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ON HYPERCENTRAL SUBGROUPS OF INFINITE GROUPS

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ABSTRACT. In this article the hypercentre of infinite groups satisfying certain finiteness conditions is described. Moreover, some criteria for a subgroup to be contained in the hypercentre are obtained.

1. Introduction

Let G be a group. The *upper central series* of G is the ascending normal series whose terms $Z_\alpha(G)$ are defined by the positions

$$Z_0(G) = \{1\}, \qquad Z_{\alpha+1}(G)/Z_\alpha(G) = Z\big(G/Z_\alpha(G)\big)$$

for each ordinal α and

$$Z_\lambda(G) = \bigcup_{\beta<\lambda} Z_\beta(G)$$

if λ is a limit ordinal.

In particular, $Z_1(G)$ is the centre of G, and the last term $\bar{Z}(G)$ of the upper central series of G is the *hypercentre* of G.

The group G is said to be *hypercentral* if it coincides with its hypercentre.

Conditions under which a given nilpotent subgroup of a finite group G is contained in the hypercentre of G have been obtained by several authors. In particular, R. Baer [1] observed that if G is a finite group and x is an element of G whose order is a power of a prime number p, then x belongs to the hypercentre of G if and only if $xy = yx$ for each element y of G with order prime to p. More recently, T. A. Peng ([4], [5]) proved that a (nilpotent) subgroup H of odd order of a finite group G is contained in the hypercentre of G if and only if H lies in the hypercentre of every soluble subgroup of G

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containing H. The assumption that the subgroup H has odd order cannot be omitted here, as for instance the Sylow 2-subgroups of the simple group $PSL(2,31)$ are maximal subgroups.

The aim of this article is to extend the above results to infinite groups. In particular, the hypercentre of an arbitrary group will be described in terms of the behaviour of its elements, and Peng's theorem will be generalized to certain classes of groups satisfying suitable finiteness conditions; as a special case we will obtain in particular that such results holds for groups with Černikov conjugacy classes. Moreover, it will be proved that some embedding properties pertaining to subgroups are countably recognizable.

Most of our notation is standard and can be found in [6].

2. Hypercentral subgroups

If G is any group, the *FC-centre* of G is the subgroup $F(G)$ consisting of all elements of G with finitely many conjugates.

Clearly the FC-centre of a group G contains the centre of G, and so it is natural to introduce the *upper FC-central series* of G as the ascending normal series whose terms $F_\alpha(G)$ are defined by the positions

$$F_0(G) = \{1\}, \quad F_{\alpha+1}(G)/F_\alpha(G) = F\big(G/F_\alpha(G)\big)$$

for each ordinal α and

$$F_\lambda(G) = \bigcup_{\beta<\lambda} F_\beta(G)$$

if λ is a limit ordinal.

The last term $\bar{F}(G)$ of the upper FC-central series of G is called the *FC-hypercentre* of G, and the group G is said to be *FC-hypercentral* if it coincides with its FC-hypercentre. Thus the hypercentre of G is a subgroup of the FC-hypercentre of G, and some properties of hypercentral groups can be carried over to FC-hypercentral groups. Note also that a periodic group G is FC-hypercentral if and only if it is hyperfinite, i.e. if G has an ascending normal series with finite factors. For the main properties of FC-hypercentral groups see [6; Part 1, Chapter 4].

Clearly, the result of Baer quoted in the introduction can be equivalently stated assuming that the product $\langle x\rangle^G\langle y\rangle$ is a nilpotent subgroup for each element y of the finite group G. Therefore our first result provides an extension of Baer's theorem to arbitrary groups.

