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ON BERNSTEIN TYPE RATIONAL FUNCTIONS OF TWO VARIABLES

ÇIĞDEM ATAKUT* — NURHAYAT İSPİR**

(*Communicated by Pavel Kostyrko*)

ABSTRACT. In this paper the authors define Bernstein type rational functions of two variables and prove the approximation theorems for them. Moreover, asymptotic approximation theorem is proved for Bernstein type rational functions of two variables.

1. Introduction

K. Balazs [1] introduced and considered some approximation properties of the Bernstein type rational functions

$$R_n(f,x) = \frac{1}{(1+a_n x)^n} \sum_{k=0}^{n} \binom{n}{k} (a_n x)^k f\left(\frac{k}{b_n}\right), \tag{1}$$

and proved that if f is continuous in $[0,\infty)$, $f(x) = O(\mathrm{e}^{\alpha x})$ $(x \to \infty)$ with some real numbers α, then in any interval $[0,A]$ $(A > 0)$ the estimate

$$|f(x) - R_n(f,x)| \le c_0 \omega_{2A}(n^{-1/3}) \qquad (0 \le x \le A) \tag{2}$$

holds for sufficiently large n's provided $a_n = n^{-1/3}$, $b_n = n^{2/3}$. Here c_0 depends only on A and α, and $\omega_{2A}(\cdot)$ is the modulus of continuity of f on the interval $[0,2A]$. As it was noted in [1], the convergence of $R_n(f,x)$ holds under the more general conditions $a_n = \frac{b_n}{n} \to 0$, $b_n \to \infty$ $(n \to \infty)$ as well.

The positive linear operator R_n was investigated by J. Szabados [2], K. Balazs [3] and V. Totik [4].

K. Balazs and J. Szabados [2] improved the estimate (2) by an appropriate choice of a_n and b_n whenever f is uniformly continuous on $[0,\infty)$.

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Furthermore they showed that these results could been applied to approximate certain improper integrals by quadrature sums of positive coefficients based on finite number of equidistant nodes. In [4], they settled the saturation properties of $R_n(f,x)$ and proved a general convergence theorem for R_n-like rational functions.

K. Balazs [3] investigated approximation by modified Bernstein type rational functions R_n on the real axis.

In the present paper we define Bernstein type rational functions of two variables and give an estimate, in therm of the first usual moduli of continuity, for the approximation of functions by means of the operators $R_{n,m}$ defined at (3). Moreover, we prove an asymptotic approximation theorem.

Let $f(x,y)$ be a function of two variables, defined in $[0,\infty)\times[0,\infty)$. By Bernstein type rational functions of two variables corresponding to $f(x,y)$ we mean the following:

$$R_{n,m}(f,x,y)=\frac{1}{(1+a_nx)^n}\frac{1}{(1+c_my)^m}\sum_{k=0}^{n}\sum_{j=0}^{m}P_{n,k}(x)P_{m,j}(y)f\left(\frac{k}{b_n},\frac{j}{d_m}\right)$$
$$(n,m=1,2,\dots) \tag{3}$$

where a_n, b_n, c_m, d_m are suitably chosen real numbers, independent of x and y and $P_{n,k}(x)=\binom{n}{k}(a_nx)^k$ and therefore $P_{m,j}(y)=\binom{m}{j}(c_my)^j$.

2. Convergence theorem

Let $R_{n,m}(f,x,y)$ be the functions defined by (3) with $a_n=\frac{b_n}{n}$, $b_n=n^{2/3}$, $c_m=\frac{d_m}{m}$, $d_m=m^{2/3}$ $(n,m=1,2,\dots)$ and let $\omega_{2A}(\delta)$ be the modulus of continuity of the function $f(x,y)$ in $[0,2A]\times[0,2A]$. We shall prove the following theorem:

THEOREM 1. *Let $f(x,y)$ be a continuous function defined in $[0,\infty)\times[0,\infty)$ such that $f(x,y)=O(\mathrm{e}^{\alpha(x+y)})$ $(x\to\infty,\ y\to\infty)$ for some real number α. Then in any square $0\le x\le A$, $0\le y\le A$, $(A\ge 0)$ the inequality*

$$\left|f(x,y)-R_{n,m}(f,x,y)\right|\le c_0\omega_{2A}\left(\sqrt{\frac{1}{n^{2/3}}+\frac{1}{m^{2/3}}}\right) \tag{4}$$

is valid if n,m are sufficiently large, where c_0 is a constant depending on A and α only.

