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ON SOME UNARY ALGEBRAS AND THEIR SUBALGEBRA LATTICES

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ABSTRACT. We first define lattices, called normal, which are uniquely represented by directed graphs. Secondly, we describe all unary algebras (called normal, too) such that their subalgebra lattices are normal. Next, we characterize pairs $\langle \mathbf{A}, \mathbf{L} \rangle$ such that the subalgebra lattice of $\mathbf{A}$ is isomorphic to $\mathbf{L}$, where $\mathbf{A}$ is a normal unary algebra and $\mathbf{L}$ is a normal lattice. Further, we describe pairs of normal unary algebras with isomorphic subalgebra lattices. We use these results in the second part of the paper to find necessary and sufficient conditions for pairs of lattices to be isomorphic to a pair of the weak and strong subalgebra lattices of one normal unary algebra.

1

In [10] we have investigated unary algebras and their subalgebra lattices. To this purpose we used connections between partial unary algebras and graphs given in [9]. Moreover, we did not restrict our attention to total algebras only, and we consider the more general case of partial algebras. This approach to unary algebras by partiality, and also this graph-algebraic language turned out to be very useful in such investigations. Recall, we first characterized all the pairs $\langle \mathbf{A}, \mathbf{L} \rangle$ such that the strong subalgebra lattice of $\mathbf{A}$ is isomorphic to $\mathbf{L}$, where $\mathbf{A}$ is an unary algebra and $\mathbf{L}$ is a lattice. Secondly, necessary and sufficient conditions were found for pairs of unary algebras to have isomorphic strong subalgebra lattices.

Now we show that these problems have much simpler solutions for special kinds of lattices and unary algebras. We first define special lattices, called normal. Secondly, we prove that every normal lattice $\mathbf{L}$ can be uniquely represented

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by a digraph (directed graph); and it is "the least digraph" in the class of all digraphs without cycles having their strong subdigraph lattices isomorphic to $\mathbf{L}$. Thirdly, we describe all unary algebras such that their strong subalgebra lattices are normal; such algebras will be called normal, too. Further, we prove that with every normal unary algebra $\mathbf{A}$ we can associate a digraph in such a way that for a normal lattice $\mathbf{L}$, the strong subalgebra lattice of $\mathbf{A}$ is isomorphic to $\mathbf{L}$ iff the digraphs corresponding to $\mathbf{A}$ and $\mathbf{L}$ are isomorphic. We also show that normal unary algebras have isomorphic strong subalgebra lattices iff their digraphs are isomorphic.

These results will be applied in the second part of the paper to find necessary and sufficient conditions for pairs of lattices to be isomorphic to a pair of the weak and strong subalgebra lattices of one normal unary algebra.

We assume knowledge of basic concepts and results from the theory of partial and total algebras, and also from lattice theory (see e.g. [4], [5], [6] and [7]). We use notations and definitions from [10] (and also from [9]).

1.1. It is well known (see [7; Theorem 3.8.8]) that a complete lattice $\mathbf{L}$ is isomorphic to the strong subalgebra lattice of a (partial) unary algebra iff

($*$) $\mathbf{L}$ is algebraic and distributive,
($**$) each element of $\mathbf{L}$ is a join of completely join-irreducible elements.

Note that Theorem 3.8.8 concerns only total algebras, but it is also true for the partial case. Because, any partial unary algebra $\mathbf{A} = \langle A, (f^{\mathbf{A}})_{f \in A} \rangle$ can be modified to a total algebra $\overline{\mathbf{A}}$ with the same subalgebra lattice (more precisely, we extend each partial unary operation f of $\mathbf{A}$ to total, by adding all pairs $\langle a, a \rangle$, where $a \in A$ and $f^{\mathbf{A}}$ is not defined on a).

For a given lattice $\mathbf{L}$ satisfying ($*$) and ($**$), there are, in general, many different unary algebras (digraphs) having strong subalgebra (subdigraph) lattices isomorphic to $\mathbf{L}$ (see [10]). In this section we show that if a lattice $\mathbf{L}$ satisfies some additional condition, then $\mathbf{L}$ can be uniquely represented by a digraph $\mathbf{D}(\mathbf{L})$. Further, the lattice $\mathbf{S}_s(\mathbf{D}(\mathbf{L}))$ is isomorphic to $\mathbf{L}$. And $\mathbf{D}(\mathbf{L})$ is "the least digraph" in the class of all digraphs without non-trivial cycles and with the strong subdigraph lattices isomorphic to $\mathbf{L}$.

We start with the following auxiliary definition: A partially ordered set $\langle P, \leq_P \rangle$ is said to satisfy the *finite cover chain condition* or briefly FC, if for all $p, q \in P$, $p <_P q$ implies that there are $p_1, \dots, p_k \in P$ such that $p = p_1 \prec_P p_2 \prec_P \cdots \prec_P p_k = q$ (where $\prec_P$ is the covering relation, see e.g. [6]). (The infinite set $\{1, 2, 3, \dots\} \cup \{z\}$ ordered by the relation $\leq$ as follows: $1 \leq 2 \leq 3 \dots$ and $n \leq z$ for $n = 1, 2, 3, \dots$, is an example of a partially ordered set which does not satisfy FC.)

THEOREM 1.1.1. *Let $\langle P, \leq_P \rangle$ be a partially ordered set. Then the following conditions are equivalent:*

(a) *$\langle P, \leq_P \rangle$ is a partially ordered set with FC.*

(b) *$\langle P, \leq_P \rangle$ is a partially ordered set such that*

(i) *for each infinite chain $p_1 <_P p_2 <_P \dots$ and $q \in P$, if $p_i <_P q$ for $i = 1, 2, 3, \dots$, then there are elements $r_1, \dots, r_n$ such that $p_1 = r_1 \prec_P r_2 \prec_P \cdots \prec_P r_n = q$,*

(ii) *for each infinite chain $\dots <_P p_2 <_P p_1$ and $q \in P$, if $q <_P p_i$ for $i = 1, 2, 3, \dots$, then there are elements $r_1, \dots, r_n$ such that $q = r_1 \prec_P r_2 \prec_P \cdots \prec_P r_n = p_1$.*

In particular, if $\langle P, \leq_P \rangle$ satisfies both the ascending and the descending chain condition, then $\langle P, \leq_P \rangle$ satisfies FC.

P r o o f . Assume first, $\langle P, \leq_P \rangle$ satisfies both the ascending and the descending chain conditions. Then $\langle P, \leq_P \rangle$ has no infinite chains. By Zorn's lemma, for any $p, q \in P$, if $p < q$, then there is a maximal chain Q with the least element p and the greatest element q. Of course, Q is finite, let $Q = \{r_1, \dots, r_n\}$ and $p = r_1 \leq_P r_2 \leq_P \cdots \leq_P r_n = q$. Then the maximality of Q implies $r_i \prec_P r_{i+1}$ for $i = 1, \dots, n-1$.

The implication (a) $\Longrightarrow$ (b) is obvious, so it is sufficient to show (b) $\Longrightarrow$ (a). Assume that $\langle P, \leq_P \rangle$ does not satisfy FC. Then for some $p, q \in P$, $p <_P q$ and there is not a finite sequence of elements $r_1, \dots, r_n$ connecting p and q such that $r_i \prec_P r_{i+1}$. Hence we deduce also that $Q = \{r \in P : p \leq_P r \leq_P q\}$ has an infinite ascending chain or an infinite descending chain. These two facts imply that (b) does not hold. □

Let $\langle P, \leq_P \rangle$ be a partially ordered set. Then $\mathbf{D}^{\mathrm{pos}}(P, \leq_P)$ is the digraph with P as its vertex set, and $\{\langle p, q \rangle \in P \times P : q \prec_P p\}$ as its edge set. For example, for the real numbers $\mathbb{R}$ with the natural less-or-equal order, $\mathbf{D}^{\mathrm{pos}}(\mathbb{R})$ has not edges. But the facts below are easy to verify. (Recall that $\leq_{\mathbf{G}}$ is the natural, reflexive and transitive, relation on the vertex set $V^{\mathbf{G}}$ of a digraph $\mathbf{G}$ (see e.g. [12]), i.e. $v \leq_{\mathbf{G}} w$ iff there is a (directed) chain from w to v or $v = w$.

Recall also that an edge e is said to be an *isthmus* (see e.g. [3]) if e is regular (i.e. it is not a loop) and each chain from the initial vertex of e to the final vertex of e contains the edge e (or equivalently, e forms the only one path connecting these points, because a path does not contain the same vertex twice).)

PROPOSITION 1.1.2. *Let $\langle P, \leq_P \rangle$ and $\langle Q, \leq_Q \rangle$ be partially ordered sets with FC. Then*

(a) $\langle P, \leq_P \rangle \simeq \langle Q, \leq_Q \rangle \iff \mathbf{D}^{\mathrm{pos}}(P, \leq_P) \simeq \mathbf{D}^{\mathrm{pos}}(Q, \leq_Q)$.