THEOREM 2.1. *Let G be a group, and let x be an element of G. Then x belongs to the hypercentre of G if and only if x belongs to the FC-hypercentre of G and the subgroup $\langle x\rangle^G\langle y\rangle$ is locally nilpotent for every element y of G.*

Proof. The conditions of the statement are obviously necessary. Conversely, assume that the theorem is false, and let G be a counterexample; replacing G by the factor group $G/\bar{Z}(G)$, we may suppose without loss of generality that G has trivial centre. Let α be the least ordinal such that the subgroup $F_\alpha(G)$ contains a non-trivial element x satisfying the hypotheses. Clearly α is not a limit ordinal, and the coset $xF_{\alpha-1}(G)$ has finitely many conjugates in $G/F_{\alpha-1}(G)$. Let $C/F_{\alpha-1}(G)$ be the centralizer of $\langle x\rangle^G F_{\alpha-1}(G)/F_{\alpha-1}(G)$ in $G/F_{\alpha-1}(G)$, and let c be any element of C. Then $[x,c]$ belongs to $F_{\alpha-1}(G)$ and $\langle [x,c]\rangle^G\langle y\rangle$ is locally nilpotent for all $y\in G$, so that $[x,c]$ lies in the hypercentre of G, and hence $[x,c]=1$. It follows that C is the centralizer of x in G, and so the conjugacy class of x is finite (i.e. $\alpha=1$). Suppose first that the nilpotent group $\langle x\rangle^G$ is not torsion-free, so that it contains a finite non-trivial subgroup N which is normal in G, and N can be chosen to be a p-group for some prime p. As $Z(G)=\{1\}$, the finite group $G/C_G(N)$ must contain an element $yC_G(N)$ of prime order $q\neq p$, and hence the group $N\langle y\rangle$ is not nilpotent. This contradiction shows that $\langle x\rangle^G$ is torsion-free. Let z be any element of $G\setminus C_G(\langle x\rangle^G)$, and let n be a positive integer such that $z^n\in C_G(\langle x\rangle^G)$. Then $\langle z^n\rangle$ is a normal subgroup of the finitely generated nilpotent group $X=\langle x\rangle^G\langle z\rangle$, so that the index $|\langle z\rangle^X:\langle z\rangle|$ is finite, and z belongs to the FC-centre of X. Therefore X is an FC-group, so that its commutator subgroup X' is periodic; on the other hand X' is contained in $\langle x\rangle^G$, so that $X'=\{1\}$ and X is abelian. This last contradiction completes the proof of the theorem. □

Corollary 2.2. *Let G be an FC-hypercentral group, and let x be an element of G. Then x belongs to the hypercentre of G if and only if the subgroup $\langle x\rangle^G\langle y\rangle$ is locally nilpotent for every element y of G.*

Let G be a group. Recall that an element x of G is said to be a *right Engel* element if for each element g of G there is a positive integer n such that $[x,g^n]=1$.

The set $R(G)$ of all right Engel elements of G contains the hypercentre of G, and many authors have obtained conditions under which $R(G)=\bar{Z}(G)$. In particular, this is the case if the group G satisfies either the minimal or the maximal condition on abelian subgroups (for details on this subject see [6; Part 2, Chapter 7]). As a consequence of Theorem 2.1, we prove here that a similar result holds for FC-hypercentral groups.

Corollary 2.3. *Let G be an FC-hypercentral group. Then $R(G)=\bar{Z}(G)$.*

Proof. Since G has an ascending series whose factors locally satisfies the maximal condition, the set $R(G)$ is a locally nilpotent normal subgroup (see [6; Part 2, p. 59]). If x is any right Engel element of G, it follows that the normal closure $\langle x\rangle^G$ is locally nilpotent, and so even hypercentral (see [6; Part 1, Theorem 4.38]). Let y be any element of G, and put $X = \langle x\rangle^G\langle y\rangle$. Suppose first that G is hyperabelian. Then $\langle y\rangle$ is an ascendant subgroup of X (see [6; Part 2, Corollary 1 to Theorem 7.34]), and hence X is locally nilpotent, so that x belongs to $\bar{Z}(G)$ by Corollary 2.2. In the general case, as the subgroup X is hyperabelian, the above argument shows that $\bar{Z}(X) = R(X)$, so that $\langle x\rangle^G$ is contained in $\bar{Z}(X)$, and X is hypercentral. Therefore it follows again from Corollary 2.2 that x lies in the hypercentre of G. □

Let $\mathfrak{X}$ be a class of groups. A normal subgroup N of a group G is said to be *G-hyper-$\mathfrak{X}$* if it has an ascending series consisting of normal subgroups of G whose factors belong to the class $\mathfrak{X}$.