The inequality (4) shows that $R_{n,m}(f,x,y)\to f(x,y)$ when $x\ge 0$, $y\ge 0$ if $n,m\to\infty$, and this convergence is uniform in every finite square.

To prove the theorem some lemmas are needed.

Lemma 1. *If $x \geq 0$, $y \geq 0$, then the following identities hold:*

$$\frac{1}{(1+a_n x)^n} \frac{1}{(1+c_m y)^m} \sum_{k=0}^{n} \sum_{j=0}^{m} P_{n,k}(x) P_{m,j}(y) = 1, \tag{5}$$

$$\frac{1}{(1+a_n x)^n} \frac{1}{(1+c_m y)^m} \sum_{k=0}^{n} \sum_{j=0}^{m} P_{n,k}(x) P_{m,j}(y)(k - b_n x) = -\frac{a_n b_n x^2}{1+a_n x}, \tag{6}$$

$$\frac{1}{(1+a_n x)^n} \frac{1}{(1+c_m y)^m} \sum_{k=0}^{n} \sum_{j=0}^{m} P_{n,k}(x) P_{m,j}(y)(j - d_m y) = -\frac{c_m d_m y^2}{1+c_m y}, \tag{7}$$

$$\frac{1}{(1+a_n x)^n} \frac{1}{(1+c_m y)^m} \sum_{k=0}^{n} \sum_{j=0}^{m} P_{n,k}(x) P_{m,j}(y)(k - b_n x)^2 = \frac{a_n^2 b_n^2 x^4 + b_n x}{(1+a_n x)^2}, \tag{8}$$

$$\frac{1}{(1+a_n x)^n} \frac{1}{(1+c_m y)^m} \sum_{k=0}^{n} \sum_{j=0}^{m} P_{n,k}(x) P_{m,j}(y)(j - d_m y)^2 = \frac{d_m^2 c_m^2 y^4 + d_m y}{(1+c_m y)^2}, \tag{9}$$

$$\begin{aligned} &\frac{1}{(1+a_n x)^n} \frac{1}{(1+c_m y)^m} \sum_{k=0}^{n} \sum_{j=0}^{m} P_{n,k}(x) P_{m,j}(y)(k - b_n x)(j - d_m y) \\ &= \frac{a_n b_n c_m d_m x^2 y^2}{(1+a_n x)(1+c_m y)}, \end{aligned} \tag{10}$$

where $a_n = \frac{b_n}{n}$, $c_m = \frac{d_m}{m}$.

Lemma 2. *If $x \geq 0$, $y \geq 0$, then the inequality*

$$\frac{1}{(1+a_n x)^n} \frac{1}{(1+c_m y)^m} \sum_{\left|\frac{k}{b_n} - x\right| \geq \delta} \sum_{j=0}^{m} P_{n,k}(x) P_{m,j}(y)\, e^{\gamma \frac{k}{b_n}} \leq c_1 \frac{a_n^2 x^4 + \frac{x}{b_n}}{(1+a_n x)^2} \tag{11}$$

holds for sufficiently large n, m, where $\delta > 0$ and γ are arbitrary fixed real numbers, $a_n = \frac{b_n}{n} \to 0$, $b_n \to \infty$, $c_m = \frac{d_m}{m} \to 0$, $d_m \to \infty$ as $n, m \to \infty$.

The proofs of Lemma 1 and Lemma 2 are evident from [1].

It is well known that if λ and δ are arbitrary positive numbers, then

$$\omega_{2A}(\lambda\delta) \leq \omega_{2A}(\delta)(\lambda + 1). \tag{12}$$

Now we prove the convergence theorem.

