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SETS OF POSITIVE INTEGERS WITH PRESCRIBED VALUES OF DENSITIES

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ABSTRACT. Two generalizations of a theorem in [STRAUCH, O.—TÓTH, J. T.: *Asymptotic density of* $A \subset \mathbb{N}$ *and density of the ratio set* $R(A)$, Acta Arith. **87** (1998), 67–78] are presented in the paper. A set with prescribed lower and upper asymptotic and logarithmic densities is constructed for every possible values. Also a set with prescribed lower and upper densities with respect to a general function under some conditions is constructed.

Preliminaries

Denote by $\mathbb{N}$ and $\mathbb{R}^+$ the set of all positive integers and positive real numbers, respectively. For a set $A \subset \mathbb{N}$ denote by $R(A) = \left\{\frac{a}{b} : a \in A,\ b \in A\right\}$ the *ratio set of* A and say that a set A is (R)-*dense* if $R(A)$ is (topologically) dense in the set $\mathbb{R}^+$. The density of $R(A)$ was first investigated in [Š]. The set of all accumulation points of $R(A)$ is characterized in [B-T]. In the paper [S-T2] distribution functions of ratio sequences $\left(\frac{a_n}{a_m}\right)_{n,m=1}^{\infty} = R(A)$, where $A = \{a_1 < a_2 < \dots < a_n < \dots\} \subset \mathbb{N}$ are studied. Let us notice that the density of $R(A)$ in the interval $(0,1)$ is sufficient for the (R)-density of a set A.

For $A \subset \mathbb{N}$ and $x > 0$ let $A(x) = \operatorname{card}\{a \in A : a \leq x\}$. We will use this notation also in cases when $A \cap \{1,2,\dots,k\}$, $k \geq x$, is already defined although the whole set A is undetermined yet.

Define

$$\underline{d}(A) = \liminf_{x\to\infty} \frac{A(x)}{x}, \qquad \overline{d}(A) = \limsup_{x\to\infty} \frac{A(x)}{x},$$
$$d(A) = \lim_{x\to\infty} \frac{A(x)}{x}$$

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the *lower asymptotic density*, *upper asymptotic density*, and *asymptotic density* (if defined), respectively.

Similarly, define

$$\underline{\delta}(A) = \liminf_{x\to\infty} \frac{\sum\limits_{a\in A,\, a\le x} \frac{1}{a}}{\ln x}, \qquad \overline{\delta}(A) = \limsup_{x\to\infty} \frac{\sum\limits_{a\in A,\, a\le x} \frac{1}{a}}{\ln x}$$

$$\delta(A) = \lim_{x\to\infty} \frac{\sum\limits_{a\in A,\, a\le x} \frac{1}{a}}{\ln x}$$

the *lower logarithmic density*, *upper logarithmic density*, and *logarithmic density* (if defined), respectively.

These values are related by the well-known inequalities holding for each set $A \subset \mathbb{N}$

$$\underline{d}(A) \le \underline{\delta}(A) \le \overline{\delta}(A) \le \overline{d}(A)\,.$$

In [S-T1] mainly the relation between (R)-density and asymptotic density is studied. Among others the following theorem is proved.

THEOREM. (Strauch-Tóth) *Let $0 \le \alpha \le \beta \le 1$. Then there is an (R)-dense set $A \subset \mathbb{N}$ such that*

$$\underline{d}(A) = \alpha\,, \qquad \overline{d}(A) = \beta\,.$$

The purpose of the present paper is to generalize this result in two different directions. Related problems and constructions by induction similar to these used in the present paper appear also in the paper [G].