In particular, a normal subgroup N of a group G is *G-hyperfinite* if it has an ascending G-invariant series with finite factors.

COROLLARY 2.4. *Let G be a periodic group, and let x be an element of G whose order is a power of a prime number p. Then x belongs to the hypercentre of G if and only if the normal closure $\langle x\rangle^G$ is a G-hyperfinite subgroup and $xy = yx$ for each element y of G whose order is prime to p.*

P r o o f . The condition of the statement is obviously necessary. Conversely, suppose that the subgroup $\langle x\rangle^G$ is G-hyperfinite and $xy = yx$ for all elements y of G whose order is prime to p. Let X be any finite set of conjugates of x. Since the result holds in the case of finite groups (see [1]), it follows that the finite subgroup $\langle X\rangle$ is nilpotent, so that $\langle x\rangle^G$ is locally nilpotent, and hence it is a p-group. Therefore $\langle x\rangle^G\langle g\rangle$ is locally nilpotent for every element g of G, and x belongs to the hypercentre of G by Theorem 2.1. □

In the above statement the assumption that the normal subgroup $\langle x\rangle^G$ is G-hyperfinite cannot be omitted, even if the group G is locally finite. In fact, the famous example by H e i n e k e n and M o h a m e d [3] shows that there exist infinite periodic metabelian primary groups with trivial centre. The same remark proves that also Theorem 2.1 cannot be improved.

COROLLARY 2.5. *Let G be a hyperfinite group, and let x be an element of G whose order is a power of a prime number p. Then x belongs to the hypercentre of G if and only if $xy = yx$ for each element y of G whose order is prime to p.*

COROLLARY 2.6. *Let G be a locally finite group, and let P be a normal p-subgroup of G (where p is a prime number). Then P is contained in the hypercentre $Z(G)$ of G if and only if P is G-hyperfinite and it is a direct factor of every subgroup S of G such that $P \leq S$ and S/P is a p'-group.*

P r o o f . Suppose first that P lies in $Z(G)$, so that obviously P is G-hyperfinite. Let S/P be any p'-subgroup of G/P; then S is a subgroup of $PC_G(P)$ by Corollary 2.4, so that S splits over P and hence P is a direct factor of

Conversely, let x be any element of P. Then $\langle x\rangle^G$ is contained in P and so it is G-hyperfinite. Moreover, if y is any element of G with order prime to p, we have that P is a direct factor of $\langle P,y\rangle$ and hence $xy=yx$. Therefore x belongs to $\bar{Z}(G)$ by Corollary 2.4, and so P is a subgroup of $\bar{Z}(G)$. □

COROLLARY 2.7. *Let G be a hyperfinite group, and let P be a normal p-subgroup of G (where p is a prime number). Then P is contained in the hypercentre $\bar{Z}(G)$ of G if and only if it is a direct factor of every subgroup S of G such that $P\leq S$ and S/P is a p'-group.*

It was also proved by Peng [4] that the size of hypercentre of a finite group G essentially depends on the behaviour of the subgroups of G whose order is divisible by at most two prime numbers. Our next corollary deals with the same question in the case of hyperfinite groups.

