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WEAK*-NORM SEQUENTIALLY CONTINUOUS OPERATORS

ALIMOHAMMADY MOHSEN

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ABSTRACT. J. Bourgain in 1979 proved that T^*, the adjoint of a operator $T\colon c_0 \to E^*$, is weak*-norm sequentially continuous. Moreover J. Bourgain and J. Diestel in 1984 showed a bounded operator $T\colon E \to F$ is limited if and only if the adjoint of T is weak*-norm sequentially continuous. They also proved that if the adjoint of T is weak*-norm sequentially continuous, then T is strictly cosingular. Here we study some properties of $W^*(E^*,F)$, the space of all bounded weak*-norm sequentially continuous linear maps from E^* to F equipped with norm topology. We give characterizations of Grothendieck spaces and Mazur spaces by comparing $W^*(E^*,F)$ and different spaces of operators.

1. Introduction

Throughout this note E, F will denote Banach spaces and E^* the dual of E. The unit ball of the Banach space E will be denoted by B_E, and the term operator will mean a bounded linear function.

Let $L(E,F)$, $L_{w^*}(E^*,F)$, $K(E,F)$ and $K_{w^*}(E^*,F)$ denote the Banach space of operators, weak*-weak continuous operators, compact operators and weak*-weak continuous compact operators between the two mentioned Banach spaces.

A Banach space E is said to be a *Mazur space* if weak* sequentially continuous functionals Λ on E^* are actually weak* continuous, i.e. Λ belongs to E.

A Banach space E is said to be a *Grothendieck space* whenever, in the dual E^* of E, weak* and weak convergence of sequences coincide.

An operator $T\colon E \to F$ is said to be *strictly cosingular* if $LT\colon E \to G$ fails to be a surjection for every infinite dimensional Banach space G and for all operators $L\colon F \to G$.

The reader may consult [5], [6] or [17] for unexplained notations.

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2. $W^*(E^*, F)$

$W^*(E^*, F)$ is here meant to denote the linear space of all weak*-norm sequentially continuous operators from E^* to F equipped with the norm topology. It is easy to see that $W^*(E^*, F)$ is a Banach subspace of $L(E^*, F)$, and $W^*(E^*, F)$ forms an ideal in $L(E^*, F)$. Moreover, when F is a Schur space (when weak and norm convergence of sequences coincide) and E is separable, then $W^*(E^*, F) = K_{w^*}(E^*, F)$. The following two results emphasize the operator theoretic aspects of Grothendieck and Mazur spaces and will prove useful in our considerations.

THEOREM 1. *The space E is a Grothendieck space if and only if $W^*(E^*, F)$ contains $K(E^*, F)$ for any Banach space F.*

P r o o f . Suppose E is a Grothendieck space, and $T \in K(E^*, F) \backslash W^*(E^*, F)$. Therefore there is a weak* null sequence $(x_n^*) \subseteq B_{E^*}$ such that $\|Tx_n^*\| \geq \varepsilon$ $(n \in \mathbb{N})$ (by passing to a subsequence if necessary). From the compactness of T, (Tx_n^*) is a norm null sequence, which is a contradiction. Conversely, if $K(E^*, F) \subseteq W^*(E^*, F)$, then $x^{**} \otimes y \in K(E^*, F)$, where $0 \neq y \in F$ and $x^{**} \in E^{**}$. Then $x^{**} \otimes y(x_n^*) = x^{**}(x_n^*)y \to 0$ (norm), where (x_n^*) is an arbitrary weak*-null sequence in E^*. This shows (x_n^*) is a weak null sequence in E, i.e. E is a Grothendieck space. □

The following result gives an analogous characterization for Mazur spaces.

