* of a formal measure and of a phrase. As we already know, any score-atom of a formal measure $\mathbf{FM}^{[\mathbf{r}]}$ has the following form:

$$A_{ij}^{[r]} = \left(\mathrm{Inf}_{ij}^{[r]}, \, \mathit{Notes}_{ij}^{[r]}, \, \mathit{Perform}_{ij}^{[r]}\right).$$

By neglecting $\mathrm{Inf}_{ij}^{[r]}$ and $\mathit{Perform}_{ij}^{[r]}$, one can define the *note-component* of the atom $A_{ij}^{[r]}$ as follows:

$$\mathit{Note}\big(A_{ij}^{[r]}\big) := \mathit{Notes}_{ij}^{[r]}.$$

On this basis, $\mathit{Note}\big(\mathbf{FM}^{[\mathbf{r}]}\big)$ (the *note-component* of the formal measure $\mathbf{FM}^{[\mathbf{r}]}$) can be represented in the following way:

$$\begin{pmatrix} \mathit{Note}\big(A_{11}^{[r]}\big) \ldots \mathit{Note}\big(A_{1n}^{[r]}\big) \\ \mathit{Note}\big(A_{21}^{[r]}\big) \ldots \mathit{Note}\big(A_{2n}^{[r]}\big) \\ \vdots \qquad\qquad \vdots \\ \mathit{Note}\big(A_{m1}^{[r]}\big) \ldots \mathit{Note}\big(A_{mn}^{[r]}\big) \end{pmatrix}.$$

Apparently, $Note(\mathbf{FM}^{[\mathbf{r}]})$ is a matrix-like configuration whose elements are note-terms.

If we represent our measure as a column sequence

$$\mathbf{FM}^{[\mathbf{r}]} = \left(\chi_1^{[r]}, \ldots, \chi_n^{[r]}\right),$$

we will have:

$$Note(\mathbf{FM}^{[\mathbf{r}]}) = \left(Note(\chi_1^{[r]}), \ldots, Note(\chi_n^{[r]})\right),$$

where $Note(\chi_1^{[r]}), \ldots, Note(\chi_n^{[r]})$ are defined in the expected way.

In a similar way, we can define the *note-component* of a whole phrase. Let

$$\text{Phr} = \left(\mathbf{FM}^{[\mathbf{r}]}, \ldots, \mathbf{FM}^{[\mathbf{r+u}]}\right).$$

We can naturally put:

$$Note(\text{Phr}) := \left(Note(\mathbf{FM}^{[\mathbf{r}]}), \ldots, Note(\mathbf{FM}^{[\mathbf{r+u}]})\right).$$

Suppose now that Phr represents a monodic phrase. In such a case, $Note(\text{Phr})$ can be easily *linearized*, giving rise to a sequence of note-terms:

$$\mathfrak{N} = \left({}^{v_1}\underline{\tau_1}, {}^{v_2}\underline{\tau_2}, \ldots, {}^{v_t}\underline{\tau_t}\right),$$

where each ${}^{v_i}\underline{\tau_i}$ is either a note-name or a pause-symbol.

Consider the note-components of two monodic phrases (in a linearized form):

$$\mathfrak{N} = \left({}^{v_1}\underline{\tau_1}, {}^{v_2}\underline{\tau_2}, \ldots, {}^{v_t}\underline{\tau_t}\right) \quad \text{and} \quad \mathfrak{N}' = \left({}^{v_1'}\underline{\tau_1'}, {}^{v_2'}\underline{\tau_2'}, \ldots, {}^{v_{t'}'}\underline{\tau_{t'}'}\right).$$

An isomorphism-relation between $\mathfrak{N}$ and $\mathfrak{N}'$ can be defined in a natural way.

