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CONVEX MAPPINGS OF ARCHIMEDEAN MV-ALGEBRAS

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ABSTRACT. In the present paper we prove a theorem of Cantor-Bernstein type for a class of archimedean MV-algebras.

A theorem of Cantor-Bernstein type for complete MV-algebras has been presented in [12]. Another form of a theorem of Cantor Bernstein type was proven in [18] for σ-complete MV-algebras. The authors of [18] remark that their result and the mentioned result from [12] are incomparable.

In the present paper we prove a theorem of Cantor-Bernstein type for a class of archimedean MV-algebras. This generalizes the main result of [12] concerning complete MV-algebras.

For related results dealing with Boolean algebras and lattice ordered groups, see [16], [17], [19], [7], [9], [13].

1. Preliminaries

For MV-algebras we apply the terminology and notation as in [6]. In this setting, an MV-algebra is an algebraic system

$$\mathcal{A} = (A; \oplus, *, \neg, 0, 1),$$

where A is a nonempty set, $\oplus$ and $*$ are binary operations, $\neg$ is a unary operation and 0, 1 are nullary operations on the set A such that the identities (m1) – (m8) from [6] are satisfied.

In [2], a different (but equivalent) system of axioms for defining the notion of MV-algebra has been applied; archimedean MV-algebras are called semi-simple MV-algebras in [2].

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Let $x, y \in A$. We put ([1])

$$x \vee y = (x * \neg y) \oplus y\,, \qquad x \wedge y = \neg(\neg x \vee \neg y)\,.$$

Then (cf. Mundici [14]) $(A, \vee, \wedge, 0, 1)$ is a distributive lattice with the least element 0 and the greatest element 1. We denote this lattice by $\ell(\mathcal{A})$; the corresponding partial order is denoted by $\leqq$.

For the lattice ordered groups we use the notation as in Conrad [3].

If G is an abelian lattice ordered group with a strong unit u, then the notation $\mathcal{A}_0(G,u)$ is applied in the same sense as in [12]. Thus $\mathcal{A}_0(G,u)$ is an MV-algebra with $1 = u$. For each MV-algebra $\mathcal{A}$ there exists an abelian lattice ordered group G with a strong unit u such that $\mathcal{A} = \mathcal{A}_0(G,u)$ (cf. Mundici [14], where the notation $\Gamma(G,u)$ has been used).

Archimedean MV-algebras were dealt with, e.g., in [10]. We denote by $D(\mathcal{A})$ the maximal completion of the MV-algebra $\mathcal{A}$ (in the sense of [11]).

An element $x \in A$ is called *singular* if the interval $[0,x]$ of $\ell(\mathcal{A})$ is a Boolean algebra.

Consider the following condition for an MV-algebra $\mathcal{A}$:

(a) The set of all singular elements of $\mathcal{A}$ has a greatest element which possesses a complement in the lattice $\ell(\mathcal{A})$.

An injective morphism φ of a lattice L_1 into a lattice L_2 is called *convex* if $\varphi(L_1)$ is a convex sublattice of L_2.

In this paper we prove the following theorem:

(A_1) Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be archimedean MV-algebras satisfying condition (a). Suppose that
- (i) there exists a convex injective morphism of the lattice $\ell(\mathcal{A}_1)$ into $\ell(\mathcal{A}_2)$;
- (ii) there exists a convex injective morphism of the lattice $\ell(\mathcal{A}_2)$ into $\ell(\mathcal{A}_1)$.

Then the MV-algebras $D(\mathcal{A}_1)$ and $D(\mathcal{A}_2)$ are isomorphic.

This generalizes Theorem (A) of [12].

Let us conclude this section by some remarks concerning the notion of singular element.

It is clear that the element 0 is singular in each MV-algebra. Further, it is well known that for each Boolean algebra B there exists an archimedean MV-algebra $\mathcal{A}$ such that $\ell(\mathcal{A}) = B$. Then each element of the underlying set of $\mathcal{A}$ is singular and $\mathcal{A}$ satisfies condition (a). For an MV-algebra $\mathcal{A}_1$ the following conditions are equivalent:

(i) $\mathcal{A}_1$ has at least two singular elements;

(ii) there exists $0 < a_1 \in A_1$ such that the interval $[0, a_1]$ of $\ell(\mathcal{A}_1)$ is a Boolean algebra.