Proof of Theorem 1. By (3) and (5)

$$
\begin{aligned}
&\Delta_{n,m}(f,x,y) = \\
&= \left| f(x,y) - R_{n,m}(f,x,y) \right| \\
&\leq \frac{1}{(1+a_n x)^n} \frac{1}{(1+c_m y)^m} \sum_{k=0}^{n} \sum_{j=0}^{m} P_{n,k}(x) P_{m,j}(y) \left| f(x,y) - f\left(\frac{k}{b_n}, \frac{j}{d_m}\right) \right| \\
&\leq \frac{1}{(1+a_n x)^n} \frac{1}{(1+c_m y)^m} \left\{ \sum_{\left|\frac{k}{b_n}-x\right| \leq 2A} \sum_{\left|\frac{j}{d_m}-y\right| \leq 2A} + \sum_{\left|\frac{k}{b_n}-x\right| > 2A} \sum_{\left|\frac{j}{d_m}-y\right| > 2A} \right. \qquad (13) \\
&\qquad \left. + \sum_{\left|\frac{k}{b_n}-x\right| \leq 2A} \sum_{\left|\frac{j}{d_m}-y\right| > 2A} + \sum_{\left|\frac{k}{b_n}-x\right| > 2A} \sum_{\left|\frac{j}{d_m}-y\right| \leq 2A} \right\} \\
&= S_1 + S_2 + S_3 + S_4 .
\end{aligned}
$$

We obtain by (12)

$$
\begin{aligned}
\left| f(x,y) - f\left(\frac{k}{b_n}, \frac{j}{d_m}\right) \right| &\leq \omega_{2A}\left(\sqrt{\left(x - \frac{k}{b_n}\right)^2 + \left(y - \frac{j}{d_m}\right)^2} \right) \\
&= \omega_{2A}\left(\delta_{n,m} \cdot \frac{1}{\delta_{n,m}} \sqrt{\left(x - \frac{k}{b_n}\right)^2 + \left(y - \frac{j}{d_m}\right)^2} \right) \\
&\leq \left(\frac{1}{\delta_{n,m}} \sqrt{\left(x - \frac{k}{b_n}\right)^2 + \left(y - \frac{j}{d_m}\right)^2} + 1 \right) \omega_{2A}(\delta_{n,m}) .
\end{aligned}
\tag{14}
$$

By (13), (14) and (5)

$$
\begin{aligned}
S_1 &\leq \omega_{2A}(\delta_{n,m}) \frac{1}{\delta_{n,m}} \frac{1}{(1+a_n x)^n} \frac{1}{(1+c_m y)^m} \sum_{\left|\frac{k}{b_n}-x\right| \leq 2A} \sum_{\left|\frac{j}{d_m}-y\right| \leq 2A} P_{n,k}(x) P_{m,j}(y) \cdot \\
&\quad \cdot \sqrt{\left(x - \frac{k}{b_n}\right)^2 + \left(y - \frac{j}{d_m}\right)^2} + \omega_{2A}(\delta_{n,m}) \\
&= S_1' + \omega_{2A}(\delta_{n,m}) .
\end{aligned}
\tag{15}
$$

Using the Schwarz inequality, then considering (5), (8) and (9) we obtain

$$S_1' \le \omega_{2A}(\delta_{n,m}) \frac{1}{\delta_{n,m}} \left\{ \frac{1}{b_n^2} \frac{1}{(1+a_n x)^n} \frac{1}{(1+c_m y)^m} \cdot \right.$$

$$\cdot \sum_{k=0}^{n} \sum_{j=0}^{m} P_{n,k}(x) P_{m,j}(y) (k - b_n x)^2$$

$$\left. + \frac{1}{d_m^2} \frac{1}{(1+a_n x)^n} \frac{1}{(1+c_m y)^m} \sum_{k=0}^{n} \sum_{j=0}^{m} P_{n,k}(x) P_{m,j}(y) (j - d_m y)^2 \right\}^{1/2}$$

$$= \omega_{2A}(\delta_{n,m}) \frac{1}{\delta_{n,m}} \left\{ \frac{1}{b_n^2} \frac{a_n^2 b_n^2 x^4 + b_n x}{(1+a_n x)^2} + \frac{1}{d_m^2} \frac{d_m^2 c_m^2 y^4 + d_m y}{(1+c_m y)^2} \right\}^{1/2} . \tag{16}$$