Sets with prescribed densities with respect to general functions

Let $f\colon \mathbb{N} \to \mathbb{R}^+$ be an arbitrary function. For a set $A \subset \mathbb{N}$ denote by χ_A the characteristic function of A and for a positive integer n let us denote

$$S_f(A,n) = \sum_{i-1}^{n} f(i)\chi_A(i)\,, \qquad S_f(n) = S_f(\mathbb{N},n)\,, \qquad \phi_f(A,n) = \frac{S_f(A,n)}{S_f(n)}$$

Densities with respect to an arbitrary function are defined as follows:

$$\underline{d}_f(A) = \liminf_{x\to\infty} \phi_f(A,n)\,, \qquad \overline{d}_f(A) = \limsup_{x\to\infty} \phi_f(A,n)\,,$$

$$d_f(A) = \lim_{n\to\infty} \phi_f(A,n)$$

are the *lower f density*, *upper f density* and *f-density* (if defined), respectively

Remark. The asymptotic density is the f-density corresponding to $f(n) = 1$ for all $n \in \mathbb{N}$. The logarithmic density is the f-density corresponding to $f(n) = \frac{1}{n}$ for all $n \in \mathbb{N}$.

The following theorem is a generalization of the above mentioned Theorem by Strauch and Tóth.

THEOREM 1. *Let $f\colon \mathbb{N} \to \mathbb{R}^+$ be a function fulfilling the conditions*

$$\sum_{n=1}^{\infty} f(n) = \infty\,, \tag{D}$$

$$\lim_{n\to\infty} \frac{f(n)}{S_f(n)} = 0 \tag{Z}$$

and let $0 \le \alpha \le \beta \le 1$ be arbitrary. Then there exists an (R)-dense set $A \subset \mathbb{N}$ such that

$$\underline{d}_f(A) = \alpha \qquad \textit{and} \qquad \overline{d}_f(A) = \beta\,.$$

Before proving Theorem 1 we will prove the following lemma.

LEMMA 1. *Let $f\colon \mathbb{N} \to \mathbb{R}^+$ be a function fulfilling both conditions* (D) *and* (Z). *Then there is an (R)-dense set $D \subset \mathbb{N}$ with $d_f(D) = 0$.*

Proof. Let $\{r_n\}_{n=1}^{\infty}$ be a sequence of all rational numbers in the interval $(0,1)$. We will construct the set D by induction.

By (Z) there exists a $k_1 \in \mathbb{N}$ such that for all $n > k_1$ the inequality $\frac{f(n)}{S_f(n)} < \frac{1}{3}$ holds. Let $q_1 > p_1 > k_1$ be such that $\frac{p_1}{q_1} = r_1$. Put $D_1 = \{p_1, q_1\}$. Then

$$\begin{aligned}
\phi_f(D_1, n) &= 0 && \text{if} \quad n < p_1\,,\\
\phi_f(D_1, n) &< \tfrac{1}{3} && \text{if} \quad p_1 \le n < q_1\,,\\
\phi_f(D_1, n) &= \tfrac{f(p_1)+f(q_1)}{S_f(n)} && \text{if} \quad q_1 \le n\,.\\
&< \tfrac{f(p_1)}{S_f(p_1)} + \tfrac{f(q_1)}{S_f(q_1)} < \tfrac{2}{3}
\end{aligned}$$

Thus $\phi_f(D_1, n) < 1$ for all $n \in \mathbb{N}$.

Induction step:

Suppose that the sets $D_1 \subset D_2 \subset \dots \subset D_{n-1}$ and positive integers $p_1 < p_2 < \dots < p_{n-1}$, $q_1 < q_2 < \dots < q_{n-1}$, $k_1 < k_2 < \dots < k_{n-1}$ are defined such that

(i) $\frac{p_i}{q_i} = r_i$, $i = 1, 2, \dots, n-1$;

(ii) $D_i \subset \{1, 2, \dots, k_{i+1}\}$, $i = 1, 2, \dots, n-2$;

(iii) $D_{i+1} \cap \{1, 2, \dots, k_{i+1}\} = D_i$, $i = 1, 2, \dots, n-2$;

(iv) $\phi_f(D_{n-1}, m) < \frac{1}{i}$ for all $m > k_i$, $i = 1, 2, \dots, n-1$.