COROLLARY 2.8. *Let G be a hyperfinite group, and let H be a subgroup of G such that $H\cap E$ is contained in $\bar{Z}(E)$ for every finite subgroup E of G with $|\pi(E)|=2$. Then H is contained in the hypercentre of G.*

Proof. Let x be an element of H with order a power of a prime number p, and let y be any element of G whose order is prime to p. The hypotheses are clearly inherited by the finite group $\langle x,y\rangle$ and its subgroup $H\cap\langle x,y\rangle$, so that $H\cap\langle x,y\rangle$ is contained in $\bar{Z}(\langle x,y\rangle)$ (see [4]) and $xy=yx$. Therefore x belongs to $\bar{Z}(G)$ by Corollary 2.5, and hence $H\leq\bar{Z}(G)$. □

As we mentioned in the introduction, the main result of [5] proves that a subgroup H of odd order of a finite group G is contained in $\bar{Z}(G)$ if and only if H is a subgroup of $\bar{Z}(K)$ for every soluble subgroup K of G containing H. We will now obtain some extensions of this theorem to infinite groups; in our statements the hypothesis that the subgroup H has no elements of order 2 will be weakened assuming that H is $\mathfrak{F}_2$-perfect.

Recall that, if p is a prime number, a group is called *$\mathfrak{F}_p$-perfect* if it has no homomorphic images that are finite non-trivial p-groups; in particular, the class of $\mathfrak{F}_p$-perfect groups contains all p-radicable groups and all periodic groups with no elements of order p.

THEOREM 2.9. *Let G be a periodic hyperabelian-by-finite group, and let H be a hypercentral $\mathfrak{F}_2$-perfect subgroup of G such that $H\leq\bar{Z}(K)$ for every hyperabelian subgroup K of G containing H. Then H is contained in the hypercentre of G.*

Proof. Clearly every hyperabelian normal subgroup of finite index of G has an ascending series with abelian factors consisting of normal subgroups of G, so that in particular there exists a largest normal subgroup L of G with such property, and the finite group G/L has trivial centre. Let U/L be any soluble subgroup of G/L containing HL/L. Then U is hyperabelian, so that H lies in $\bar{Z}(U)$ and hence HL/L is a subgroup of $\bar{Z}(U/L)$. As HL/L is a nilpotent group of odd order, it follows from Peng's theorem that HL/L is also contained in the hypercentre of G/L, so that $H \leq L$. Thus H is a subgroup of $\bar{Z}(L)$, and so $H^G \leq \bar{Z}(L)$. In particular, H^G is hypercentral and it is also G-hyperfinite, as the index $|G : L|$ is finite. Let x be any element of H whose order is a power of a prime number p, and let y be an element of G with order prime to p. Since the group $H^G\langle y\rangle$ is hyperabelian, we have that H is contained in the hypercentre of $\langle H, y\rangle$, and so $xy = yx$ by Corollary 2.4. The same result yields now that x belongs to the hypercentre of G, so that $H \leq \bar{Z}(G)$ and the theorem is proved. □

Lemma 2.10. *Let G be a hyper- (abelian or finite) group, and let J be the subgroup generated by all hyperabelian normal subgroups of G. Then J has an ascending series with abelian factors consisting of normal subgroups of G.*

P r o o f. It is clearly enough to prove that J contains an abelian non-trivial normal subgroup of G, provided that $J \neq \{1\}$. Let N be a hyperabelian non-trivial normal subgroup of G. Since G is hyper- (abelian or finite), every non-trivial normal subgroup of G must contain a non-trivial normal subgroup of G which is either abelian or finite; in particular, N contains an abelian non-trivial normal subgroup A of G, and clearly A is a subgroup of J. The lemma is proved. □

A group G is said to be a *ZAF-group* if it has an ascending normal series whose factors are either finite or central.

Clearly the class of ZAF-groups is closed with respect to forming subgroups and homomorphic images; moreover, all hypercentral and all hyperfinite groups have the property ZAF. Note also that every ZAF-group is FC-hypercentral, and so periodic ZAF-groups are hyperfinite.

Our next lemma shows in particular that in any ZAF-group the set of all elements of finite order is a subgroup, and that torsion-free ZAF-groups are locally nilpotent, and so even hypercentral.