Let $\mathcal{C}$ be the class of all MV-algebras satisfying condition (i) and let $\mathcal{V}$ be a variety of MV-algebras. If $\mathcal{C} \subseteq \mathcal{V}$, then $\mathcal{V}$ is the class of all MV-algebras. In fact, assume that $\mathcal{C} \subseteq \mathcal{V}$. Let $\mathcal{A}$ be an arbitrary MV-algebra and let $\mathcal{A}_0$ be an MV-algebra belonging to $\mathcal{C}$. Then the direct product $\mathcal{A} \times \mathcal{A}_0 = \mathcal{A}'$ belongs to $\mathcal{C}$, whence $\mathcal{A}' \in \mathcal{V}$. Further, $\mathcal{A}$ is isomorphic to a homomorphic image of $\mathcal{A}'$ and thus $\mathcal{A}$ belongs to $\mathcal{V}$. Therefore the class $\mathcal{C}$ cannot be characterized by identities.

2. Condition $(*)$

In [12], the following condition for an MV-algebra $\mathcal{A}$ was investigated:

$(*)$ Each singular element of $\mathcal{A}$ has a complement in $\ell(\mathcal{A})$.

2.1. Lemma. *Let $\mathcal{A}$ be an MV-algebra. Then* (a) $\implies$ $(*)$.

Proof. Let (a) be valid. Thus there exists a greatest singular element s^0 in $\ell(\mathcal{A})$ and s^0 has a complement s^1 in the lattice $\ell(\mathcal{A})$. For each element x of A we put

$$x_1 = x \wedge s^0, \qquad x_2 = x \wedge s^1 .$$

Consider the mapping

$$\varphi(x) = (x_1, x_2)$$

of $\ell(\mathcal{A})$ into the direct product

$$[0, s^0] \times [0, s^1] \tag{1}$$

of intervals $[0, s^0]$ and $[0, s^1]$ of the lattice $\ell(\mathcal{A})$. Since $\ell(\mathcal{A})$ is distributive, the mapping φ is an isomorphism of $\ell(\mathcal{A})$ onto the direct product (1).

Let s be a singular element of $\mathcal{A}$. Then $s \in [0, s^0]$. Therefore there exists s' in $[0, s^0]$ such that $s \wedge s' = 0$ and $s \vee s' = s^0$. Denote $s^* = s' \vee s^1$. The distributivity of $\ell(\mathcal{A})$ and the isomorphism φ yield

$$s \wedge s^* = 0, \qquad s \vee s^* = 1 .$$

Hence condition $(*)$ holds. □

An MV-algebra is called *complete* if the lattice $\ell(\mathcal{A})$ is complete.

2.2. Lemma. *Let $\mathcal{A}$ be a complete MV-algebra. Then* $(*)$ $\implies$ (a).

Proof. Let $(*)$ be satisfied. We denote by S the set of all singular elements of $\mathcal{A}$. The set S is nonempty, since 0 belongs to S. There exists $x^0 \in A$ such that $x^0 = \sup S$ is valid in $\ell(\mathcal{A})$. According to [12; Lemmas 2.8, 2.9], the element x^0 is singular. Hence x^0 is the greatest singular element in $\mathcal{A}$. Moreover, in view of $(*)$, x^0 has a complement in $\ell(\mathcal{A})$. Hence condition (a) holds. □

From 2.1 and 2.2 we infer:

2.3. COROLLARY. *Let $\mathcal{A}$ be a complete MV-algebra. Then* $(*) \iff$ (a).

We recall the following result which will be applied below.

2.4. THEOREM. ([12; Theorem (A)]) *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be complete MV-algebras satisfying condition* $(*)$. *Further, suppose that conditions* (i) *and* (ii) *from* (A_1) *hold. Then, the MV-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are isomorphic.*

For each complete MV-algebra we have $D(\mathcal{A}) = \mathcal{A}$. Thus, from 2.3, we conclude that 2.4 is a consequence of (A_1).

3. Archimedean property and maximal completion

Assume that G and $\mathcal{A}$ are as in Section 1. Then the group operation $+$ on G can be considered as a partial binary operation on A. Namely, for $a_1, a_2 \in A$ we consider $a_1 + a_2$ to be defined in A if $a_1 + a_2$ belongs to A; otherwise $a_1 + a_2$ is said to be non-defined in A.