By (D) and (Z), there exists a positive integer $k_n > \max D_{n-1}$ such that

$$\phi_f(D_{n-1}, k_n) < \frac{1}{3n}, \tag{a}$$

$$\frac{f(i)}{S_f(i)} < \frac{1}{3n}, \qquad i > k_n. \tag{b}$$

Choose positive integers p_n, q_n such that $q_n > p_n > k_n$ and $\frac{p_n}{q_n} = r_n$. Set $D_n = D_{n-1} \cup \{p_n, q_n\}$. Then relations in (i) and (iv) hold also for n and relations in (ii) and (iii) hold also for $n-1$.

To verify (iv) for n calculate using (a) and (b)

$$\phi_f(D_n, m) \le \frac{S_f(D_{n-1}, k_n)}{S_f(k_n)} = \phi_f(D_{n-1}, k_n) < \frac{1}{3n}, \qquad k_n \le m < p_n,$$

$$\phi_f(D_n, m) \le \frac{S_f(D_{n-1}, k_n)}{S_f(k_n)} + \frac{f(p_n)}{S_f(p_n)} < \frac{2}{3n}, \qquad p_n \le m < q_n,$$

$$\phi_f(D_n, m) \le \frac{S_f(D_{n-1}, k_n)}{S_f(k_n)} + \frac{f(p_n)}{S_f(p_n)} + \frac{f(q_n)}{S_f(q_n)} < \frac{1}{n}, \qquad m \ge q_n.$$

After completing induction the relations in (i)–(iv) hold for every $n \in \mathbb{N}$. Put $D = \bigcup_{n=1}^{\infty} D_n$ to get the required set. It is (R)-dense by (i) and $d_f(D) = 0$ holds by (iv). □

P r o o f o f T h e o r e m 1 . First, we will construct by induction a set $B = \bigcup_{i=1}^{\infty} \langle p_i, q_i) \cap \mathbb{N}$ such that $\underline{d}_f(B) = \alpha$ and $\overline{d}_f(B) = \beta$.

Let $p_1 = 1$, $q_1 = 2$.

Induction step.

Let $p_1 < q_1 < p_2 < q_2 < \cdots < p_{n-1} < q_{n-1}$ be defined. Let p_n be the first positive integer greater than q_{n-1} such that

$$\phi_f\left(\bigcup_{i=1}^{n-1} \langle p_i, q_i) \cap \mathbb{N},\ p_n\right) < \alpha + \frac{1}{n}$$

and let q_n be the first positive integer greater than p_n such that

$$\phi_f\left(\bigcup_{i=1}^{n} \langle p_i, q_i) \cap \mathbb{N},\ q_n\right) > \beta - \frac{1}{n}.$$

Both p_n and q_n are guaranteed by property (D). Then $\underline{d}_f(B) = \alpha$ and $\overline{d}_f(B) = \beta$ by property (Z).

To get the set A in Theorem 1, set $A = B \cup D$ where D is the set guaranteed by Lemma 1. Then the set A, as a superset of (R)-dense set, is (R)-dense as well. Also

$$\phi_f(A,n) \le \phi_f(B,n) + \phi_f(D,n)$$

where the limit of the last term as n tends to infinity is zero. Thus

$$\underline{d}_f(A) = \underline{d}_f(B) = \alpha\,, \qquad \overline{d}_f(A) = \overline{d}_f(B) = \beta\,.$$

□

Sets with prescribed both asymptotic and logarithmic densities

In this section we will use the notation $S_0(A,n)$ and $\phi_0(A,n)$ as defined in the previous section for the function $f(n) = 1$ corresponding to the asymptotic density and $S_1(A,n)$ and $\phi_1(A,n)$ as defined in the previous section for the function $f(n) = \frac{1}{n}$ corresponding to the logarithmic density. The following theorem provides another generalization of the above mentioned theorem by Strauch and Tóth.