(b) $\left\langle V^{\mathbf{D}^{\mathrm{pos}}(P, \leq_P)}, \leq_{\mathbf{D}^{\mathrm{pos}}(P, \leq_P)} \right\rangle \simeq \langle P, \leq_P \rangle$.

(c) $\mathbf{D}^{\mathrm{pos}}(P,\leq_P)$ *is a simple digraph without cycles, and each of its edges is an isthmus.*

(d) $\mathbf{D}^{\mathrm{pos}}\big(V^{\mathbf{D}^{\mathrm{pos}}(P,\leq_P)}, \leq_{\mathbf{D}^{\mathrm{pos}}(P,\leq_P)}\big) \simeq \mathbf{D}^{\mathrm{pos}}(P,\leq_P)$.

Note, (c) and the implication $\Longrightarrow$ in (a) hold for any partially ordered set (with or without FC).

Let $\mathbf{L} = \langle L, \leq_{\mathbf{L}}\rangle$ be a lattice and $\mathrm{Ir}(\mathbf{L})$ the set of all completely join-irreducible elements of $\mathbf{L}$. Then

$$\mathbf{D}(\mathbf{L}) = \mathbf{D}^{\mathrm{pos}}\big(\mathrm{Ir}(\mathbf{L}), \leq_{\mathbf{L}}\big).$$

Further, $\mathbf{L}$ is said to be a *normal lattice* if $\mathbf{L}$ satisfies $(*)$, $(**)$, and $\big\langle \mathrm{Ir}(\mathbf{L}), \leq_{\mathbf{L}}\big\rangle$ satisfies FC.

For instance, the set of real numbers $\mathbb{R}$ with the natural less-or-equal order is a complete lattice which has no completely join-irreducible elements. Thus $\mathbf{D}(\mathbb{R})$ is the empty digraph. However, every normal lattice $\mathbf{L}$ is indeed represented by its digraph.

THEOREM 1.1.3. *Let* $\mathbf{L}$ *and* $\mathbf{K}$ *be normal lattices. Then*

(a) $\mathbf{K} \simeq \mathbf{L} \iff \mathbf{D}(\mathbf{K}) \simeq \mathbf{D}(\mathbf{L})$.

(b) $\mathbf{S}_s\big(\mathbf{D}(\mathbf{L})\big) \simeq \mathbf{L}$.

(c) $\mathbf{D}(\mathbf{L})$ *is a simple digraph without cycles, and each of its edges is an isthmus.*

Proof. It is proved in [6; p. 83] that if a lattice $\mathbf{L}$ satisfies $(*)$ and $(**)$, then $\mathbf{L}$ is isomorphic to the lattice of all order-ideals of $\big\langle \mathrm{Ir}(\mathbf{L}), \leq_{\mathbf{L}}\big\rangle$ with set inclusion. Thus (because $\Longleftarrow$ is obvious)

$$\big\langle \mathrm{Ir}(\mathbf{K}), \leq_{\mathbf{K}}\big\rangle \simeq \big\langle \mathrm{Ir}(\mathbf{L}), \leq_{\mathbf{L}}\big\rangle \iff \mathbf{K} \simeq \mathbf{L}.$$

This equivalence and Proposition 1.1.2(a) complete the proof of (a).

Next, (b) follows from Proposition 1.1.2(b) and [10; Corollary 3.10]; and (c) is just a reformulation of Proposition 1.1.2(c). □

Now we show that for every normal lattice $\mathbf{L}$, its digraph $\mathbf{D}(\mathbf{L})$ is "the least digraph" in the class of all digraphs $\mathbf{G}$ such that $\mathbf{G}$ does not contain non-trivial cycles and $\mathbf{S}_s(\mathbf{G}) \simeq \mathbf{L}$. We first prove the following analogous result for partially ordered sets.

THEOREM 1.1.4. *Let* $\langle P, \leq_P\rangle$ *be a partially ordered set satisfying FC and let* $\mathbf{G}$ *be a digraph. Then the following conditions are equivalent:*

(a) $\langle V^{\mathbf{G}}, \leq_{\mathbf{G}}\rangle \simeq \langle P, \leq_P\rangle$.

(b) $\mathbf{D}^{\mathrm{pos}}(P,\leq_P)$ *is (up to isomorphism) a weak subdigraph of* $\mathbf{G}$ *having the same vertex set and for each regular edge* e *of* $\mathbf{G}$*, there is a chain in* $\mathbf{D}^{\mathrm{pos}}(P,\leq_P)$ *going from the initial vertex* $I_1^{\mathbf{G}}(e)$ *of* e *to the final vertex* $I_2^{\mathbf{G}}(e)$ *of* e.

Proof.

(a) $\implies$ (b). To simplify notation we assume that partially ordered sets $\langle V^{\mathbf{G}}, \leq_{\mathbf{G}} \rangle$ and $\langle P, \leq_P \rangle$ are equal. It is easy to see that for any vertices v, w of $\mathbf{G}$, if $w \prec_{\mathbf{G}} v$, then there is an edge from v to w. Hence we obtain that $\mathbf{D}^{\mathrm{pos}}(P, \leq_P)$ is a weak subdigraph of $\mathbf{G}$; with the same vertex set, of course.

Next, if e is a regular edge, then $I_2^{\mathbf{G}}(e) <_{\mathbf{G}} I_1^{\mathbf{G}}(e)$, so there are elements $p_1, \dots, p_n \in P$ such that

$$I_2^{\mathbf{G}}(e) = p_1 \prec_P p_2 \prec_P \dots \prec_P p_n = I_1^{\mathbf{G}}(e),$$

because $\langle P, \leq_P \rangle$ satisfies FC.

Obviously the sequence $\big(\langle p_n, p_{n-1} \rangle, \langle p_{n-1}, p_{n-2} \rangle, \dots, \langle p_2, p_1 \rangle\big)$ is the desired path.

(b) $\implies$ (a). Again, to simplify notation, we assume that $\mathbf{D}^{\mathrm{pos}}(P, \leq_P)$ is just a weak subdigraph of $\mathbf{G}$. Then by Proposition 1.1.2(b) we have that

$$v \leq_P w \implies v \leq_{\mathbf{G}} w .$$

On the other hand, let $v, w \in V^{\mathbf{G}}$ and $(e_1, \dots, e_n)$ be a chain in $\mathbf{G}$ going from w to v. Then for each $1 \leq i \leq n$, there is a chain in $\mathbf{D}^{\mathrm{pos}}(P, \leq_P)$ going from $I_1^{\mathbf{G}}(e_i)$ to $I_2^{\mathbf{G}}(e_i)$. Obviously the sum of these chains is a chain in $\mathbf{D}^{\mathrm{pos}}(P, \leq_P)$ going from w to v. Thus for each $v, w \in V^{\mathbf{G}}$,

$$v \leq_{\mathbf{G}} w \implies v \leq_P w .$$

□

By Theorem 1.1.4 and [10; Corollary 3.10] we obtain our result for normal lattices and digraphs.

Corollary 1.1.5. *Let* $\mathbf{L}$ *be a normal lattice and* $\mathbf{G}$ *a digraph. Then the following conditions are equivalent:*

(a) $\mathbf{S}_s(\mathbf{G}) \simeq \mathbf{L}$ *and* $C_n(\mathbf{G}) = \emptyset$.

(b) $\mathbf{D}(\mathbf{L})$ *and* $\mathbf{G}$ *satisfies* (b) *of Theorem* 1.1.4.

In the above result, and also in the rest of the paper, by $C_n(\mathbf{G})$ we denote the family of all non-trivial (i.e. with at least two different vertices) directed cycles of a digraph $\mathbf{G}$.

Note that if we want to omit in (a) the condition $C_n(\mathbf{G}) = \emptyset$, then in (b) it is necessary and sufficient to replace $\mathbf{G}$ by the quotient digraph $\mathbf{G}/\theta(\mathbf{G})$. (Recall that the quotient digraph with respect to an equivalence relation θ on $V^{\mathbf{G}}$ is obtained by the contraction each equivalence class of θ to one vertex (for precise definition see [10; Definition 2.1]). Moreover, $\theta(\mathbf{G})$ is the equivalence relation on $V^{\mathbf{G}}$ containing all pairs $\langle v, w \rangle$ such that $v = w$ or v and w lie on some cycle (see [10; Definition 3.1]).)

1.2. In this section we characterize digraphs with normal strong subdigraph lattices. Such digraphs will be called normal, too. Next we prove that with any normal digraph $\mathbf{G}$ we can associate a digraph $\mathbf{TQ}(\mathbf{G})$ in such a way that for a normal lattice $\mathbf{L}$, the strong subdigraph lattice $\mathbf{S}_s(\mathbf{G})$ is isomorphic to $\mathbf{L}$ iff $\mathbf{TQ}(\mathbf{G})$ and $\mathbf{D}(\mathbf{L})$ are isomorphic. Moreover, we show that for any normal digraphs $\mathbf{G}$ and $\mathbf{H}$, their strong subdigraph lattices $\mathbf{S}_s(\mathbf{G})$ and $\mathbf{S}_s(\mathbf{H})$ are isomorphic iff $\mathbf{TQ}(\mathbf{G})$ and $\mathbf{TQ}(\mathbf{H})$ are isomorphic. Using these digraph results, we will solve our algebraic problems.