The set of all positive integers will be denoted by $\mathbb{N}$. Let $a \in A$. We put $1 \cdot a = a$. If $a + a$ is defined in A, then we set $2 \cdot a = a + a$; otherwise, $2 \cdot a$ is not defined. By induction we define the meaning of the symbol $n \cdot a$.

3.1. DEFINITION. ([10]) An MV-algebra $\mathcal{A}$ is said to be *archimedean* if, whenever a, b are elements of A such that for each $n \in \mathbb{N}$, $n \cdot a$ is defined and $n \cdot a \leqq b$, then $a = 0$.

3.2. LEMMA. ([10]) *The following conditions are equivalent:*

(i) *$\mathcal{A}$ is archimedean.*
(ii) *G is archimedean.*

From the relations between $\mathcal{A}$ and G we immediately infer that for $a, b \in A$ we have $a + b \in A$ if and only if $a + b = a \oplus b$. Also, $a \leqq a \oplus b$ and $b \leqq a \oplus b$. This yields that the following conditions are equivalent:

(i) $a + b$ is defined in A.
(ii) $(a \oplus b) - a = b$.

If $p, q \in A$ and $p \leqq q$, then the element $q - p$ of G belongs to A and it can be expressed by the operations of $\mathcal{A}$ as follows:

$$q - p = \neg(p \oplus \neg q)$$

(cf. [8; Lemma 1.10]).

For each $x, y \in A$ we denote

$$f(x, y) = \neg(x \oplus \neg y)\,.$$

Hence we have:

3.3. Lemma. *Let $a, b \in A$. Then the following conditions are equivalent:*

(i) *$a+b$ is defined in $\mathcal{A}$.*

(ii) *$f(a, a \oplus b) = b$.*

For $a \in A$ we put $1a = a$, $2a = a \oplus a$ and $(n+1)a = na \oplus a$ (see [1]).

An element $a \in A$ will be called *regular* if $n \cdot a$ is defined in A for each $n \in \mathbb{N}$. From 3.3 we infer that a is regular if and only if for each $n \in \mathbb{N}$ the following relation is valid:

$$f\big(na, (n+1)a\big) = a\,. \tag{1}$$

By applying 3.3 and (1) we obtain the following characterization of archimedean MV-algebras, which uses merely the operations of $\mathcal{A}$:

3.4. Lemma. *An MV-algebra $\mathcal{A}$ is archimedean if and only if whenever a is a regular element of $\mathcal{A}$ and b is an element of $\mathcal{A}$ such that $na \leqq b$ for each $n \in \mathbb{N}$, then $a = 0$.*

In the remaining part of this section we assume that the MV-algebra $\mathcal{A}$ is archimedean. Hence, in view of 3.2, G is archimedean as well.

Let us briefly recall (mainly for fixing the notation) the basic steps in constructing the maximal completion $D(\mathcal{A})$ of $\mathcal{A}$ and the Dedekind completion of G. The relations between these two types of completions were investigated in [11; Sect. 3] (without the assumption that G and $\mathcal{A}$ are archimedean).

For a nonempty subset X of A we denote by X^u (or X^ℓ) the set of all upper bounds (or the set of all lower bounds, respectively) of the set X in the lattice $\ell(\mathcal{A})$. Put $X^* = X^{u\ell}$ and let $d(\mathcal{A})$ be the system of all sets X^* which can be constructed in this way. The system $d(\mathcal{A})$ is partially ordered by the set-theoretical inclusion. Then $d(\mathcal{A})$ is a complete lattice.

If $a \in A$, then the element a will be identified with the set $\{a\}^*$; thus we can consider $\ell(\mathcal{A})$ to be a sublattice of the lattice $d(\mathcal{A})$.

Similarly, for a nonempty upper bounded subset Y of G we denote by Y^{u_1} (and Y^{ℓ_1}) the set of all upper bounds (or the set of all lower bounds, respectively) of the set Y in G. Further we put $Y^{*_1} = Y^{u_1\ell_1}$. We define $d(G)$ to be the system of all sets Y^{*_1}, where Y runs over the system of all nonempty upper bounded subsets of G. Then $d(G)$ is a conditionally complete lattice (under the partial order defined by the set-theoretical inclusion).