THEOREM 2. *Let $0 \le \alpha \le \beta \le \gamma \le \delta \le 1$ be given numbers. Then there exists an (R)-dense set $A \subset \mathbb{N}$ such that*

$$\underline{d}(A) = \alpha\,, \qquad \underline{\delta}(A) = \beta\,, \qquad \overline{\delta}(A) = \gamma\,, \qquad \overline{d}(A) = \delta\,.$$

Proof. By Lemma 1, there exists a set $D \subset \mathbb{N}$ such that D is (R)-dense and $d(D) = 0$. So, as in the proof of Theorem 1, it suffices to find a set $B \subset \mathbb{N}$ with $\underline{d}(B) = \alpha$, $\underline{\delta}(B) = \beta$, $\overline{\delta}(B) = \gamma$, $\overline{d}(B) = \delta$ and put $A = B \cup D$ to prove the theorem. We will construct the set B by induction such that in the nth step a positive integer m_n will be defined and the set $B \cap \{1, 2, \dots, m_n\}$ will be determined.

Let $1 \in B$, let $2 \notin B$, let $3 \in B$ if and only if $\frac{B(2)}{2} = \phi_0(B,2) < \beta$ and let $4 \in B$ if and only if $\frac{B(3)}{3} = \phi_0(B,3) < \gamma$. Put $m_1 = 4$.

Induction step:
Suppose that $m_1 < m_2 < \dots < m_{n-1}$ and $B \cap \{1, 2, \dots, m_k\}$ are determined such that for each $1 \le k \le n-1$

$$m_k > (k+1)^{k+1} \qquad \text{and} \qquad \phi_1(B, m_k) \in \left(\gamma - \tfrac{1}{k},\, \gamma + \tfrac{1}{k}\right). \tag{1}$$

The number m_n and the elements of the set $B \cap \{m_{n-1}+1, m_n\}$ will be constructed in four substeps.

a) Let $a_n > m_{n-1}$ be the first positive integer such that $\frac{B(m_{n-1}) + (a_n - m_{n-1})}{a_n} > \delta - \frac{1}{n}$ and include all integers $k \in (m_{n-1}, a_n\rangle$ into the set B.

b) Let $b_n > a_n$ be the first positive integer such that $\frac{B(m_{n-1})+(a_n-m_{n-1}}{b_n} < \alpha + \frac{1}{n}$ and include no $k \in (a_n, b_n\rangle$ into the set B.

c) For $k = b_n + 1, b_n + 2, b_n + 3, \dots$ set inductively $k \in B$ if and only if $\phi_0(B, k-1) < \beta$. Continue this process until $\phi_1(B, c_n) \in \left(\beta - \frac{1}{n}, \beta + \frac{1}{n}\right)$ for some $c_n > b_n$. Such a c_n exists because if this process was infinite, we would have $d(B) = \delta(B) = \beta$.

d) For $k = c_n + 1, c_n + 2, c_n + 3, \dots$ set inductively $k \in B$ if and only if $\phi_0(B, k-1) < \gamma$. Continue this process until $\phi_1(B, m_n) \in \left(\gamma - \frac{1}{n}, \gamma + \frac{1}{r}\right)$ for some $m_n > (n+1)^{n+1}$. The reason for the existence of such an m_n is similar to that of the existence of c_n.

Now, by the construction, it is obvious that

$$\underline{d}(B) = \alpha, \qquad \overline{d}(B) = \delta.$$

It suffices to prove the corresponding equalities for the logarithmic densities. Let us notice that by a) for all $k \in (m_{n-1}, a_n)$

$$\frac{B(m_{n-1}) + (k - m_{n-1})}{k} \le \delta - \frac{1}{n} \le 1 - \frac{1}{n}$$

and consequently

$$B(m_{n-1}) + k - m_{n-1} \le k - \frac{k}{n},$$

what implies

$$k < n m_{n-1}. \tag{2}$$

Also, by b) for all $k \in (a_n, b_n)$

$$\frac{B(m_{n-1}) + (a_n - m_{n-1})}{k} \ge \alpha + \frac{1}{n} \ge \frac{1}{n}$$

and consequently

$$a_n - \left(m_{n-1} - B(m_{n-1})\right) \ge \frac{k}{n},$$

what implies

$$k < n a_n. \tag{3}$$

Now, let us estimate the bounds of possible changes of the values $\phi_1(B, k)$ for $k \in (m_{n-1}, m_n\rangle$ for $n > 1$.