We start with some technical notation. For a given digraph $\mathbf{G}$, by $\mathbf{G}_{\mathrm{sm}}$ we denote the unique (up to isomorphism) weak subdigraph of $\mathbf{G}$ which is simple and $V^{\mathbf{G}_{\mathrm{sm}}} = V^{\mathbf{G}}$, and for each $v, w \in V^{\mathbf{G}}$ with $v \neq w$, if there is an edge of $\mathbf{G}$ from v to w, then there exists an edge of $\mathbf{G}_{\mathrm{sm}}$ from v to w. First, such a simple weak subdigraph really exists (it easily follows from the axiom of choice). Secondly, there are, in general, many such simple digraphs, but they are all isomorphic. Thirdly,

$$\langle V^{\mathbf{G}}, \leq_{\mathbf{G}} \rangle \simeq \langle V^{\mathbf{G}_{\mathrm{sm}}}, \leq_{\mathbf{G}_{\mathrm{sm}}} \rangle \qquad \text{and} \qquad \mathbf{S}_s(\mathbf{G}_{\mathrm{sm}}) \simeq \mathbf{S}_s(\mathbf{G}) . \tag{SM}$$

Note that here $\leq_{\mathbf{G}}$ need not be a partial order, because $\mathbf{G}$ may contain non-trivial cycles.

Now we define an auxiliary kind of digraphs. A digraph $\mathbf{G}$ is called to be *critical* if $\mathbf{G}$ is simple and it does not contain cycles and each of its edges is an isthmus. Note that for a partially ordered set $\langle P, \leq_P \rangle$ *and a lattice* $\mathbf{L}$, *digraphs* $\mathbf{D}^{\mathrm{pos}}(P, \leq_P)$ and $\mathbf{D}(\mathbf{L})$ are critical.

Lemma 1.2.1. *Let* $\mathbf{G}$ *and* $\mathbf{H}$ *be critical digraphs. Then*

(a) $\langle V^{\mathbf{G}}, \leq_{\mathbf{G}} \rangle$ *is a partially ordered set satisfying FC.*

(b) $\mathbf{G} \simeq \mathbf{D}^{\mathrm{pos}}(V^{\mathbf{G}}, \leq_{\mathbf{G}})$.

(c) $\langle V^{\mathbf{G}}, \leq_{\mathbf{G}} \rangle \simeq \langle V^{\mathbf{H}}, \leq_{\mathbf{H}} \rangle \iff \mathbf{G} \simeq \mathbf{H}$.

Proof.

(a) $\langle V^{\mathbf{G}}, \leq_{\mathbf{G}} \rangle$ is partially ordered, since $\mathbf{G}$ contains no cycles. Take $v, w \in V^{\mathbf{G}}$ such that $v <_{\mathbf{G}} w$. Then there is a path $(e_1, \dots, e_n)$ going from w to v. Hence, since $e_1, \dots, e_n$ are isthmi,

$$v = I_2^{\mathbf{G}}(e_n) \prec_{\mathbf{G}} I_1^{\mathbf{G}}(e_n) = I_2^{\mathbf{G}}(e_{n-1}) \prec_{\mathbf{G}} \cdots = I_2^{\mathbf{G}}(e_1) \prec_{\mathbf{G}} I_1^{\mathbf{G}}(e_1) = w .$$

(b) It follows from (a) and Theorem 1.1.4 (applying to $\mathbf{G}$ and $\langle V^{\mathbf{G}}, \leq_{\mathbf{G}} \rangle$) that $\mathbf{D}^{\mathrm{pos}}(V^{\mathbf{G}}, \leq_{\mathbf{G}})$ is, up to isomorphism, a weak subdigraph of $\mathbf{G}$ satisfying conditions from the point (b) of Theorem 1.1.4. Hence it easily follows that $\mathbf{D}^{\mathrm{pos}}(V^{\mathbf{G}}, \leq_{\mathbf{G}})$ is isomorphic to $\mathbf{G}$, because each edge of $\mathbf{G}$ is an isthmus.

(c) The implication $\Longleftarrow$ is obvious, so it is sufficient to show the second. Take any order isomorphism $\phi \colon \langle V^{\mathbf{G}}, \leq_{\mathbf{G}} \rangle \longrightarrow \langle V^{\mathbf{H}}, \leq_{\mathbf{H}} \rangle$ and let

$$A = \{ \langle v, w \rangle \in V^{\mathbf{G}} \times V^{\mathbf{G}} : \; w \prec_{\mathbf{G}} v \} , \qquad B = \{ \langle v, w \rangle \in V^{\mathbf{H}} \times V^{\mathbf{H}} : \; w \prec_{\mathbf{H}} v \} .$$

Then $\phi \times \phi$ is a bijection from A onto B. It is easy to see that for each pair $\langle v, w\rangle \in A$, there is an edge from v to w, and such edge is exactly one (because $\mathbf{G}$ is simple); moreover the inverse fact also holds, i.e. for any edge e of $\mathbf{G}$, its endpoints form a pair belonging to A (because each edge is an isthmus). Analogous facts for $\mathbf{H}$ are also satisfied. Thus ϕ forms a digraph isomorphism from $\mathbf{G}$ onto $\mathbf{H}$. □

Let $\mathbf{G}$ be a simple digraph. Then $\mathbf{T}(\mathbf{G})$ is the weak subdigraph of $\mathbf{G}$ containing all vertices of $\mathbf{G}$ and all isthmi of $\mathbf{G}$. Next, for an arbitrary digraph $\mathbf{G}$ we define $\mathbf{T}(\mathbf{G}) = \mathbf{T}(\mathbf{G}_{\mathrm{sm}})$ and $\mathbf{TQ}(\mathbf{G}) = \mathbf{T}\big(\mathbf{G}/\theta(\mathbf{G})\big)$. Finally, $\mathbf{T}(\mathbf{A}) = \mathbf{T}\big(\mathbf{G}(\mathbf{A})\big)$ and $\mathbf{TQ}(\mathbf{A}) = \mathbf{TQ}\big(\mathbf{G}(\mathbf{A})\big)$ for any partial unary algebra $\mathbf{A}$ (where $\mathbf{G}(\mathbf{A})$ is the digraph representing $\mathbf{A}$, see [9] and [10], which is obtained from the algebra by omitting the names of all operations). Note (see also Theorem 3.4 from [10]) that $\theta\big(\mathbf{TQ}(\mathbf{G})\big)$ is the identity relation, $\mathbf{TQ}(\mathbf{G})$ is a critical digraph without cycles and $\mathbf{TQ}\big(\mathbf{TQ}(\mathbf{G})\big) = \mathbf{TQ}(\mathbf{G})$. Further, $\mathbf{TQ}(\mathbf{G}) = \mathbf{T}(\mathbf{G})$ if $\mathbf{G}$ contains only trivial cycles, and $\mathbf{T}(\mathbf{G}) = \mathbf{G}$ if $\mathbf{G}$ is critical.

Obviously $\mathbf{TQ}(\mathbf{G})$ is, in general, completely different from $\mathbf{G}$, but for special digraphs this construction preserves some graph properties. First, we say that a simple digraph $\mathbf{G}$ without cycles is *normal* if for each regular edge e of $\mathbf{G}$, there is a path $(f_1, \dots, f_n)$ from $I_1^{\mathbf{G}}(e)$ to $I_2^{\mathbf{G}}(e)$ and $f_1, \dots, f_n$ are isthmi. Secondly, a digraph $\mathbf{G}$ without non-trivial cycles is normal if $\mathbf{G}_{\mathrm{sm}}$ is normal. Thirdly, a digraph $\mathbf{G}$ is normal if $\mathbf{G}/\theta(\mathbf{G})$ is normal. And finally, partial unary algebra $\mathbf{A}$ is normal if its digraph $\mathbf{G}(\mathbf{A})$ is normal. The third definition is correct, because for a digraph $\mathbf{G}$, $\mathbf{G}/\theta(\mathbf{G})$ contains no non-trivial cycles (see [10; Theorem 3.4]). Note also that every critical digraph is, in particular, normal. Moreover, we have:

LEMMA 1.2.2. *Let $\mathbf{G}$ be a normal digraph and $\theta = \theta(\mathbf{G})$. Then*

$$\big\langle V^{\mathbf{G}/\theta}, \leq_{\mathbf{G}/\theta}\big\rangle \simeq \big\langle V^{\mathbf{TQ}(\mathbf{G})}, \leq_{\mathbf{TQ}(\mathbf{G})}\big\rangle .$$

The proof follows directly from (SM) and the definition of normal digraphs.