Analogously as in the case of $d(\mathcal{A})$, each $g \in G$ will be identified with the set $\{g\}^{*_1}$. Then the underlying lattice of G is a sublattice of $d(G)$.

If we put

$$Z_1 + Z_2 = \{z_1 + z_2 :\ z_1 \in Z_1 \text{ and } z_2 \in Z_2\}^{*_1} \tag{2}$$

for each $Z_1, Z_2 \in d(G)$, then $d(G)$ turns out to be a complete lattice ordered group such that G is an ℓ-subgroup of $d(G)$.

The lattice ordered group $d(G)$ is said to be the *Dedekind completion* of G. (For details, see e.g., D a r n e l [4].)

Analogously as in (2), we put

$$T_1 \oplus T_2 = \{t_1 \oplus t_2 : t_1 \in T_1 \text{ and } t_2 \in T_2\}^* \tag{3}$$

for each $T_1, T_2 \in d(\mathcal{A})$.

Further, for each $Z \in d(G)$ define $\varphi\colon d(G) \to d(\mathcal{A})$ by

$$\varphi(Z) = Z \cap [0,u],$$

where by $[0,u]$ we mean the interval in G with the endpoints 0 and u. It is easy to verify that $\varphi(Z)$ belongs to $d(\mathcal{A})$.

Consider the MV-algebra $\mathcal{A}^1 = \mathcal{A}_0(d(G),u)$. Thus the underlying set A_1 of $\mathcal{A}^1$ is the system of all elements Z of $d(G)$ such that (under the identification defined above) the relation

$$0 \leqq Z \leqq u$$

is satisfied.

Let us denote by φ_0 the mapping φ reduced to the set A_1.

The results of [11] yield:

(i) There exists an MV-algebra $\mathcal{B} = (B; \oplus, *, \neg, 0, 1)$ such that $\ell(\mathcal{B}) = d(\mathcal{A})$ and the operation $\oplus$ is as in (3).

(ii) The mapping φ_0 is an isomorphism of the MV-algebra $\mathcal{A}_0(d(G),u)$ onto the MV-algebra $\mathcal{B}$.

We call $\mathcal{B}$ the *maximal completion* (or the *Dedekind completion*) of $\mathcal{A}$ and we write $\mathcal{B} = D(\mathcal{A})$.

Let us here remark that in the non-archimedean case the above result fails to hold. Namely, if $\mathcal{A}$ is non-archimedean (and hence G is non-archimedean), then the underlying lattice of the Dedekind completion is a proper sublattice of the above constructed lattice $d(G)$. Similarly, the underlying lattice of the maximal completion of $\mathcal{A}$ is a proper sublattice of the lattice $d(\mathcal{A})$.

4. Proof of (A_1)

For an archimedean MV-algebra $\mathcal{A}$, let $\mathcal{B}$ be as in the preceeding section.

4.1. LEMMA. *Let $\mathcal{A}$ be an archimedean MV-algebra. Suppose that $x \in A$, $h \in d(\mathcal{A})$, $x \leqq h$ and that x is not singular in $\mathcal{A}$. Then h is not singular in $\mathcal{B}$.*

P r o o f . By way of contradiction, suppose that h is singular in $\mathcal{B}$. Since x fails to be singular in $\mathcal{A}$, there exists $y \in A$ such that $0 < y < x$ and y has no complement in the interval $[0,x]$ of $\ell(\mathcal{A})$.

From the assumption that h is singular in $\mathcal{B}$ and from the relation $x \leqq h$ we conclude that the element x is singular in $\mathcal{B}$ as well. Hence the interval $[0,x]$ of $\ell(\mathcal{B})$ is a Boolean algebra. From this we infer that there exists $h_1 \in \ell(\mathcal{B})$ such that the relations

$$y \wedge h_1 = 0\,, \qquad y \vee h_1 = x \tag{1}$$

are valid in $\ell(\mathcal{B})$. Consider the lattice ordered group $d(G)$. We have (cf. [3])

$$(y \vee h_1) - y = h_1 - (y \wedge h_1)\,,$$

thus according to (1) we get

$$x - y = h_1\,.$$

Since $0 \leqq x-y \leqq x$, we obtain $y-x \in A$, whence $h_1 \in A$, and then h_1 is a complement of y in the interval $[0,x]$ of $\ell(\mathcal{A})$. Thus we arrived at a contradiction.