Let $k \in (m_{n-1}, a_n\rangle$. Then using (2) and (1)

$$\begin{aligned}
0 &< \phi_1(B,k) - \phi_1(B,m_{n-1}) \\
&\le \frac{S_1(B,m_{n-1}) + \ln \frac{k}{m_{n-1}} + 1}{\ln k} - \frac{S_1(B,m_{n-1})}{\ln m_{n-1}} \\
&= S_1(B,m_{n-1})\left(\frac{1}{\ln k} - \frac{1}{\ln m_{n-1}}\right) + \frac{\ln \frac{k}{m_{n-1}}}{\ln k} + \frac{1}{\ln k} \\
&\le \frac{\ln n}{\ln k} + \frac{1}{\ln k} \le \frac{\ln n}{\ln n^n} + \frac{1}{\ln n^n} \\
&\le \frac{1}{n} + \frac{1}{n} = \frac{2}{n}
\end{aligned}$$

and consequently by (1)

$$\phi_1(B,k) \le \gamma + \frac{1}{n-1} + \frac{2}{n}\,. \tag{4}$$

Let $k \in (a_n, b_n\rangle$. Then using (3), (1) and the fact

$$S_1(B,a_n) \le \sum_{k=1}^{a_n} \frac{1}{k} \le \ln a_n + 1$$

we have

$$\begin{aligned}
0 &< \phi_1(B,a_n) - \phi_1(B,k) = \frac{S_1(B,a_n)}{\ln a_n} - \frac{S_1(B,a_n)}{\ln k} \\
&\le S_1(B,a_n)\left(\frac{1}{\ln a_n} - \frac{1}{\ln k}\right) \le S_1(B,a_n)\left(\frac{1}{\ln a_n} - \frac{1}{\ln n a_n}\right) \\
&= S_1(B,a_n)\frac{\ln n}{\ln a_n(\ln n + \ln a_n)} \\
&\le \frac{\ln n(\ln a_n + 1)}{\ln a_n(\ln n + \ln a_n)} \le \frac{2\ln n}{\ln n + \ln n^n} \\
&= \frac{2}{n+1}
\end{aligned}$$

and consequently by (1)

$$\phi_1(B,k) \ge \gamma - \frac{1}{n-1} - \frac{2}{n+1}\,. \tag{5}$$

This shows that the value of $\phi_1(B,k)$ is sufficiently close to the γ for all $k \in (m_n, b_n)$.

The way of construction also implies that the value of $\phi_1(B,k)$ tends to β for $k \in (b_n, c_n)$ and it tends to γ for $k \in (c_n, m_n)$. This together with (4) and (5) implies the equations

$$\underline{\delta}(B) = \beta\,, \qquad \overline{\delta}(B) = \gamma$$

hold. $\square$

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Problem of common generalization

Theorem 1 and Theorem 2 provide generalizations of the mentioned theorem by Strauch and Tóth in two different directions. A natural question arises whether a common generalization of both theorems holds as follows.

QUESTION. Let $f_1 \colon \mathbb{N} \to \mathbb{R}^+$ and $f_2 \colon \mathbb{N} \to \mathbb{R}^+$ be two functions fulfilling (D) and (Z) and such that the inequalities

$$\underline{d}_{f_1}(A) \le \underline{d}_{f_2}(A) \le \overline{d}_{f_2}(A) \le \overline{d}_{f_1}(A)$$

hold for each $A \subset \mathbb{N}$. Further let $0 \le \alpha \le \beta \le \gamma \le \delta \le 1$ be given numbers. Does there exist an (R)-dense set $A \subset \mathbb{N}$ such that

$$\underline{d}_{f_1}(A) = \alpha\,, \qquad \underline{d}_{f_2}(A) = \beta\,, \qquad \overline{d}_{f_2}(A) = \gamma\,, \qquad \overline{d}_{f_1}(A) = \delta\,?$$

We conjecture that this is not true in general. More precisely, we conjecture the following.

CONJECTURE. *Let $0 \le p \le q \le 1$ and $f_1(n) = n^{-p}$, $f_2(n) = n^{-q}$. Then the answer to the above question is always yes if and only if $q = 1$.*

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