THEOREM 1.2.3. *Let $\mathbf{G}$ be a digraph and $\mathbf{A}$ a partial unary algebra. Then*

(a) *$\mathbf{G}$ is normal iff the partially ordered set $\Big\langle V^{\mathbf{G}/\theta(\mathbf{G})}, \leq_{\mathbf{G}/\theta(\mathbf{G})}\Big\rangle$ satisfies FC.*

(b) *$\mathbf{G}$ is normal iff $\mathbf{S}_s(\mathbf{G})$ is a normal lattice.*

(c) *$\mathbf{A}$ is normal iff $\mathbf{S}_s(\mathbf{A})$ is a normal lattice.*

P r o o f . By [10; Theorem 3.9] we get (it is sufficient to take $\mathbf{L} = \mathbf{S}_s(\mathbf{G})$)

$$\Big\langle V^{\mathbf{G}/\theta(\mathbf{G})}, \leq_{\mathbf{G}/\theta(\mathbf{G})}\Big\rangle \simeq \Big\langle \mathrm{Ir}\big(\mathbf{S}_s(\mathbf{G})\big), \leq_{\mathbf{S}_s(\mathbf{G})}\Big\rangle .$$

We also know that $\mathbf{S}_s(\mathbf{G})$ satisfies the conditions $(*)$ and $(**)$. By these facts and (a) we obtain (b). Thus also (c), because $\mathbf{S}_s(\mathbf{A}) \simeq \mathbf{S}_s\big(\mathbf{G}(\mathbf{A})\big)$ (see [9; Theorem 2.2.4]).

(a) $\Longleftarrow$. Let $\mathbf{H} = \big(\mathbf{G}/\theta(\mathbf{G})\big)_{\mathrm{sm}}$. Then $\langle V^{\mathbf{H}}, \leq_{\mathbf{H}}\rangle$ also satisfies FC. Thus for any regular edge e of $\mathbf{H}$, there are vertices $v_1, v_2, \dots, v_n$ such that

$$I_2^{\mathbf{H}}(e) = v_1 \prec_{\mathbf{H}} \cdots \prec_{\mathbf{H}} v_n = I_1^{\mathbf{H}}(e)\,.$$

It follows from the definition of the covering relation (since $\mathbf{H}$ is simple) that from v_{i+1} to v_i (for $i = 1, \dots, n-1$), there is the exactly one path consisting of the exactly one edge, say f_i. Hence, $f_1, \dots, f_{n-1}$ are isthmi, and they form a path going from v_n to v_1.

The implication $\Longrightarrow$ follows from Lemmas 1.2.1(a) and 1.2.2. □

Theorem 1.1.1 can be reformulated for digraphs to obtain a characterization of normal simple digraphs without cycles. It is sufficient to replace the covering relation by the concept of isthmus, and ascending ordered chains by infinite paths (in a digraph) with the final vertices and descending ordered chains by infinite paths with the initial vertices. Note also that the fact from the end of Theorem 1.1.1 can be translated for arbitrary (not only simple and without cycles) digraphs, i.e.

if a digraph $\mathbf{G}$ does not contain infinite paths, then $\mathbf{G}$ is normal.

It is sufficient to show that $\mathbf{G}/\theta(\mathbf{G})$ does not also contain infinite paths. This follows from the fact that two vertices belong to the same equivalence class if they are equal, or there are paths from the one to the other and conversely (see [10; Definition 3.1, Proposition 3.2]). More formally, if $\mathbf{G}/\theta(\mathbf{G})$ contained an infinite path, then it would be sufficient to substitute between edges of this path finite paths connecting the suitable endpoint of one edge with the suitable endpoint of the other. In this way we would obtain an infinite path in $\mathbf{G}$, because equivalence classes are disjoint.

Now we can formulate and prove our main results. First, for digraphs, and next, for algebras.

THEOREM 1.2.4. *Let $\mathbf{G}$ be a normal digraph and $\mathbf{L}$ a normal lattice. Then*

$$\mathbf{S}_s(\mathbf{G}) \simeq \mathbf{L} \iff \mathbf{TQ}(\mathbf{G}) \simeq \mathbf{D}(\mathbf{L})\,.$$

P r o o f. By [10; Theorem 3.9] (where $\theta = \theta(\mathbf{G})$),

$$\mathbf{S}_s(\mathbf{G}) \simeq \mathbf{L} \iff \langle V^{\mathbf{G}/\theta}, \leq_{\mathbf{G}/\theta}\rangle \simeq \langle \mathrm{Ir}(\mathbf{L}), \leq_{\mathbf{L}}\rangle\,.$$

Secondly, by Proposition 1.1.2(a), and also Theorem 1.2.3(a), we have

$$\langle V^{\mathbf{G}/\theta}, \leq_{\mathbf{G}/\theta}\rangle \simeq \langle \mathrm{Ir}(\mathbf{L}), \leq_{\mathbf{L}}\rangle \iff \mathbf{D}^{\mathrm{pos}}(V^{\mathbf{G}/\theta}, \leq_{\mathbf{G}/\theta}) \simeq \mathbf{D}(\mathbf{L})\,.$$

Now it is remained to show

$$\mathbf{TQ}(\mathbf{G}) \simeq \mathbf{D}^{\mathrm{pos}}(V^{\mathbf{G}/\theta}, \leq_{\mathbf{G}/\theta})\,. \tag{1}$$

By Lemma 1.2.2,

$$\left\langle V^{\mathbf{G}/\theta}, \leq_{\mathbf{G}/\theta}\right\rangle \simeq \left\langle V^{\mathbf{TQ}(\mathbf{G})}, \leq_{\mathbf{TQ}(\mathbf{G})}\right\rangle .$$

Hence,

$$\mathbf{D}^{\mathrm{pos}}\left(V^{\mathbf{G}/\theta}, \leq_{\mathbf{G}/\theta}\right) \simeq \mathbf{D}^{\mathrm{pos}}\left(V^{\mathbf{TQ}(\mathbf{G})}, \leq_{\mathbf{TQ}(\mathbf{G})}\right) .$$

This fact and Lemma 1.2.1(b) imply

$$\mathbf{D}^{\mathrm{pos}}\left(V^{\mathbf{G}/\theta}, \leq_{\mathbf{G}/\theta}\right) \simeq \mathbf{TQ}(\mathbf{G}) ,$$

since $\mathbf{TQ}(\mathbf{G})$ is critical. □

COROLLARY 1.2.5. *For each normal digraph* $\mathbf{G}$*,*

$$\mathbf{TQ}(\mathbf{G}) \simeq \mathbf{D}\left(\mathbf{S}_s(\mathbf{G})\right), \qquad \mathbf{S}_s(\mathbf{G}) \simeq \mathbf{S}_s\left(\mathbf{TQ}(\mathbf{G})\right),$$
$$\mathbf{TQ}(\mathbf{G}) \simeq \mathbf{D}^{\mathrm{pos}}\left(V^{\mathbf{G}/\theta(\mathbf{G})}, \leq_{\mathbf{G}/\theta(\mathbf{G})}\right) .$$

P r o o f. The third part has been shown in the previous proof.

By Theorem 1.2.3(b), $\mathbf{S}_s(\mathbf{G})$ is a normal lattice. Thus by Theorem 1.2.4, applying to $\mathbf{L} = \mathbf{S}_s(\mathbf{G})$,

$$\mathbf{TQ}(\mathbf{G}) \simeq \mathbf{D}\left(\mathbf{S}_s(\mathbf{G})\right) .$$

Hence and by Theorem 1.1.3(b) we obtain also

$$\mathbf{S}_s\left(\mathbf{TQ}(\mathbf{G})\right) \simeq \mathbf{S}_s\left(\mathbf{D}\left(\mathbf{S}_s(\mathbf{G})\right)\right) \simeq \mathbf{S}_s(\mathbf{G}) .$$

□

THEOREM 1.2.6. *Let* $\mathbf{G}$ *and* $\mathbf{H}$ *be normal digraphs. Then*

$$\mathbf{S}_s(\mathbf{G}) \simeq \mathbf{S}_s(\mathbf{H}) \iff \mathbf{TQ}(\mathbf{G}) \simeq \mathbf{TQ}(\mathbf{H}) .$$

P r o o f. By Corollary 1.2.5,

$$\mathbf{TQ}(\mathbf{G}) \simeq \mathbf{TQ}(\mathbf{H}) \iff \mathbf{D}^{\mathrm{pos}}\left(V^{\mathbf{G}/\theta(\mathbf{G})}, \leq_{\mathbf{G}/\theta(\mathbf{G})}\right) \simeq \mathbf{D}^{\mathrm{pos}}\left(V^{\mathbf{H}/\theta(\mathbf{H})}, \leq_{\mathbf{H}/\theta(\mathbf{H})}\right) .$$