Lemma 2.11. *Every ZAF-group is locally finite-by-nilpotent.*

P r o o f. Assume by contradiction that there exists a finitely generated ZAF-group G which is not finite-by-nilpotent. Let

$$\{1\} = G_0 < G_1 < \cdots < G_\alpha < G_{\alpha+1} < \cdots < G_\tau = G$$

be an ascending normal series whose factors are either finite or central, and consider the least ordinal $\alpha \leq \tau$ such that G/G_α is finite-by-nilpotent. Then G_α is the normal closure of a finite subset of G, and hence α is not a limit ordinal. Since either $G_\alpha/G_{\alpha-1}$ is finite or it is contained in the centre of $G/G_{\alpha-1}$, it follows that also $G/G_{\alpha-1}$ is finite-by-nilpotent (see [6; Part 1, Theorem 4.25, Corollary 2 to Theorem 4.21]). This contradiction proves the lemma. $\square$

It is well-known that locally nilpotent FC-hypercentral groups are hypercentral (see [6; Part 1, Theorem 4.38]). We will now prove that the hypercentre of an FC-hypercentral group has a "local" characterization.

LEMMA 2.12. *Let G be an FC-hypercentral group, and let x be an element of G which belongs to the hypercentre of every finitely generated subgroup of G. Then x also belongs to the hypercentre of G.*

Proof. Assume by contradiction that x does not belong to $\bar{Z}(G)$. Replacing G by the factor group $G/\bar{Z}(G)$, it can be assumed without loss of generality that G has trivial centre. Since G is FC-hypercentral, it follows that the normal closure $\langle x\rangle^G$ must contain a non-trivial element a with finitely many conjugates in G, and $\langle a\rangle^G$ is contained in $\langle X\rangle$, where X is a suitable finite set of conjugates of x. As the index $|G : C_G(\langle a\rangle^G)|$ is finite, we have $G = EC_G(\langle a\rangle^G)$ for some finitely generated subgroup E of G containing X. By hypothesis X is contained in the hypercentre of E, and hence $\langle a\rangle^G \cap Z(E) \neq \{1\}$. On the other hand, $\langle a\rangle^G \cap Z(E)$ is contained in $Z(G)$, and this contradiction proves that x is an element of $\bar{Z}(G)$. $\square$

THEOREM 2.13. *Let G be a ZAF-group, and let H be a hypercentral $\mathfrak{F}_2$-perfect subgroup of G such that $H \leq \bar{Z}(K)$ for every hyperabelian subgroup K of G containing H. Then H is contained in the hypercentre of G.*

Proof. Let

$$\{1\} = G_0 < G_1 < \cdots < G_\alpha < G_{\alpha+1} < \cdots < G_\tau = G$$

be an ascending normal series of G whose factors are either finite or central. Assume by contradiction that H is not contained in the hypercentre of G, and let $\alpha \leq \tau$ be the least ordinal such that H is not a subgroup of $\bar{Z}(HG_\alpha)$. Consider an element x of $H \setminus \bar{Z}(HG_\alpha)$. If α is a limit ordinal, then