Since $b_n = n^{2/3}$ and $d_m = m^{2/3}$, in this case $a_n = \frac{b_n}{n} = n^{-1/3}$ and $c_m = \frac{d_m}{m} = m^{-1/3}$. So by (15) and (16) we have

$$S_1 \le \omega_{2A}(\delta_{n,m}) \left\{ \frac{1}{\delta_{n,m}} \left(\frac{x^4 + x}{n^{2/3}} + \frac{y^4 + y}{m^{2/3}} \right)^{1/2} + 1 \right\} . \tag{17}$$

Since $0 \le x \le A$ and $0 \le y \le A$, then by (17)

$$S_1 \le c_2 \omega_{2A}(\delta_{n,m}) \left\{ \frac{1}{\delta_{n,m}} \left(\frac{1}{n^{2/3}} + \frac{1}{m^{2/3}} \right)^{1/2} + 1 \right\} . \tag{18}$$

where $c_2 = \sqrt{A^4 + A}$.

Assuming $\delta_{n,m} = \left(\frac{1}{n^{2/3}} + \frac{1}{m^{2/3}}\right)^{1/2}$, by (18) we have

$$S_1 \le c_2' \omega_{2A} \left(\sqrt{\frac{1}{n^{2/3}} + \frac{1}{m^{2/3}}} \right) , \tag{19}$$

where $c_2' = 2c_2$.

Since $f(x,y) = O\left(\mathrm{e}^{\alpha(x+y)}\right)$ $(x, y \to \infty,\ \alpha$ fixed), the estimation of S_2 is a trivial consequence of Lemma 2:

$$\begin{aligned} S_2 &\le \frac{1}{(1+a_n x)^n} \frac{1}{(1+c_m y)^m} \sum_{\left|\frac{k}{b_n} - x\right| > 2A} \sum_{\left|\frac{j}{d_m} - y\right| > 2A} P_{n,k}(x) P_{m,j}(y) c_4 \mathrm{e}^{\alpha \frac{k}{b_n}} \mathrm{e}^{\alpha \frac{j}{d_m}} \\ &\le c_1 c_4 c_5 \left(\frac{a_n^2 x^4 + \frac{x}{b_n}}{(1+a_n x)^2} \right) \left(\frac{c_m^2 y^4 + \frac{y}{d_m}}{(1+c_m y)^2} \right) \\ &\le c_1 c_4 c_5 \left(a_n^2 x^4 + \frac{x}{b_n} \right) \left(c_m^2 y^4 + \frac{y}{d_m} \right) \\ &\le \frac{c_6}{n^{2/3} m^{2/3}} (x^4 + x)(y^4 + y) \\ &\le \frac{c_6 c_0^2}{n^{2/3} m^{2/3}} = \frac{c_7}{n^{2/3} m^{2/3}} , \end{aligned} \tag{20}$$

where $c_6 = c_1 c_4 c_5$ and $c_7 = c_6 c_0^2$, $c_0^2 = A^4 + A$.