Moreover, by Proposition 1.1.2(a) and Theorem 1.2.3(a) we obtain

$$\left\langle V^{\mathbf{G}/\theta(\mathbf{G})}, \leq_{\mathbf{G}/\theta(\mathbf{G})}\right\rangle \simeq \left\langle V^{\mathbf{H}/\theta(\mathbf{H})}, \leq_{\mathbf{H}/\theta(\mathbf{H})}\right\rangle$$
$$\iff \mathbf{D}^{\mathrm{pos}}\left(V^{\mathbf{G}/\theta(\mathbf{G})}, \leq_{\mathbf{G}/\theta(\mathbf{G})}\right) \simeq \mathbf{D}^{\mathrm{pos}}\left(V^{\mathbf{H}/\theta(\mathbf{H})}, \leq_{\mathbf{H}/\theta(\mathbf{H})}\right) .$$

These two facts imply

$$\left\langle V^{\mathbf{G}/\theta(\mathbf{G})}, \leq_{\mathbf{G}/\theta(\mathbf{G})}\right\rangle \simeq \left\langle V^{\mathbf{H}/\theta(\mathbf{H})}, \leq_{\mathbf{H}/\theta(\mathbf{H})}\right\rangle \iff \mathbf{TQ}(\mathbf{G}) \simeq \mathbf{TQ}(\mathbf{H}) .$$

Thus [10; Theorem 3.11] completes the proof. □

If we additionally assume that digraphs $\mathbf{G}$ and $\mathbf{H}$ in Theorems 1.2.4 and 1.2.6 do not contain non-trivial cycles, then $\mathbf{TQ}(\mathbf{G})$ and $\mathbf{TQ}(\mathbf{H})$ can be replaced by $\mathbf{T}(\mathbf{G})$ and $\mathbf{T}(\mathbf{H})$.

Note that it is not true for digraphs having non-trivial cycles. To see it, take digraphs $\mathbf{G}$ and $\mathbf{H}$ having integers as their vertices and with the following edge sets respectively

$$\{\langle n, n-1\rangle :\ n \in \mathbb{Z}\} \cup \{\langle -n, n\rangle :\ n = 1,2,3,\dots\} \quad \text{and} \quad \{\langle n, n-1\rangle :\ n \in \mathbb{Z}\};$$

and take the lattice $\mathbf{L} = \mathbf{S}_s(\mathbf{H})$.

It is easy to see that for any $n \in \mathbb{Z}$, $\langle n, n-1\rangle$ is an isthmus in $\mathbf{G}$, thus also in $\mathbf{H}$, and $\langle -n, n\rangle$ is not an isthmus in $\mathbf{G}$ for $n = 1,2,3,\dots$. Hence first, $\mathbf{T}(\mathbf{G}) = \mathbf{H}$. Secondly, $\mathbf{H}$ is a critical digraph. So $\mathbf{TQ}(\mathbf{H}) = \mathbf{T}(\mathbf{H}) = \mathbf{H}$ and $\mathbf{L}$ is a normal lattice. Thus also $\mathbf{H} \simeq \mathbf{D}(\mathbf{S}_s(\mathbf{H})) = \mathbf{D}(\mathbf{L})$ by Corollary 1.2.5. On the other hand, $\mathbf{S}_s(\mathbf{G})$ and $\mathbf{L}$ are not isomorphic, because it is easy to see that $\mathbf{S}_s(\mathbf{G})$ is the two-element chain (the empty digraph and $\mathbf{G}$), and $\mathbf{L}$ is an infinite lattice. Summarizing we have found the digraphs $\mathbf{G}$, $\mathbf{H}$ and the lattice $\mathbf{L}$ such that

$$\mathbf{T}(\mathbf{G}) \simeq \mathbf{D}(\mathbf{L}), \quad \mathbf{T}(\mathbf{G}) = \mathbf{T}(\mathbf{H}) \qquad \text{and} \qquad \mathbf{S}_s(\mathbf{G}) \not\simeq \mathbf{L}, \quad \mathbf{S}_s(\mathbf{G}) \not\simeq \mathbf{S}_s(\mathbf{H}).$$

Now applying the above digraph facts and [9; Theorem 2.2.4], we can formulate our algebraic results.

THEOREM 1.2.7. *Let $\mathbf{A}$ and $\mathbf{B}$ be normal partial unary algebras, and $\mathbf{L}$ a normal lattice. Then*

(a) $\mathbf{S}_s(\mathbf{A}) \simeq \mathbf{L} \iff \mathbf{TQ}(\mathbf{A}) \simeq \mathbf{D}(\mathbf{L})$.
(b) $\mathbf{S}_s(\mathbf{A}) \simeq \mathbf{S}_s(\mathbf{B}) \iff \mathbf{TQ}(\mathbf{A}) \simeq \mathbf{TQ}(\mathbf{B})$.

We know (see [10; Proof of Corollary 3.14]) that $\mathbf{G}(\mathbf{A})$ contains only trivial cycles iff $\langle a\rangle_{\mathbf{A}} \neq \langle b\rangle_{\mathbf{A}}$ for all $a, b \in A$, $a \neq b$ (where $\langle a\rangle_{\mathbf{A}}$ is the strong subalgebra generated by a). Hence and by Theorem 1.2.7, we have:

COROLLARY 1.2.8. *Let $\mathbf{A}$ and $\mathbf{B}$ be normal partial unary algebras such that*

(i) *For any $a_1, a_2 \in A$, $a_1 \neq a_2 \implies \langle a_1\rangle_{\mathbf{A}} \neq \langle a_2\rangle_{\mathbf{A}}$.*
(ii) *For any $b_1, b_2 \in B$, $b_1 \neq b_2 \implies \langle b_1\rangle_{\mathbf{B}} \neq \langle b_2\rangle_{\mathbf{B}}$.*

Then

$$\mathbf{S}_s(\mathbf{A}) \simeq \mathbf{S}_s(\mathbf{B}) \iff \mathbf{T}(\mathbf{A}) \simeq \mathbf{T}(\mathbf{B}).$$

2

Investigations of relationships between properties of algebras or properties of varieties of algebras and those of their subalgebra lattices are an important part of universal algebra (see e.g. [7], [8]). The theory of partial algebras provides additional tools for such investigation, since several different structures may be considered in this case (see e.g. [4] or [5]). The important concept of subalgebra in this theory, beside the usual (strong) subalgebra, is that of weak subalgebra. Let $\mathbf{A} = \langle A, (k^{\mathbf{A}})_{k\in K}\rangle$ and $\mathbf{B} = \langle B, (k^{\mathbf{B}})_{k\in K}\rangle$ be partial unary algebras of type K. Recall that $\mathbf{B}$ is a *weak subalgebra* of $\mathbf{A}$, written $\mathbf{B} \leq_w \mathbf{A}$, iff $B \subseteq A$ and $k^{\mathbf{B}} \subseteq k^{\mathbf{A}}$ for all $k \in K$. The set of all weak subalgebras of $\mathbf{A}$ with the (weak subalgebra) inclusion $\leq_w$ forms a complete and algebraic lattice $\mathbf{S}_w(\mathbf{A})$. It seems that the weak subalgebra lattice alone, and also together with the strong subalgebra lattice, yields a lot of interesting information on an unary algebra, also total (see e.g. [2] and [11]).

In this part we apply results from the first and some facts of the graph theory to describe pairs of lattices isomorphic to the weak and strong subalgebra lattices, respectively, of one **normal** unary algebra.

2.1. We will need Robinson's Theorem (see [13] or [3; Chap. 9, Theorem 10]) about (undirected) graphs which can be directed to a form of strongly connected digraph. First, recall that a digraph is *strongly connected* if each directed pair of different vertices is connected by a (directed) path. Secondly, for a digraph $\mathbf{G}$, the graph obtained from $\mathbf{G}$ by omitting the orientation of all edges will be denoted by $\mathbf{G}^*$. Thirdly, $V^{\mathbf{G}}$ and $E^{\mathbf{G}}$ denote the vertex and edge sets of $\mathbf{G}$, respectively, and $I^{\mathbf{G}}$ is the incident function. Finally, we assume that an undirected cycle contains pairwise different edges.

THEOREM 2.1.1. (Robinson, H. E.) *Let* $\mathbf{U}$ *be a graph. There is a strongly connected digraph* $\mathbf{G}$ *such that* $\mathbf{G}^* = \mathbf{U}$ *iff* $\mathbf{U}$ *is connected and each of its regular edges lies on an undirected cycle.*

Note that Robinson proved this theorem for the finite case only, but we consider here also infinite graphs. Therefore we now show the result for arbitrary graphs.

Proof. The implication $\Longrightarrow$ is obtained by the following fact (see [3; Chap. 3, Theorem 7], observe that the assumption in its proof about finiteness of digraph is not essential): a digraph is *strongly connected* iff it is connected and each of its regular edges lies on a (directed) cycle.