$$HG_\alpha = \bigcup_{\beta<\alpha} HG_\beta$$

and hence x belongs to the hypercentre of every finitely generated subgroup of HG_α, contradicting Lemma 2.12. Therefore α is not a limit ordinal, and H is a subgroup of the hypercentre of $HG_{\alpha-1}$. Put $X = HG_\alpha$ and $Y = HG_{\alpha-1}$. Suppose first that $G_\alpha/G_{\alpha-1}$ is contained in the centre of $G/G_{\alpha-1}$. Then Y is a normal subgroup of X, and so also the hypercentre $\bar{Z}(Y)$ of Y is normal in X. Since H is contained in $\bar{Z}(Y)$, it follows that H^X is a hypercentral normal subgroup of X. Let y be any element of X; then $H^X\langle y\rangle$ is hyperabelian and so every conjugate of H under the action of X lies in the hypercentre of $H^X\langle y\rangle$. Thus H^X is contained in $\bar{Z}(H^X\langle y\rangle)$, and in particular $\langle x\rangle^X\langle y\rangle$ is locally nilpotent, so that x belongs to the hypercentre of X by Corollary 2.2, a contradiction. Suppose now that $G_\alpha/G_{\alpha-1}$ is finite. Then the index $|X : Y$ is finite, so that X/N is finite, where $N = Y_X$ is the core of Y in X. Let J be the subgroup generated by all hyperabelian normal subgroups of X; then $H \cap N \leq \bar{Z}(N) \leq J$, and hence $H/H\cap J$ is also finite. It follows that $\bar{H} = HJ/J$ is a finite nilpotent subgroup of odd order of the group $\bar{X} = X/J$. Let $\bar{U}$ be any finite soluble subgroup of $\bar{X}$ containing $\bar{H}$. Then U is hyperabelian by Lemma 2.10, so that H is contained in $\bar{Z}(U)$, and hence H is a subgroup of $\bar{Z}(\bar{U})$. As the statement holds for finite groups, we obtain that $\bar{H}$ is contained in the hypercentre of every finite subgroup of $\bar{X}$ containing $\bar{H}$. Since $\bar{X}$ is a ZAF-group, the set $\bar{T} = T/J$ of all its elements of finite order is a hyperfinite subgroup, and so $\bar{H}$ lies in $\bar{Z}(\bar{T}) = W/J$ by Corollary 2.5. Clearly W is a hyperabelian normal subgroup of X, and hence also H^X is hyperabelian. As in the first case, we can now reach a contradiction using Corollary 2.2. The theorem is proved. □

COROLLARY 2.14. *Let G be a hyperfinite group, and let H be a hypercentral $\mathfrak{F}_2$-perfect subgroup of G such that $H \leq \bar{Z}(K)$ for every hyperabelian subgroup K of G containing H. Then H is contained in the hypercentre of G.*

A group G is said to be a CC*-group* (or to have *Černikov conjugacy classes* if $G/C_G(\langle x\rangle^G)$ is a Černikov group for each element x of G.

It is well-known that if G is a CC-group, its commutator subgroup G' i covered by Černikov normal subgroups of G (see [6; Part 1, Theorem 4.36]), so that in particular G' is G-hyperfinite, and hence G is a ZAF-group. Thus our next result follows immediately from Theorem 2.13.

COROLLARY 2.15. *Let G be a CC-group. If H is a hypercentral $\mathfrak{F}_2$-perfect subgroup of G such that $H \leq \bar{Z}(K)$ for every hyperabelian subgroup K of G containing H, then H is contained in the hypercentre of G.*

3. Countable subgroups

A group class $\mathfrak{X}$ is said to be *countably recognizable* if a group G belongs to $\mathfrak{X}$ provided that all its countable subgroups are $\mathfrak{X}$-groups.

Many countably recognizable group classes were determined by Baer [2]; in particular, he proved that hypercentral groups, FC-hypercentral groups and hyperfinite groups form countably recognizable classes of groups. This can also be obtained as a consequence of our results in this section.

THEOREM 3.1. *Let G be a group, and let H be a countable subgroup of G such that $H \leq \bar{F}(X)$ for every countable subgroup X of G containing H. Then H is a subgroup of the FC-hypercentre of G.*