DEFINITION 2.7. (*Isomorphism*)
$\mathfrak{N}$ and $\mathfrak{N}'$ are *isomorphic* ($\mathfrak{N} \equiv \mathfrak{N}'$) iff the following conditions are satisfied (for any i such that $1 \leq i \leq t$):

1) $t = t'$ (in other words, the two monodic phrases contain the same number of note-terms).
2) $v_i = v_i'$ (in other words, any two corresponding note-terms have the same value).
3) ${}^{v_i}\underline{\tau_i}$ is an onset-note (a tied note) iff ${}^{v_{i'}'}\underline{\tau_{i'}'}$ is an onset-note (a tied note).
4) ${}^{v_i}\underline{\tau_i}$ is a pause-symbol iff ${}^{v_{i'}'}\underline{\tau_{i'}'}$ is a pause-symbol. If ${}^{v_i}\underline{\tau_i}$ is a pause symbol, then ${}^{v_i}\underline{\tau_i} = {}^{v_{i'}'}\underline{\tau_{i'}'}$.
5) If ${}^{v_i}\underline{\tau_i}$ and ${}^{v_j}\underline{\tau_j}$ are note-names, then $d\left({}^{v_i}\underline{\tau_i}, {}^{v_j}\underline{\tau_j}\right) = d\left({}^{v_i'}\underline{\tau_i'}, {}^{v_j'}\underline{\tau_j'}\right)$. Moreover, ${}^{v_i}\underline{\tau_i} \preceq {}^{v_j}\underline{\tau_j}$ iff ${}^{v_i'}\underline{\tau_i'} \preceq {}^{v_j'}\underline{\tau_j'}$.

Finally, we say that two monodic phrases (Phr_1 and Phr_2) are *isomorphic* ($\mathrm{Phr}_1 \equiv \mathrm{Phr}_2$) when their note-components are isomorphic. From an intuitive point of view, two isomorphic monodic phrases "express the same melody". Apparently, $\equiv$ is an *equivalence relation* (reflexive, symmetric and transitive). On this basis, *by abstraction*, one can define a *monodic theme* as an equivalence class of isomorphic monodic phrases (in the framework of a given musical language $\mathcal{L}$). In other words, a *theme* Th is identified with a class $[\mathrm{Phr}]_{\equiv}$ that contains all the phrases (expressed in the language $\mathcal{L}$) that are isomorphic to a given monodic phrase Phr. When a phrase Phr belongs to the equivalence class that represents a theme Th, following the usual musical terminology, we will also say that Phr *expresses* the theme Th.

Themes can be also dealt with in a more direct way, as special examples of numerical structures. Consider again the note-component of a monodic phrase:

$$\mathfrak{N} = \left({}^{v_1}\underline{\tau_1}, {}^{v_2}\underline{\tau_2}, \ldots, {}^{v_t}\underline{\tau_t}\right).$$

Such a configuration can be transformed, by means of a map f, into a sequence of quadruples of numbers, giving rise to a natural musical interpretation. The basic intuitive idea can be sketched as follows. For any ${}^{v_i}\underline{\tau_i}$ (occurring in $\mathfrak{N}$), $f\left({}^{v_i}\underline{\tau_i}\right)$ is a sequence (x_i, y_i, z_i, w_i) of four (rational) numbers such that:

- x_i informs us about the nature of the corresponding term (onset-note or tied note or pause-symbol);
- y_i informs us about the tone-distance with respect to the previous note;
- z_i informs us about the value of the corresponding note;
- w_i informs us about the meter of the formal measure where the corresponding note occurs.