Since

$$\left| f(x,y) - f\left(\frac{k}{b_n}, \frac{j}{d_m}\right) \right| \leq \left| f(x,y) - f\left(\frac{k}{b_n}, y\right) \right| + \left| f\left(\frac{k}{b_n}, y\right) - f\left(\frac{k}{b_n}, \frac{j}{d_m}\right) \right|,$$

we have

$$\begin{aligned} S_3 \leq & \frac{1}{(1+a_n x)^n} \frac{1}{(1+c_m y)^m} \sum_{\left|\frac{k}{b_n} - x\right| \leq 2A} \sum_{\left|\frac{j}{d_m} - y\right| > 2A} P_{n,k}(x) P_{m,j}(y) \cdot \\ & \qquad \cdot \left| f(x,y) - f\left(\frac{k}{b_n}, y\right) \right| \\ & + \frac{1}{(1+a_n x)^n} \frac{1}{(1+c_m y)^m} \sum_{\left|\frac{k}{b_n} - x\right| \leq 2A} \sum_{\left|\frac{j}{d_m} - y\right| > 2A} P_{n,k}(x) P_{m,j}(y) \cdot \qquad (21) \\ & \qquad \cdot \left| f\left(\frac{k}{b_n}, y\right) - f\left(\frac{k}{b_n}, \frac{j}{d_m}\right) \right| \\ \leq & \, \omega_{2A}\left(\frac{1}{n^{1/3}}\right) \left[(x^4 + x)^{1/2} + 1\right] + \frac{c_8}{m^{2/3}} (y^4 + y) \\ \leq & \, c_9 \left\{ \omega_{2A}\left(\frac{1}{n^{1/3}}\right) + \frac{1}{m^{2/3}} \right\}. \end{aligned}$$

The estimate of S_4 is similar to S_3.

Since

$$\left| f(x,y) - f\left(\frac{k}{b_n}, \frac{j}{d_m}\right) \right| \leq \left| f(x,y) - f\left(x, \frac{j}{d_m}\right) \right| + \left| f\left(x, \frac{j}{d_m}\right) - f\left(\frac{k}{b_n}, \frac{j}{d_m}\right) \right|,$$

then we obtain

$$S_4 \leq c_{10} \left\{ \omega_{2A}\left(\frac{1}{m^{1/3}}\right) + \frac{1}{n^{2/3}} \right\}. \qquad (22)$$

Now, on the basis of (19), (20), (21) and (22), the inequality (13) may be written in the following way

$$\begin{aligned} \Delta_{n,m}(f,x,y) \leq & \, c_2' \omega_{2A}\left(\sqrt{\frac{1}{n^{2/3}} + \frac{1}{m^{2/3}}}\right) + \frac{c_7}{n^{2/3} m^{2/3}} \\ & + c_9 \left\{ \omega_{2A}\left(\frac{1}{n^{1/3}}\right) + \frac{1}{m^{2/3}} \right\} + c_{10} \left\{ \omega_{2A}\left(\frac{1}{m^{1/3}}\right) + \frac{1}{n^{2/3}} \right\} \\ \leq & \, c_0 \left\{ \omega_{2A}\left(\sqrt{\frac{1}{n^{2/3}} + \frac{1}{m^{2/3}}}\right) + \frac{1}{n^{2/3}} + \frac{1}{m^{2/3}} \right. \\ & \left. + \omega_{2A}\left(\frac{1}{n^{1/3}}\right) + \omega_{2A}\left(\frac{1}{m^{1/3}}\right) \right\}. \end{aligned}$$

Since $\delta \le c'_{10}\omega(\delta)$, then

$$\frac{1}{n^{2/3}} + \frac{1}{m^{2/3}} \le \sqrt{\frac{1}{n^{2/3}} + \frac{1}{m^{2/3}}} \le c'_{10}\omega_{2A}\left(\sqrt{\frac{1}{n^{2/3}} + \frac{1}{m^{2/3}}}\right).$$

Also,

$$\frac{1}{n^{1/3}} = \sqrt{\frac{1}{n^{2/3}}} < \sqrt{\frac{1}{n^{2/3}} + \frac{1}{m^{2/3}}}$$

and

$$\omega_{2A}\left(\frac{1}{n^{1/3}}\right) < \omega_{2A}\left(\sqrt{\frac{1}{n^{2/3}} + \frac{1}{m^{2/3}}}\right),$$

we obtain

$$\Delta_{n,m}(f,x,y) \le c_0\omega_{2A}\left(\sqrt{\frac{1}{n^{2/3}} + \frac{1}{m^{2/3}}}\right) \qquad (0 \le x \le A,\ \ 0 \le y \le A),$$

which gives the proof. □

3. Asymptotic approximation

In this part of the paper we prove an asymptotic approximation theorem similar to [1].