THEOREM 2. *The space E is a Mazur space if and only if $W^*(E^*, F) \subseteq L_{w^*}(E^*, F)$ for any Banach space F.*

P r o o f . Suppose E is a Mazur space, $T \in W^*(E^*, F)$ and (x_n^*) is a weak* null sequence in E^*. Then $T^*y^*(x_n^*) \to 0$ $(y^* \in F^*)$, i.e. T^*y^* is a weak* sequentially continuous functional; so by the assumption it lies in E ([5]). For $x_\alpha^* \to 0$ (weak*) in E^*, and for each $y^* \in F^*$, $(T^*y^*)(x_\alpha^*) = y^*\big(T(x_\alpha^*)\big) \to 0$; so $T \in L_{w^*}(E^*, F)$. By replacing $\mathbb{C}$ with F, the converse is straightforward. □

Remark. In general, there is no specific relation between $W^*(E^*, F)$ and the other known linear subspaces of $L(E^*, F)$:

(a) $W^*(E^*, E^*) \neq L(E^*, E^*)$, since $I \in L(E^*, E^*) \setminus W^*(E^*, F)$.

(b) Suppose $F = \mathbb{C}$ and E is not a Grothendieck space, then by Theorem 1 $W^*(E^*, F) \neq K(E^*, F)$.

(c) If $T\colon c_0 \to \ell_\infty$ is the natural inclusion map, then $T^*\colon \ell_\infty^* \to \ell_1$ is a bounded linear projection. But ℓ_1 is a Schur space and ℓ_∞ is a Grothendieck space, therefore T^* maps weak* null sequences to norm null sequences, i.e.,

$T^* \in W^*(\ell_\infty^*, \ell_1)$, but T^* is not weakly compact. This shows that $W^*(\ell_\infty^*, \ell_1) \neq W(\ell_\infty^*, \ell_1)$.

(d) In the case that F is a dual space, $W^*(E^*, F)$ contains the space of adjoints of all limited operators between the predual of F and E ([3]).

The following result is essentially due to J. Bourgain and J. Diestel [3]. It elaborates the relation between the space of all strictly cosingular operators and $W^*(E^*, F)$. We can also demonstrate its proof in a more simple way.

THEOREM 3. *An operator $T\colon E \to F$ between the Banach spaces E and F is strictly cosingular if $T^* \in W^*(F^*, E^*)$.*

Proof. Suppose $q_2 T = q_1$, where $T^* \in W^*(F^*, E^*)$, q_1 and q_2 are surjectives, then $T^* q_2^* = q_1^*$. By the Josefson-Nissenzweig Theorem ([5]), there is a normalized weak* null sequence (z_n^*) in E^*. Since $q_1^*(z_n^*) \to 0$ (weak*) and $q_2^*(z_n^*) \to 0$ (weak*), $T^* q_2^*(z_n^*) = q_1^*(z_n^*)$ goes to zero in norm, which is a contradiction, since q_1^* is an embedding. □

$\mathcal{A}$ denotes the set of all those bounded operators from E to F whose adjoints lie in $W^*(F^*, E^*)$, and $\mathcal{A}'$ its adjoint class.

The following Theorem establishes another characterization for Mazur spaces.

THEOREM 4. *The space F is a Mazur space if and only if for every Banach space E, $\mathcal{A}' = W^*(F^*, E^*)$.*

Proof. Let F be a Mazur space. It is clear that $\mathcal{A}' \subseteq W^*(F^*, E^*)$. By Theorem 2, each $T \in W^*(E^*, F)$ is weak* to weak* continuous; so $T \in \mathcal{A}'$. For the converse, set $E = \mathbb{C}$. Since $W^*(F^*, \mathbb{C}) = \mathcal{A}'$, any element $\Lambda\colon F^* \to \mathbb{C}$ which is weak*-sequentially continuous belongs to $\mathcal{A}'$. Therefore there exists a bounded operator $S\colon \mathbb{C} \to F$ such that $\Lambda = S^*$. It follows that $\Lambda \in F$, i.e. F is a Mazur space. □

There is a natural isometric isomorphism $T \mapsto T^*$ from $K_{w^*}(E^*, F)$ onto $K_{w^*}(F^*, E)$ ([4]). Here a similar result for $W^*(E^*, F)$ is given.

We say a Banach space E is *w^*-sqcu* if the unit ball of its dual is weak* sequentially compact (cf. [5; Chapter 13]).