It is sufficient to show that there is a digraph $\mathbf{G}$ such that $\mathbf{G}^* = \mathbf{U}$ and each of its regular edges lies on a (directed) cycle. To do this, we will use transfinite sequences. Recall that cardinal numbers can be defined as initial ordinal numbers, i.e. the least ordinal number in a class of equipotent ordinals. Let ξ be the cardinality of the set of all undirected cycles of $\mathbf{U}$. Define $(r_\alpha)_{\alpha<\xi}$ to be a transfinite sequence of all the undirected cycles of $\mathbf{U}$. Next, for any ordinal number $\alpha \le \xi$, let $\mathbf{U}_\alpha$ be the weak subgraph of $\mathbf{U}$ with

$$V^{\mathbf{U}_\alpha} = V^{\mathbf{U}} \qquad \text{and} \qquad E^{\mathbf{U}_\alpha} = \bigcup_{\beta<\alpha} E^{r_\beta},$$

where E^{r_β} is the set of all edges of r_β.

Then first,

$$\mathbf{U}_{\alpha_1} \le_w \mathbf{U}_{\alpha_2} \qquad \text{for any} \quad \alpha_1 \le \alpha_2 \le \xi.$$

Secondly,

$$\mathbf{U}_\xi = \mathbf{U}.$$

It follows from the facts that $\mathbf{U}_\xi$ contains all vertices and all undirected cycles of $\mathbf{U}$, and each regular edge lies on an undirected cycle, and each loop forms a trivial cycle.

Now we prove that there is a transfinite sequence of digraphs $(\mathbf{G}_\alpha)_{\alpha\le\xi}$ such that for every ordinal number $\alpha \le \xi$,

(1) each regular edge of $\mathbf{G}_\alpha$ lies on a (directed) cycle,
(2) $\mathbf{G}^*_\alpha = \mathbf{U}_\alpha$,
(3) for each ordinal number $\gamma \le \alpha$, $\mathbf{G}_\gamma \le_w \mathbf{G}_\alpha$.

Having this fact it is sufficient to take $\mathbf{G} = \mathbf{G}_\xi$. Note that (3) is only a technical condition needed to construct the sequence.

We apply transfinite induction on α. For $\alpha = 0$ we have $E^{\mathbf{U}_0} = \emptyset$, so $\mathbf{U}_0$ can be regarded as a digraph which, of course, satisfies (1), (2) and (3).

Induction step.

Take an ordinal number $1 \le \alpha \le \xi$ and assume that there is a transfinite sequence $(\mathbf{G}_\zeta)_{\zeta<\alpha}$ of digraphs satisfying (1)–(3) for each ordinal number $\zeta < \alpha$.

If α is a limit ordinal, then we take the following digraph $\mathbf{G}_\alpha$

$$V^{\mathbf{G}_\alpha} = V^{\mathbf{U}}, \qquad E^{\mathbf{G}_\alpha} = \bigcup_{\zeta<\alpha} E^{\mathbf{G}_\zeta} \qquad \text{and} \qquad I^{\mathbf{G}_\alpha} = \bigcup_{\zeta<\alpha} I^{\mathbf{G}_\zeta}.$$

First, by (3), $I^{\mathbf{G}_\alpha}$ is a well-defined function, so $\mathbf{G}_\alpha$ is indeed a digraph. Secondly, this definition implies $\mathbf{G}_\zeta \le_w \mathbf{G}_\alpha$ for each ordinal number $\zeta \le \alpha$, so (3) holds. Further, since α is a limit ordinal, we obtain

$$E^{\mathbf{G}_\alpha} = \bigcup_{\zeta<\alpha} E^{\mathbf{G}_\zeta} = \bigcup_{\zeta<\alpha} E^{\mathbf{U}_\zeta} = \bigcup_{\zeta<\alpha}\bigcup_{\beta<\zeta} E^{r_\beta} = \bigcup_{\beta<\alpha} E^{r_\beta} = E^{\mathbf{U}_\alpha}.$$

Hence, also $I^{\mathbf{G}^*_\alpha} = I^{\mathbf{U}_\alpha}$, because by (2), $I^{\mathbf{G}^*_\alpha} = \bigcup_{\zeta<\alpha} I^{\mathbf{G}^*_\zeta} = \bigcup_{\zeta<\alpha} I^{\mathbf{U}_\zeta} \subseteq I^{\mathbf{U}_\alpha}$. Thus $\mathbf{G}^*_\alpha = \mathbf{U}_\alpha$, so (2) holds for α.

Take a regular edge $e \in E^{\mathbf{G}_\alpha}$. Then $e \in E^{\mathbf{G}_\zeta} = E^{\mathbf{U}_\zeta}$ for some ordinal number $\zeta < \alpha$. Thus e lies on a (directed) cycle in $\mathbf{G}_\zeta$, which is also a cycle in $\mathbf{G}_\alpha$. Thus (1) is also satisfied.

Now assume $\alpha = \beta + 1$ for some ordinal number $\beta < \xi$ (i.e. α is a successor). Take the undirected cycle $r_\beta = \langle (f_1, \dots, f_n), (u_1, \dots, u_{n+1}) \rangle$ of $\mathbf{U}$, (where $I^{\mathbf{U}}(f_i) = \{u_i, u_{i+1}\}$), and the following digraph $\mathbf{G}_\alpha$

$$V^{\mathbf{G}_\alpha} = V^{\mathbf{U}}, \qquad E^{\mathbf{G}_\alpha} = E^{\mathbf{G}_\beta} \cup \{f_1, \dots, f_n\},$$

$$I^{\mathbf{G}_\alpha}\big|_{E^{\mathbf{G}_\beta}} = I^{\mathbf{G}_\beta} \qquad \text{and}$$
$$I^{\mathbf{G}_\alpha}(f_i) = \langle u_i, u_{i+1} \rangle \qquad \text{if} \quad f_i \notin E^{\mathbf{G}_\beta} \qquad \text{for} \quad i = 1, 2, \dots, n.$$

First, $I^{\mathbf{G}_\alpha}$ is well-defined on $\{f_1, \dots, f_n\} \setminus E^{\mathbf{G}_\beta}$, because $f_1, \dots, f_n$ are pairwise different. Thus $\mathbf{G}$ is indeed a digraph. Secondly, $\mathbf{G}^*_\beta = \mathbf{U}_\beta \leq_w \mathbf{U}_\alpha$ and $E^{\mathbf{U}_\alpha} = E^{\mathbf{U}_\beta} \cup \{f_1, \dots, f_n\} = E^{\mathbf{G}_\beta} \cup \{f_1, \dots, f_n\} = E^{\mathbf{G}_\alpha}$. Hence, $\mathbf{G}^*_\alpha = \mathbf{U}_\alpha$, so (2) is true. Thirdly, it is clear that $\mathbf{G}_\beta \leq_w \mathbf{G}_\alpha$, which implies (3) for α, because $\mathbf{G}_\beta$ satisfies induction hypotheses.

Obviously if $\{f_1, \dots, f_n\} \subseteq E^{\mathbf{G}_\beta}$ or $\{f_1, \dots, f_n\} \cap E^{\mathbf{G}_\beta} = \emptyset$, then (1) holds for α. Note that in the second case r_β is just a (directed) cycle in $\mathbf{G}_\alpha$.

Now assume $\{f_1, \dots, f_n\} \not\subseteq E^{\mathbf{G}_\beta}$ and $\{f_1, \dots, f_n\} \cap E^{\mathbf{G}_\beta} \neq \emptyset$. Of course, we can also assume $f_n \in E^{\mathbf{G}_\beta}$ and $f_1 \notin E^{\mathbf{G}_\beta}$. Let k be the greatest number such that $\{f_1, \dots, f_k\} \cap E^{\mathbf{G}_\beta} = \emptyset$ and let l be the greatest number such that $l \geq k + 1$ and $\{f_{k+1}, \dots, f_l\} \subseteq E^{\mathbf{G}_\beta}$. By our assumptions, such numbers exist and $k \leq n - 1$, $l \leq n$. Then $(f_1, \dots, f_k)$ is a (directed) chain in $\mathbf{G}_\alpha$. Further, for $i = k + 1, \dots, l$, $\{I_1^{\mathbf{G}_\alpha}(f_i), I_2^{\mathbf{G}_\alpha}(f_i)\} = \{u_i, u_{i+1}\}$ and f_i lies on a (directed) cycle, because $f_i \in E^{\mathbf{G}_\beta}$. Thus there is a (directed) chain going from u_i to u_{i+1}. This implies that there is a chain $(g_1, \dots, g_m)$ going from u_{k+1} to u_{l+1} (where $u_{n+1} = u_1$). Hence, $q = (f_1, \dots, f_k, g_1, \dots, g_m)$ is a chain going from u_1 to u_{l+1}. If $l = n$, i.e. $u_{l+1} = u_1$, then q is a (directed) cycle. If $l < n$, then this procedure can be repeated, as many times as needed, to obtain a (directed) cycle q containing all edges from $E^{r_\beta} \setminus E^{\mathbf{G}_\beta}$.