Proof. Let N be the intersection of the FC-hypercentre of G with the normal closure H^G of H, and let K/N be any countable subgroup of G/N containing HN/N. Then $K = LN$, where L is a countable subgroup of G, and H is contained in the FC-hypercentre of the countable subgroup $\langle H, L\rangle$. It follows that HN/N lies in the FC-hypercentre of K/N, so that the hypotheses are inherited by the group G/N and its countable subgroup HN/N. Assume by contradiction that the statement is false, so that H is not contained in N. Replacing G by G/N it can be assumed without loss of generality that the normal closure H^G contains no non-trivial elements having finitely many conjugates in G. Write

$$H = \{h_n : n \in \mathbb{N}_0\}$$

where $h_0 = 1$, and put $X_0 = \{1\}$. Suppose now that for some non-negative integer n a countable subgroup X_n of G has been defined containing the elements $h_0, \ldots, h_n$. Clearly there exists a countable subset W_n of G such that every non-trivial element of $H^G \cap X_n$ has infinitely many conjugates under the action of W_n. Consider the countable subgroup $X_{n+1} = \langle X_n, h_{n+1}, W_n\rangle$, and put

$$X = \bigcup_{n\in\mathbb{N}_0} X_n .$$

Thus X is a countable subgroup of G containing H, and hence H lies in the FC-hypercentre of X. Let u be a non-trivial element of $H^X \cap F_1(X)$, and let m be a positive integer such that u belongs to X_m; then u has infinitely many conjugates under the action of X_{m+1}. This contradiction proves the theorem. □

COROLLARY 3.2. *Let G be a group, and let H be a countable subgroup of G such that $H \leq \bar{Z}(X)$ for every countable subgroup X of G containing H. Then H is contained in the hypercentre of G.*

Proof. Assume by contradiction that the statement is false. As in the proof of Theorem 3.1 it can be assumed without loss of generality that the normal closure H^G has trivial intersection with the centre of G. Moreover, the same theorem yields that H is contained in the FC-hypercentre of G, so that H^G contains a non-trivial element u with finitely many conjugates in G, and u belongs to H^X, where X is a countable subgroup of G containing H. Then $G/C_G(\langle u\rangle^G)$ is finite, and there exists a finitely generated subgroup E of G such that $G = C_G(\langle u\rangle^G)E$. By hypothesis H is contained in the hypercentre of the countable subgroup $\langle X, E\rangle$, so that also $\langle u\rangle^G = \langle u\rangle^E$ lies in $\bar{Z}(\langle X, E\rangle)$. Thus $\langle u\rangle^G \cap Z(\langle X, E\rangle)$ is a non-trivial subgroup of $Z(G)$, a contradiction. □

COROLLARY 3.3. *Let G be a group, and let H be a countable subgroup of G such that H^X is X-hyperfinite for every countable subgroup X of G containing H. Then the normal closure H^G is a G-hyperfinite subgroup.*

Proof. By Theorem 3.1 the subgroup H is contained in the FC-hypercentre of G, so that also its normal closure H^G lies in $\bar{F}(G)$; moreover, H^G is clearly periodic, so that it is a G-hyperfinite subgroup by Dietzmann's lemma. □

Clearly every group G contains a largest G-hypercyclic subgroup, and so a proof similar to that of Theorem 3.1 can be used to obtain the following result.

COROLLARY 3.4. *Let G be a group, and let H be a countable subgroup of G such that H^X is X-hypercyclic for every countable subgroup X of G containing H. Then the normal closure H^G is a G-hypercyclic subgroup.*

In the case of normal subgroups, Corollary 3.2 can also be obtained as a consequence of our last result that describes the subgroups of the hypercentre of a group in terms of commutators. This is an extension of the characterization of hypercentral groups given by S. N. Černikov (see [6; Part 1, Theorem 2.19]), and we omit here the proof, since it is a slight modification of the argument used in that case. Note also that in our statement the assumption that the subgroup H is normal cannot be dropped out, as in any group each subgroup of order 2 satisfies the condition on the sequences.

THEOREM 3.5. *Let G be a group, and let H be a normal subgroup of G. Then H is contained in the hypercentre of G if and only if for all sequences $(x_n)_{n\in\mathbb{N}}$ of elements of H and $(g_n)_{n\in\mathbb{N}}$ of elements of G such that $[x_n, g_n] = x_{n+1}$ for all n, there exists a positive integer m such that $x_m = 1$.*

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