THEOREM 2. *Let $f(t,\tau)$ be a function defined in $[0,\infty) \times [0,\infty)$ such that $f(t,\tau) = O\big(e^{\alpha(t+\tau)}\big)$ $(t,\tau \to \infty$, α is a fixed real number), then at each point (x,y) in which f''_{xx}, f''_{xy} and f''_{yy} exist finitely*

$$\begin{aligned} R_{n,m}(f,x,y) = f(x,y) &+ a_n f'_x(x,y)\left(\frac{-x^2}{1+a_n x}\right) + c_m f'_y(x,y)\left(\frac{-y^2}{1+c_m y}\right) \\ &+ a_n f''_{xx}(x,y)\left(\frac{a_n b_n x^4 + \frac{x}{a_n}}{2b_n(1+a_n x)^2}\right) \\ &+ a_n c_m f''_{xy}(x,y)\left(\frac{x^2y^2}{(1+a_n x)(1+c_m y)}\right) \\ &+ c_m f''_{yy}(x,y)\left(\frac{c_m d_m y^4 + \frac{y}{c_m}}{2d_m(1+c_m y)^2}\right) + (a_n + c_m)\rho_{n,m}, \end{aligned} \tag{23}$$

where $\rho_{n,m} \to 0$, $a_n = \frac{b_n}{n} \to 0$, $c_m = \frac{d_m}{m} \to 0$, $\frac{n^{1/2}}{b_n} \to 0$ and $\frac{m^{1/2}}{d_m} \to 0$ as $n,m \to \infty$.

P r o o f . By the conditions of the theorem, f''_{xx}, f''_{xy} and f''_{yy} are finite, thus we may write

$$\begin{aligned} f(t,\tau) = f(x,y) &+ (t-x)f'_x(x,y) + (\tau-y)f'_y(x,y) + \frac{1}{2}(t-x)^2 f''_{xx}(x,y) \\ &+ (t-x)(\tau-y)f''_{xy}(x,y) + \frac{1}{2}(\tau-y)^2 f''_{yy}(x,y) \\ &+ \left[(t-x)^2+(\tau-y)^2\right]\lambda(t,\tau,x,y)\,, \end{aligned} \tag{24}$$

where $\lambda(t,\tau,x,y) \to 0$ as $t \to x$, $\tau \to y$. Hence

$$\begin{aligned} f\left(\frac{k}{b_n},\frac{j}{d_m}\right) &= f(x,y) + \left(\frac{k}{b_n}-x\right)f'_x(x,y) + \left(\frac{j}{d_m}-y\right)f'_y(x,y) \\ &+ \frac{1}{2}\left(\frac{k}{b_n}-x\right)^2 f''_{xx}(x,y) + \left(\frac{k}{b_n}-x\right)\left(\frac{j}{d_m}-y\right)f''_{xy}(x,y) \\ &+ \frac{1}{2}\left(\frac{j}{d_m}-y\right)^2 f''_{yy}(x,y) \\ &+ \left[\left(\frac{k}{b_n}-x\right)^2+\left(\frac{j}{d_m}-y\right)^2\right]\lambda\left(\frac{k}{b_n},\frac{j}{d_m},x,y\right). \end{aligned} \tag{25}$$