□

4.2. Lemma. *Let $\mathcal{A}$ and $\mathcal{B}$ be as in* 4.1. *Suppose that $\mathcal{A}$ satisfies condition* (a). *Then $\mathcal{B}$ satisfies condition* (a) *as well.*

Proof.

α) Since (a) holds for $\mathcal{A}$, there exists a greatest singular element x^0 in $\mathcal{A}$. Let h be a singular element of $\mathcal{B}$. We denote by $\{x_i\}_{i\in I}$ the set of all elements of $\mathcal{A}$ which satisfy the relation $x_i \leqq h$. In view of the construction of $\mathcal{B}$, the relation

$$\sup\{x_i\}_{i\in I} = h$$

is valid in $\ell(\mathcal{B})$. According to 4.1, all x_i are singular in $\mathcal{A}$, thus $x_i \leqq x^0$ for each $i \in I$, whence $h \leqq x^0$.

β) Let $h_1 \in \ell(\mathcal{B})$, $h_1 \leqq x^0$. By [15] (see also [5; p. 436]), there exists a system $\{x_k\}_{k\in K}$ of elements of A such that

$$\bigvee_{k\in K} x_k = h_1$$

is valid in $\ell(\mathcal{B})$. For each $k \in K$ we have $x_k \leqq x^0$, hence there exists x'_k in A such that x'_k is a relative complement of x_k in the interval $[0,x^0]$ of $\ell(\mathcal{B})$.

Since $\ell(\mathcal{B})$ is complete, there exists h_2 in $\ell(\mathcal{B})$ such that

$$\bigwedge_{k\in K} x'_k = h_2\,.$$

For each $k \in K$ we have $0 \leqq x_k \wedge h_2 \leqq x_k \wedge x'_k = 0$, thus $x_k \wedge h_2 = 0$. Similarly, for each $k \in K$, the relation $h_1 \vee x'_k = x^0$ is valid.

It is well known that each lattice ordered group is infinitely distributive. Thus, from the relation between MV-algebras and abelian lattice ordered groups we get that $\ell(\mathcal{B})$ is infinitely distributive. Hence

$$h_1 \wedge h_2 = \Big(\bigvee_{k\in K} x_k\Big) \wedge h_2 = \bigvee_{k\in K} (x_k \wedge h_2) = 0\,,$$

$$h_1 \vee h_2 = h_1 \vee \Big(\bigwedge_{k\in K} x'_k\Big) = \bigwedge_{k\in K} (h_1 \vee x'_k) = x^0\,.$$

Thus, the interval $[0, x^0]$ of $\ell(\mathcal{B})$ is a Boolean algebra. Therefore, x^0 is a singular element of $\mathcal{B}$. Then, in view of α), x^0 is the greatest singular element of $\mathcal{B}$.

γ) According to (a), there exists $y^0 \in A$ such that the relations

$$x^0 \wedge y^0 = 0\,, \qquad x^0 \vee y^0 = u \tag{2}$$

are valid in $\ell(\mathcal{A})$. Then, in view of the construction of $d(\mathcal{A})$, relations (2) remain valid for $\ell(\mathcal{B})$ as well. Hence, condition (a) is satisfied for $\mathcal{B}$. □

4.3. LEMMA. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be archimedean MV-algebras. Further, let $\mathcal{B}_i$ be the maximal completion of $\mathcal{A}_i$ $(i = 1,2)$. Suppose that there exists a convex injective morphism of $\ell(\mathcal{A}_1)$ into $\ell(\mathcal{A}_2)$. Then there exists a convex injective morphism of $\ell(\mathcal{B}_1)$ into $\ell(\mathcal{B}_2)$.*

Proof. It suffices to apply a similar argument as in the proof of [13; Lemma 2.1]. □

We are now able to prove (A_1) from Section 1.

Proof. Let the assumptions of (A_1) be satisfied. In view of 4.2, both $\mathcal{B}_1 = D(\mathcal{A}_1)$ and $\mathcal{B}_2 = D(\mathcal{A}_2)$ fulfil condition (a). Also, both $\mathcal{B}_1$ and $\mathcal{B}_2$ are complete. Then, according to 4.3 and by 2.4, the MV-algebras $\mathcal{B}_1$ and $\mathcal{B}_2$ are isomorphic. □

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