The precise definition of $f\left({}^{\iota_i}\underline{\tau_i}\right) = (x_i, y_i, z_i, w_i)$ is the following:

$$x_i = \begin{cases} 0 & \text{if } \underline{\tau_i} \text{ is an onset-note;} \\ 1 & \text{if } \underline{\tau_i} \text{ is a tied note;} \\ 100 & \text{if } \underline{\tau_i} \text{ is a pause-symbol.} \end{cases}$$

$$y_i = \begin{cases} 100 & \text{if } \underline{\tau_i} \text{ is a pause-symbol;} \\ 1000 & \text{if } \underline{\tau_i} \text{ is the first note-name occurring in } \mathfrak{N}; \\ +d(\tau_i, p(\tau_i)) & \text{if } p(\tau_i) \text{ is the last note-name} \\ & \text{that occurs before } \tau_i \text{ in } \mathfrak{N} \text{ and } p(\tau_i) \preceq \tau_i; \\ -d(\tau_i, p(\tau_i)) & \text{if } p(\tau_i) \text{ is the last note-name} \\ & \text{that occurs before } \tau_i \text{ in } \mathfrak{N} \text{ and } \tau_i \preceq p(\tau_i). \end{cases}$$

$$z_i = \frac{b}{ay} \quad \text{if } v_i = \frac{b}{ay}\left[\frac{x}{y}\right];$$
$$w_i = \frac{x}{y} \quad \text{if } v_i = \frac{b}{ay}\left[\frac{x}{y}\right].$$

Clearly, the choice of the values 0, 1, 100 for x_i and of the values 100, 1000 for y_i is purely conventional. All the other values for y_i, z_i and w_i have, instead, a genuine musical meaning.

Finally we put:

$$f(\mathfrak{N}) := \left(f\left({}^{v_1}\underline{\tau_1}\right), f\left({}^{v_2}\underline{\tau_2}\right), \ldots, f\left({}^{v_t}\underline{\tau_t}\right)\right).$$

Unlike $\mathfrak{N}$ (which contains particular note-names), $f(\mathfrak{N})$ is compatible with different possible choices of the first note to be performed. On this basis, any monodic theme can be represented as a numerical structure Th^N such that $\mathrm{Th}^N = f(\mathfrak{N})$, where $\mathfrak{N}$ is the note-component of a monodic phrase Phr of the language.

So far, we have only considered the relation between themes and monodic phrases. However, it often happens that themes are expressed by more complex, non-monodic phrases. As is well known, a number of intriguing musical effects depend on the fact that a theme may be, in a sense, "hidden" in some complex musical phrases. A formal analysis of situations of this kind naturally leads to the following definition:

DEFINITION 2.9. A phrase Phr *expresses* a theme Th, when Th is the theme determined by a monodic subphrase of Phr (i.e. a monodic phrase that is a subsegment of Phr).

3. Scales and tonalities

Tonal music is essentially concerned with scales and tonalities. We will try and discuss these concepts in our syntactical framework, by using some elementary set-theoretic tools. Consider the set $BNames_i^{Notes}$ of all basic note-names of the ith octave. For any octave-labels i and j, we will say that $\mathbf{c_i}$ represents a *twin-note* of $\mathbf{c_j}$, that $\mathbf{c_i^\sharp}$ represents a *twin-note* of $\mathbf{c_j^\sharp}$, and so on. More precisely, the *twin-relation* can be defined as follows.

DEFINITION 3.1. (*Twin-relation*)
The basic note-name $\nu^{[h]}$ is a *twin-note* of the basic note-name $\nu^{[k]}$ iff $d(\nu^{[h]}, \nu^{[k]}) = n12$, where $0 \le n < 7$.

In other words, the interval between $\nu^{[h]}$ and $\nu^{[k]}$ is a multiple of an octave-interval. One can easily realize that the twin-relation is an equivalence relation.

We will now introduce the notion of *scale-instance*, which represents a note-sequence (ordered according to the the natural order) where the distance between any two elements is less than 12 (the octave-interval).

DEFINITION 3.2. (*Scale-instance*)
A *scale-instance* is a basic-note name sequence $\mathcal{S} = (\nu_1, \ldots, \nu_t)$ that satisfies the following conditions:

1) $4 \leq t \leq 12$;[5]
2) $\nu_1, \ldots, \nu_t$ are basic note-names such that $\nu_1 \prec \nu_2 \prec \cdots \prec \nu_t$;
4) $d(\nu_1, \nu_t) < 12$.