Substituting this expression in $R_{n,m}(f,x,y)$ and taking into account the identities (5)–(10), we get

$$\begin{aligned} &R_{n,m}(f,x,y) = \\ &= f(x,y)\frac{1}{(1+a_nx)^n}\frac{1}{(1+c_my)^m}\sum_{k=0}^{n}\sum_{j=0}^{m}P_{n,k}(x)P_{m,j}(y) \\ &+ \frac{f'_x(x,y)}{b_n}\frac{1}{(1+a_nx)^n}\frac{1}{(1+c_my)^m}\sum_{k=0}^{n}\sum_{j=0}^{m}P_{n,k}(x)P_{m,j}(y)(k-b_nx) \\ &+ \frac{f'_y(x,y)}{d_m}\frac{1}{(1+a_nx)^n}\frac{1}{(1+c_my)^m}\sum_{k=0}^{n}\sum_{j=0}^{m}P_{n,k}(x)P_{m,j}(y)(j-d_my) \\ &+ \frac{f''_{xx}(x,y)}{2b_n^2}\frac{1}{(1+a_nx)^n}\frac{1}{(1+c_my)^m}\sum_{k=0}^{n}\sum_{j=0}^{m}P_{n,k}(x)P_{m,j}(y)(k-b_nx)^2 \\ &+ \frac{f''_{xy}(x,y)}{b_nd_m}\frac{1}{(1+a_nx)^n}\frac{1}{(1+c_my)^m}\sum_{k=0}^{n}\sum_{j=0}^{m}P_{n,k}(x)P_{m,j}(y)(k-b_nx)(j-d_my) \\ &+ \frac{f''_{yy}(x,y)}{2d_m^2}\frac{1}{(1+a_nx)^n}\frac{1}{(1+c_my)^m}\sum_{k=0}^{n}\sum_{j=0}^{m}P_{n,k}(x)P_{m,j}(y)(j-d_my)^2 \end{aligned}$$

$$+\frac{1}{(1+a_n x)^n}\frac{1}{(1+c_m y)^m}\sum_{k=0}^{n}\sum_{j=0}^{m}P_{n,k}(x)P_{m,j}(y)\cdot$$
$$\cdot\left[\left(\frac{k}{b_n}-x\right)^2+\left(\frac{j}{d_m}-y\right)^2\right]\lambda\left(\frac{k}{b_n},\frac{j}{d_m},x,y\right) \tag{26}$$
$$=f(x,y)+f'_x(x,y)\left(-\frac{a_n b_n x^2}{b_n(1+a_n x)}\right)+f'_y(x,y)\left(-\frac{c_m d_m y^2}{d_m(1+c_m y)}\right)$$
$$+f''_{xx}(x,y)\left(\frac{a_n^2 b_n^2 x^4+b_n x}{2b_n^2(1+a_n x)^2}\right)+f''_{xy}(x,y)\left(\frac{a_n b_n c_m d_m x^2 y^2}{b_n d_m(1+a_n x)(1+c_m y)}\right)$$
$$+f''_{yy}(x,y)\left(\frac{d_m^2 c_m^2 y^4+d_m y}{2d_m^2(1+c_m y)^2}\right)+r_{n,m},$$

where

$$r_{n,m}=\frac{1}{(1+a_n x)^n}\frac{1}{(1+c_m y)^m}\sum_{k=0}^{n}\sum_{j=0}^{m}P_{n,k}(x)P_{m,j}(y)\cdot$$
$$\cdot\left[\left(\frac{k}{b_n}-x\right)^2+\left(\frac{j}{d_m}-y\right)^2\right]\lambda\left(\frac{k}{b_n},\frac{j}{d_m},x,y\right). \tag{27}$$

For given an $\varepsilon>0$, let us choose $\delta>0$ so small that $|\lambda(t,\tau,x,y)|<\varepsilon$ when $|t-x|<\delta$ and $|\tau-y|<\delta$. For this δ, decompose the sum (27) into four parts:

$$r_{n,m}=\frac{1}{(1+a_n x)^n}\frac{1}{(1+c_m y)^m}\left\{\sum_{|\frac{k}{b_n}-x|\le\delta}\sum_{|\frac{j}{d_m}-y|<\delta}+\sum_{|\frac{k}{b_n}-x|\ge\delta}\sum_{|\frac{j}{d_m}-y|\ge\delta}\right.$$
$$\left.+\sum_{|\frac{k}{b_n}-x|<\delta}\sum_{|\frac{j}{d_m}-y|\ge\delta}+\sum_{|\frac{k}{b_n}-x|\ge\delta}\sum_{|\frac{j}{d_m}-y|<\delta}\right\}$$
$$=A_1+A_2+A_3+A_4. \tag{28}$$