Thus each regular edge of $\mathbf{G}_\alpha$ lies on a (directed) cycle, i.e. (1) also holds for α. This completes the proof of the induction step. □

2.2. It is proved in [1] that a complete lattice $\mathbf{L} = \langle L, \leq_{\mathbf{L}} \rangle$ is isomorphic to the weak subalgebra lattice of some partial unary algebra iff

(i) $\mathbf{L}$ is algebraic and distributive,
(ii) each element of $\mathbf{L}$ is a join of join-irreducible elements,
(iii) each non-zero and non-atomic join-irreducible element of $\mathbf{L}$ contains exactly one or two atoms,
(iv) the set of all non-zero and non-atomic join-irreducible elements of $\mathbf{L}$ is an antichain with respect to the lattice order $\leq_{\mathbf{L}}$.

Recall (see [9; Definition 2.3.7]) that with any lattice $\mathbf{L}$ satisfying (i)–(iv) we can associate the graph $\mathbf{G}(\mathbf{L})$ containing the set of all atoms of $\mathbf{L}$ as its vertex set and the set of all non-zero and non-atomic join-irreducible elements of $\mathbf{L}$ as its edge set, and a vertex v and an edge e are incident if $v \leq_{\mathbf{L}} e$.

In this section we describe pairs of the weak and strong subalgebra lattices of one **normal** partial unary algebra. By results from [9] we first have that an algebra can be replaced by a digraph. Secondly, we can replace the weak subalgebra lattice by a graph ([9; Theorem 2.3.11]), and the strong subalgebra lattice by a critical digraph (Theorem 1.2.7). Summarizing it is sufficient to find necessary and sufficient conditions for a graph $\mathbf{U}$ and a critical digraph $\mathbf{H}$ to exist a **normal** digraph $\mathbf{G}$ such that $\mathbf{G}^* \simeq \mathbf{U}$ and $\mathbf{TQ}(\mathbf{G}) \simeq \mathbf{H}$. The desired conditions are given by the following main result. (Recall that the quotient graph $\mathbf{G}/\theta$ of a graph $\mathbf{G}$ with respect to an equivalence relation θ on $V^{\mathbf{G}}$ is obtained by the contraction of each equivalence class of θ to one point (see [10]). Further, for any $X \subseteq V^{\mathbf{G}}$, $[X]_{\mathbf{G}}$ denotes the subgraph consisting of X and all edges with endpoints in X.)

THEOREM 2.2.1. *Let $\mathbf{U}$ be a graph and $\mathbf{H}$ a critical digraph. Then the following conditions are equivalent:*

(a) *There is a* **normal** *digraph $\mathbf{G}$ such that $\mathbf{G}^* \simeq \mathbf{U}$ and $\mathbf{TQ}(\mathbf{G}) \simeq \mathbf{H}$.*

(b) *There is an equivalence relation θ on $V^{\mathbf{U}}$ such that*

  (b.1) *for any equivalence class $W \in V^{\mathbf{U}}/\theta$, $[W]_{\mathbf{U}}$ is a connected graph such that each of its regular edges lies on an undirected cycle,*

  (b.2) *$\mathbf{H}^*$ is a weak subgraph (up to isomorphism) of $\mathbf{U}/\theta$ having the same vertex set and if $\mathbf{H}^*$ is identified with this weak subgraph, then for each regular edge e of $\mathbf{U}/\theta$, there is a (directed) path in $\mathbf{H}$ going from one endpoint of e to the other.*

Proof.

(a) $\Longrightarrow$ (b). Let $\mathbf{G}$ be a normal digraph such that

$$\mathbf{G}^* \simeq \mathbf{U} \qquad \text{and} \qquad \mathbf{TQ}(\mathbf{G}) \simeq \mathbf{H}.$$

The orientation of all edges of $\mathbf{G}$ can be transported onto $\mathbf{U}$. Thus we can assume $\mathbf{G}^* = \mathbf{U}$. Observe

$$[X]_{\mathbf{U}} = \big([X]_{\mathbf{G}}\big)^* \qquad \text{for any} \quad X \subseteq V^{\mathbf{U}}.$$

Hence it is easy to see that the equivalence relation $\theta = \theta(\mathbf{G})$ (see [10]) satisfies (b.1).

Now take $\mathbf{K} = \big(\mathbf{G}/\theta\big)_{\mathrm{sm}}$. Since $\mathbf{K}$ is a simple digraph without cycles, between any two different vertices there is at most one undirected edge in the graph $\mathbf{K}^*$ (otherwise we would have a directed cycle in $\mathbf{K}$). Hence and by the equalities

$$\mathbf{U}/\theta = \mathbf{G}^*/\theta = \big(\mathbf{G}/\theta\big)^*$$

we deduce that $\mathbf{K}^*$ is a weak subgraph of $\mathbf{U}/\theta$ having the same vertex set and such that for any pair of different vertices, if there is an edge between them in $\mathbf{U}/\theta$, then there is an edge between them in $\mathbf{K}^*$. This fact and the definition of the digraph $\mathbf{TQ}(\mathbf{G})$, since $\mathbf{G}$ is normal, imply that $\mathbf{TQ}(\mathbf{G})$, thus also $\mathbf{H}$ satisfies (b.2).

(b) $\Longrightarrow$ (a). Let θ be an equivalence relation on $V^{\mathbf{U}}$ satisfying (b.1) and (b.2). Take the graph $\mathbf{U}/\theta$ and assume that $\mathbf{H}^*$ is its weak subgraph satisfying (b.2) (it is sufficient to take an isomorphic copy of $\mathbf{H}^*$ by any digraph isomorphism forced by (b.2)). Then all edges of $\mathbf{U}/\theta$ which belong to $\mathbf{H}$ can be directed as in $\mathbf{H}$. Obviously all loops of $\mathbf{U}/\theta$ can be regarded as directed edges. Finally, observe that all regular edges of $\mathbf{U}/\theta$ outside $\mathbf{H}$ can be directed according to the orientation of $\mathbf{H}$. More formally, take an arbitrary regular edge e of $\mathbf{U}/\theta$ such that e does not belong to $\mathbf{H}$. Then there is a (directed) path in $\mathbf{H}$ from one endpoint v of e to the other endpoint w of e. Thus v can be taken as the initial vertex of e and w can be defined as the final vertex of e. Since $\mathbf{H}$ has no (directed) cycles, it follows that any path in $\mathbf{H}$ connecting the endpoints of e must go from v to w. This implies that the above procedure uniquely direct e. In this way we construct the digraph $\mathbf{D}$ such that

$$\mathbf{D}^* = \mathbf{U}/\theta . \tag{1}$$

$$\mathbf{H} \leq_w \mathbf{D} \quad \text{and} \quad V^{\mathbf{H}} = V^{\mathbf{D}} . \tag{2}$$

Since $\mathbf{H}$ has not (directed) cycles and the orientation of all edges of $\mathbf{D}$ is according to $\mathbf{H}$, we deduce that $\mathbf{D}$ does not contain non-trivial cycles, i.e. (see the definition at the end of Section 1.2)

$$C_n(\mathbf{D}) = \emptyset . \tag{3}$$

Take the simple digraph $\mathbf{D}_{\mathrm{sm}}$. Since $\mathbf{H}$ is a simple digraph and $\mathbf{H} \leq_w \mathbf{D}$, $\mathbf{H}$ can be assumed to be a weak subdigraph of $\mathbf{D}_{\mathrm{sm}}$. Then by the construction of $\mathbf{D}$ and $\mathbf{D}_{\mathrm{sm}}$ we infer that for any regular edge e of $\mathbf{D}_{\mathrm{sm}}$, there is a (directed) path in $\mathbf{H}$ going from the initial vertex $I_1^{\mathbf{D}_{\mathrm{sm}}}(e)$ of e to the final vertex $I_2^{\mathbf{D}_{\mathrm{sm}}}(e)$ of e. Hence, all isthmi of $\mathbf{D}_{\mathrm{sm}}$ belong to $\mathbf{H}$.

On the other hand, take an isthmus e of $\mathbf{H}$ and assume that $p = (f_1, \dots, f_k)$ is a (directed) path in $\mathbf{D}_{\mathrm{sm}}$ going from $I_1^{\mathbf{D}_{\mathrm{sm}}}(e)$ to $I_2^{\mathbf{D}_{\mathrm{sm}}}(e)$. Then there are (directed) paths $p_1, \dots, p_k$ in $\mathbf{H}$ such that p_i goes from $I_1^{\mathbf{D}_{\mathrm{sm}}}(f_i)$ to $I_2^{\mathbf{D}_{\mathrm{sm}}}(f_i)$ for $i = 1, \dots, k$. Since $\mathbf{H}$ does not contain cycles, the sum of these k paths form another (directed) path in $\mathbf{H}$ going from $I_1^{\mathbf{D}_{\mathrm{sm}}}(e)$ to $I_2^{\mathbf{D}_{\mathrm{sm}}}(e)$. But e is an isthmus in $\mathbf{H}$, so this path contains only e. This implies that p has also only one edge. Hence, $p = (e)$, because $\mathbf{D}_{\mathrm{sm}}$ is simple. Thus e is also an isthmus in $\mathbf{D}_{\mathrm{sm}}$.