Following the usual terminology, we will say that ν_1 is the *first degree* of the scale, ν_2 is the *second degree* of the scale, and so on.

An isomorphism-relation can be naturally defined on the set of all scale-instances.

DEFINITION 3.3. (*Scale-isomorphism*)
Two scale-instances $(\nu_1, \ldots, \nu_t)$ and $(\nu'_1, \ldots, \nu'_t)$ are called *isomorphic* ($(\nu_1, \ldots, \nu_t) \approx (\nu'_1, \ldots, \nu'_t)$) iff

1) $t = t'$;
2) for any k (with $1 \leq k \leq t$) ν_k and ν'_k are twin-notes.

From an intuitive point of view, two isomorphic scale-instances represent "the same scale" realized in the framework of two (possibly) different segments of the note-system. Apparently, $\approx$ is an equivalence-relation. By abstraction, we can now define the notion of *syntactical scale*.

DEFINITION 3.4. (*Syntactical scale*)
A *syntactical scale* is an equivalence class of isomorphic scale-instances.

In other words, a syntactical scale is a class that contains precisely all the scale-instances that are isomorphic to a given scale-instance. Hence, any syntactical scale can be represented in the following form:

$$\mathcal{SC} = \left[(\nu_1, \ldots, \nu_t)\right]_{\approx},$$

where $(\nu_1, \ldots, \nu_t)$ is a scale-instance. We will also say that $(\nu_1, \ldots, \nu_t)$ is an instance of $\mathcal{SC}$.

[5]The upper bound (12) is obviously determined by the choice of the tempered system. The choice of the lower bound (4) seems to be reasonable according to the historical musical tradition.

EXAMPLE. The G-major scale can be represented as follows:

$$G^{>} = \left[(\mathbf{g_3}, \mathbf{a_3}, \mathbf{b_3}, \mathbf{c_4}, \mathbf{d_4}, \mathbf{e_4}, \mathbf{f_4^\sharp})\right]_{\approx}.$$

Any scale $\mathcal{SC} = \left[(\nu_1, \ldots, \nu_t)\right]_{\approx}$ naturally gives rise to a corresponding set (which is a subset of $BNames^{Notes}$). This set, indicated by $\mathrm{Set}^{\mathcal{SC}}$ and called the *scale-set* of $\mathcal{SC}$, is defined as follows:

$$\mathrm{Set}^{\mathcal{SC}} = \bigcup \{\{\nu_1, \ldots, \nu_t\} : (\nu_1, \ldots, \nu_t) \in \mathcal{SC}\}.$$

In other words, $\mathrm{Set}^{\mathcal{SC}}$ contains precisely all the basic note-names that occur in an instance of the scale $\mathcal{SC}$.

On this basis, one can give a natural definition for a syntactical notion of *tonality*.

DEFINITION 3.5. (*Syntactical tonality*)
A *syntactical tonality* is a subset $\mathcal{T}$ of $BNames^{Notes}$ that is the scale-set of a given scale.

Notice that, according to our definition, the set of all basic note-names is a tonality, determined by the twelve-note scale. In the framework of the tonal tempered system, the 12 major tonalities and the corresponding 12 minor tonalities give rise to 24 different syntactical tonalities that satisfy our definition.

For concrete compositions of the tonal musical tradition, a crucial and intriguing question concerns the relation that holds between a given fragment of a score and "its tonality". Such a relation often appears *ambiguous* and *undecidable*. Needless to recall how such tonal ambiguities play a deep expressive role in a number of musical works. How to describe formally relations of this kind?

An useful tool might be provided by some elementary fuzzy set-theoretic methods, which have often been used to deal with uncertain and ambiguous conceptual situations. Consider a formal score $\mathbf{FS}$. Given a tonality $\mathcal{T}$, for any basic note-name ν occurring in $\mathbf{FS}$, it is well determined whether ν belongs to $\mathcal{T}$. Hence, one is dealing with a classical dichotomic membership relation. What about musical phrases? Let us first refer to a possible *sharp* definition (which might appear *prima facie* reasonable):

> *A phrase* Phr *belongs to a tonality* $\mathcal{T}$ *iff all the basic note-names occurring in* Phr *belong to* $\mathcal{T}$.