By (8) and (9) and considering the property of $\lambda(t,\tau,x,y)$ we obtain

$$|A_1|<\varepsilon\left\{\frac{a_n^2x^4+\frac{x}{b_n}}{(1+a_n x)^2}+\frac{c_m^2y^4+\frac{y}{d_m}}{(1+c_m y)^2}\right\}. \tag{29}$$

Now we give upper estimation for $|A_2|$. Henceforth c_i, $i=11,12,\dots$, are positive numbers depending only on x,y and α.

By $f(t,\tau)=O\big(\mathrm{e}^{\alpha(t+\tau)}\big)$ $(t,\tau\to\infty,\ \alpha$, fixed), it follows from (25) for some c_{11} that

$$\left|\left[\left(\frac{k}{b_n}-x\right)^2+\left(\frac{j}{d_m}-y\right)^2\right]\lambda\left(\frac{k}{b_n},\frac{j}{d_m},x,y\right)\right|<c_{11}\,\mathrm{e}^{\alpha\frac{k}{b_n}}\,\mathrm{e}^{\alpha\frac{j}{d_m}}. \tag{30}$$

Using (27), (28), (30) and (11) we get

$$|A_2| < c_{12}\left(\frac{a_n^2x^4 + \frac{x}{b_n}}{(1+a_nx)^2}\right)\left(\frac{c_m^2y^4 + \frac{y}{d_m}}{(1+c_my)^2}\right), \tag{31}$$

$$|A_3| < c_{13}\left(\frac{c_m^2y^4 + \frac{y}{d_m}}{(1+c_my)^2}\right), \tag{32}$$

$$|A_4| < c_{14}\left(\frac{a_n^2x^4 + \frac{x}{b_n}}{(1+a_nx)^2}\right). \tag{33}$$

Let now

$$\rho_{n,m} = \frac{r_{n,m}}{a_n + c_m}. \tag{34}$$

By (34), (28), (29), (31), (32) and (33), the relation

$$\begin{aligned}
&\lim_{m\to\infty}\lim_{n\to\infty}|\rho_{n,m}| < \\
&< \lim_{m\to\infty}\lim_{n\to\infty}\Bigg\{\varepsilon\Bigg[\frac{a_n^2x^4}{(a_n+c_m)(1+a_nx)^2} + \frac{x}{b_n(a_n+c_m)(1+a_nx)^2} \\
&+ \frac{c_m^2y^4}{(a_n+c_m)(1+c_my)^2} + \frac{y}{d_m(a_n+c_m)(1+c_my)^2}\Bigg] \\
&+ c_{12}\left(\frac{a_n^2x^4}{(a_n+c_m)(1+a_nx)^2} + \frac{x}{b_n(a_n+c_m)(1+a_nx)^2}\right)\cdot \\
&\cdot\left(\frac{c_m^2y^4}{(1+c_my)^2} + \frac{y}{d_m(1+c_my)^2}\right) \\
&+ c_{13}\left(\frac{c_m^2y^4}{(a_n+c_m)(1+c_my)^2} + \frac{y}{d_m(a_n+c_m)(1+c_my)^2}\right) \\
&+ c_{14}\left(\frac{a_n^2x^4}{(a_n+c_m)(1+a_nx)^2} + \frac{x}{b_n(a_n+c_m)(1+a_nx)^2}\right)\Bigg\} = 0
\end{aligned} \tag{35}$$

holds, because $a_n = \frac{b_n}{n} \to 0$, $\frac{n^{1/2}}{b_n} \to 0$, $c_m = \frac{d_m}{m} \to 0$ and $\frac{m^{1/2}}{d_m} \to 0$ as $n, m \to \infty$.

(26), (27), (34) and (35) give the proof of Theorem 2. □

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