Summarizing, $\mathbf{H}$ consists of all vertices and all its isthmi of $\mathbf{D}_{\mathrm{sm}}$. Hence, $\mathbf{T}(\mathbf{D}_{\mathrm{sm}}) = \mathbf{H}$, so

$$\mathbf{T}(\mathbf{D}) = \mathbf{H} . \tag{4}$$

Moreover, by the definition of $\mathbf{D}$ and (3) we obtain

(5) D is a normal digraph.

Now take an equivalence class $X \in V^{\mathbf{U}}/\theta$. Then by (b.1) and Theorem 2.1.1, all (undirected) edges of $[X]_{\mathbf{U}}$ can be directed to a form of strongly connected digraph $\mathbf{K}_X$, i.e.

$$\mathbf{K}_X^* = [X]_{\mathbf{U}}\,.$$

Let e be an arbitrary edge of $\mathbf{U}$. If e is a loop in $\mathbf{U}/\theta$, then $I^{\mathbf{U}}(e) \subseteq X$ for some equivalence class $X \in V^{\mathbf{U}}/\theta$ (recall, see [10], that endpoints of e in $\mathbf{U}/\theta$ are equal to equivalence classes of endpoints of e from $\mathbf{U}$, i.e. $I^{\mathbf{U}/\theta}(e) = I^{\mathbf{U}}(e)/\theta$). Then we direct e in the same way as in $\mathbf{K}_X$. If e is not a loop in $\mathbf{U}/\theta$, then by (1), there are vertices v and w such that $I^{\mathbf{U}}(e) = \{v, w\}$ and $v/\theta = I_1^{\mathbf{D}}(e)$, $w/\theta = I_2^{\mathbf{D}}(e)$. These two vertices are uniquely determined, so v can be defined as the initial vertex of e and w can be defined as the final vertex of e. In this way we construct the digraph $\mathbf{G}$ such that

$$\mathbf{G}^* = \mathbf{U}\,. \tag{6}$$

$$\mathbf{G}/\theta = \mathbf{D}\,. \tag{7}$$

$$\mathbf{K}_X \leq_w \mathbf{G} \qquad \text{for any} \quad X \in V^{\mathbf{U}}/\theta\,. \tag{8}$$

Now take the equivalence relation $\theta(\mathbf{G})$. Then

$$\theta = \theta(\mathbf{G})\,.$$

To see this, take vertices v and w such that $v\theta w$ and $v \neq w$. Then $v, w \in X$ for some equivalence class $X \in V^{\mathbf{U}}/\theta$. So $v, w \in E^{\mathbf{K}_X}$. Since $\mathbf{K}_X$ is strongly connected, there is a path going from v to w, and there is a path going from w to v. These two paths are also paths in $\mathbf{G}$ by (8). Obviously their sum form a cycle in $\mathbf{G}$ containing v and w. Thus $v\theta(\mathbf{G})w$. On the other hand, take two different vertices v and w such that $v\theta(\mathbf{G})w$. Then there is a (directed) cycle r containing these vertices. Obviously the image of r is also a cycle in $\mathbf{G}/\theta = \mathbf{D}$ containing v/θ and w/θ. By (3), it is a trivial cycle in $\mathbf{D}$, so $v/\theta = w/\theta$, i.e. $v\theta w$.

By the above equality

$$\mathbf{G}/\theta(\mathbf{G}) = \mathbf{D}\,.$$

Hence and by (5), $\mathbf{G}$ is normal. Next, by (4), $\mathbf{TQ}(\mathbf{G}) = \mathbf{H}$, which completes the proof of (b) $\Longrightarrow$ (a). □

COROLLARY 2.2.2. *Let* $\mathbf{U}$ *be a graph and* $\mathbf{H}$ *a critical digraph. Then the following conditions are equivalent:*

(a) *There is a* **normal** *digraph* $\mathbf{G}$ *such that* $C_n(\mathbf{G}) = \emptyset$, $\mathbf{G}^* \simeq \mathbf{U}$, $\mathbf{TQ}(\mathbf{G}) \simeq \mathbf{H}$.

(b) $\mathbf{H}^*$ *is a weak subgraph (up to isomorphism) of* $\mathbf{U}$ *having the same vertex set and for each regular edge* e *of* $\mathbf{U}$, *there is a (directed) path in* $\mathbf{H}$ *going from one endpoint of* e *to the other.*

Proof. Observe that (b) is the particular case of the condition (b) from Theorem 2.2.1 for the identity relation. Moreover, $\theta(\mathbf{G})$ is the identity relation iff $\mathbf{G}$ does not contain non-trivial cycles. Thus our corollary follows from the proof of Theorem 2.2.1. □

Using Theorem 2.2.1 we obtain the following solution of our algebraic problem.

THEOREM 2.2.3. *Let* $\mathbf{K}$ *and* $\mathbf{L}$ *be lattices. Then the following conditions are equivalent:*

(a) *There is a* **normal** *partial unary algebra* $\mathbf{A}$ *such that*
$$\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{K} \text{ and } \mathbf{S}_s(\mathbf{A}) \simeq \mathbf{L}.$$
(b) $\mathbf{K}$ *and* $\mathbf{L}$ *are lattices such that*
  (b.1) $\mathbf{K}$ *satisfies conditions* (i)–(iv),
  (b.2) $\mathbf{L}$ *is a normal lattice,*
  (b.3) *the graph* $\mathbf{G}(\mathbf{K})$ *and the digraph* $\mathbf{D}(\mathbf{L})$ *satisfy* (b) *of Theorem* 2.2.1.

Proof.

(a) $\Longrightarrow$ (b). First, (b.2) follows from Theorem 1.2.3. Secondly, by [9; Theorem 2.3.12],
$$\mathbf{G}(\mathbf{A})^* \simeq \mathbf{G}(\mathbf{K});$$
and also by Theorem 1.2.7,
$$\mathbf{TQ}\big(\mathbf{G}(\mathbf{A})\big) = \mathbf{TQ}(\mathbf{A}) \simeq \mathbf{D}(\mathbf{L}).$$
Hence and by Theorem 2.2.1 we obtain (a) $\Longrightarrow$ (b), because $\mathbf{G}(\mathbf{A})$ is normal and $\mathbf{D}(\mathbf{L})$ is critical.

(b) $\Longrightarrow$ (a). By assumptions, $\mathbf{G}(\mathbf{K})$ and $\mathbf{D}(\mathbf{L})$ satisfy (b) of Theorem 2.2.1, and $\mathbf{D}(\mathbf{L})$ is a critical digraph. Thus there is a normal digraph $\mathbf{G}$ such that
$$\mathbf{G}^* \simeq \mathbf{G}(\mathbf{K}) \qquad \text{and} \qquad \mathbf{TQ}(\mathbf{G}) \simeq \mathbf{D}(\mathbf{L}).$$
Then by [9; Theorem 2.3.12],
$$\mathbf{S}_w(\mathbf{G}) \simeq \mathbf{K}.$$
Moreover, by Theorem 1.2.4,
$$\mathbf{S}_s(\mathbf{G}) \simeq \mathbf{L},$$
because $\mathbf{L}$ is a normal lattice.

Now, having [9; Theorem 2.2.4], it is sufficient to construct (see [9] for simple details of this construction) a partial unary algebra $\mathbf{A}$ such that $\mathbf{G}(\mathbf{A}) \simeq \mathbf{G}$. Of course, $\mathbf{A}$ is, in particular, normal. □

COROLLARY 2.2.4. *Let* $\mathbf{K}$ *and* $\mathbf{L}$ *be lattices. Then the following conditions are equivalent:*

(a) *There is a* **normal** *partial unary algebra* $\mathbf{A} = \langle A, (k^{\mathbf{A}})_{k \in K} \rangle$ *such that*

(a.1) *for any* $a, b \in A$, $a \neq b \implies \langle a \rangle_{\mathbf{A}} \neq \langle b \rangle_{\mathbf{A}}$.

(a.2) $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{K}$ *and* $\mathbf{S}_s(\mathbf{A}) \simeq \mathbf{L}$.

(b) $\mathbf{K}$ *and* $\mathbf{L}$ *are lattices such that*

(b.1) $\mathbf{K}$ *satisfies conditions* (i)–(iv),

(b.2) $\mathbf{L}$ *is a normal lattice,*

(b.3) *the graph* $\mathbf{G}(\mathbf{K})$ *and the digraph* $\mathbf{D}(\mathbf{L})$ *satisfy* (b) *of Corollary* 2.2.2.

The proof is obtained in the same way as the proof of Theorem 2.2.3. It is sufficient to use Corollary 2.2.2 and the fact (see the end of the first part) that $\mathbf{G}(\mathbf{A})$ does not contain non-trivial cycles iff (a.1) holds.

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