Such a definition turns out to describe an extreme case of tonal stability. However, as is well known, tonal stability is generally boring! In concrete cases, a number of "beautiful" musical effects seem to be essentially based on tonal ambiguities. One should render, formally, the following intuitive idea: a phrase Phr is a subset of a tonality $\mathcal{T}$, "with some possible exceptions". How shall we *count* the number of "exceptions" in the case of scores? The number of notes that are outside a given tonality does not seem significant, because this number obviously depends on the instrumental complexity of the score. More interestingly, we can refer to the number of columns that represent "exceptions" with respect to a given tonality.

Consider a phrase represented as a sequence of n (non-empty) columns:

$$\mathrm{Phr} = (\chi_1, \ldots, \chi_n)\,.$$

Let $\mathcal{T}$ be a tonality and let χ_i be a column of Phr. We say that χ_i belongs to the tonality $\mathcal{T}$ iff all basic note-names occurring in χ_i belong to $\mathcal{T}$. We can now define the following *fuzzy relation*:

DEFINITION 3.6. (*Fuzzy tonality membership relation*)
A phrase $\mathrm{Phr} = (\chi_1, \ldots, \chi_n)$ belongs to the tonality $\mathcal{T}$ with determinacy-degree r ($\mathrm{Phr} \in_r \mathcal{T}$) iff $r = \frac{n-m}{n}$, where m is the number of columns of Phr that do not belong to $\mathcal{T}$.

One immediately obtains:

- $\mathrm{Phr} \in_1 \mathcal{T}$ iff all columns of Phr belong to $\mathcal{T}$;
- $\mathrm{Phr} \in_0 \mathcal{T}$ iff no column of Phr belongs to $\mathcal{T}$;
- r tends to 1 (0), when the number m of columns that do not belong to $\mathcal{T}$ decreases (increases).

In many interesting situations, a musical phrase belongs to different tonalities with (possibly) different determinacy-degrees. We need only think of the characteristic tonal ambiguity that arises when the third degree of the scale is missing in a given phrase. Hence the phrase may belong (with determinacy degree 1) to a given major tonality and to the corresponding minor tonality at the same time. As expected, all *modulations* admit a natural description in terms of this fuzzy tonality membership relation.

Other important concepts, which are concerned with harmonic and rhythmical correlations, can be analyzed in a similar way. What about the relation between a score and the class of all its (actual and possible) interpretations? This relation seems to be similar to the well-known relation that holds between the *formal system* of a given scientific theory (say, arithmetics or classical mechanics) and the class of all *semantic interpretations* (or *models*) of the theory in question. In both cases, the syntactical component (either the score of a musical composition or the formal system of a scientific theory) turns out to be compatible with a number of different interpretations. As happens in the case of scientific theories, one could claim that musical compositions can be represented as special *pairs*, consisting of a *score* and of a class of *possible interpretations*. Of course, the notion of *musical interpretation* turns out to be much more complex than the corresponding notion of semantic interpretation of a formal system. We will discuss this question elsewhere.

Finally let us ask: what might be the interest of formalizing musical languages? As is well known, in the case of scientific theories, formalization is not aimed at providing some *perfect* languages that should substitute the "old rough" languages used by the scientific community. Formal languages are heavy and unreadable (if not accompanied by some translation-rules into the natural language). In a similar way, any attempt to substitute a traditional score **S** with its formal version **FS** would be absolutely unreasonable! In both cases (science and music), the basic aim of formalizing languages is bringing into light some deep linguistic structures that represent significant *invariants* in a variety of different kinds of expressions and notation-systems. Identifying the elements that have a fundamental role in our information encoding process represents the first step for any successful semantic analysis.

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