THEOREM 5. *Let E be w^*-sqcu Banach space. Then $h \mapsto h^*$ from $W^*(E^*, F)$ into $W^*(F^*, E)$ is a linear isometry. If in addition F is a w^*-sqcu Banach space, then this isomorphism is a surjection.*

Proof. By the assumption, $W^*(E^*, F) \subseteq K_{w^*}(E^*, F)$. For $h \in W^*(E^*, F)$ we have $h^* \in K_{w^*}(F^*, E)$. We show h^* is in fact weak*-norm sequentially continuous. Suppose on the contrary, there is a weak*-null sequence (y_n^*) in E^* (by the Banach Steinhaus Theorem we can assume $(y_n^*) \subseteq B_{F^*}$) such that

$\|h^*y_n^*\| > \varepsilon$ $(n \in \mathbb{N})$ for some $\varepsilon > 0$. There exists $(x_n^*) \subseteq B_{E^*}$, such that $|x_n^* h^* y_n^*| > \varepsilon$ $(n \in \mathbb{N})$. But E is w^*-sqcu, so there exists a subsequence $(x_{n_k}^*)_k$ weak* convergent to x^*. Then $h(x_{n_k}^*) \to h(x^*)$ (norm), and for a suitable $k_1 \in \mathbb{N}$, we get $\|h(x_{n_k}^* - x^*)\| < \frac{\varepsilon}{3}$ (for all $k \geq k_1$). Hence $|y_{n_k}^* h(x_{n_k}^* - x^*)| < \frac{\varepsilon}{3}$. But $y_{n_k}^* \to 0$ (weak*) therefore

$$\exists\, k_2 \in \mathbb{N} \quad \forall\, k \geq k_2 \qquad \left|y_{n_k}^*\big(h(x^*)\big)\right| < \frac{\varepsilon}{3}\,.$$

Set $k_3 = \max\{k_1, k_2\}$; then for $k \geq k_3$

$$\varepsilon < |y_{n_k}^* h x_{n_k}^*| \leq |y_{n_k}^* h(x_{n_k}^* - x^*)| + |y_{n_k}^* h(x^*)| < \frac{2\varepsilon}{3}\,,$$

which is a contradiction. Thus $h^* \in W^*(F^*, E)$, and $h \mapsto h^*$ is a linear isometry from $W^*(E^*, F)$ to $W^*(F^*, E)$. A similar argument in case F is also w^*-sqcu. completes the proof. □

DEFINITION. A subset L of E is said to be a (V^*)-*set* if

$$\lim \sup_{x \in L} |x_n^*(x)| = 0\,,$$

where Σx_n^* is w.u.c in E^*.

The Banach space E has the (V^*)-*property* if all its (V^*)-subsets are relatively weakly compact ([1], [4]).

THEOREM 6. *Let E^* be a separable Banach space and let $W^*(E^*, F)$ be weakly sequentially complete. Then F has the (V^*)-property if and only if $W^*(E^*, F)$ has the (V^*)-property.*

Proof. Certainly if $W^*(E^*, F)$ has the (V^*)-property then F does so. Now suppose $M \subseteq W^*(E^*, F)$ is a (V^*)-set, $(h_n)_n$ is an arbitrary sequence in M and $A = \{x_n^* :\ n \in \mathbb{N}\}$ is a dense subset of E^*. Since $\big(h_n(x^*)\big)$ is a (V^*)-set for all $x^* \in E^*$, therefore there is a subsequence $\big(k(n)\big)_n$ of $\mathbb{N}$ such that, $\big(h_{k(n)}(x^*)\big)_n$ is weakly Cauchy in F for all $x^* \in A$. By density of A in E^*, $\big(h_{k(n)}(x^*)\big)_n$ is weakly Cauchy in F for all $x^* \in E^*$. A characterization of extreme points of linear subspaces of $K_{w^*}(E^*, F)$ that contains $E \otimes F$ due to W. Ruess and C. P. Stegall [20] together with the theorem of Rainwater ([5]) and our assumption show that $(h_{k(n)})_n$ is weakly convergent. □

In the two next theorems we will show that weakly sequentially completeness of $W^*(E^*, F)$ in the above result can hold.

THEOREM 7. *Suppose E and F are weakly sequentially complete Banach spaces, E is w^*-sqcu and F is a Schur space. Then $W^*(E^*, F)$ is weakly sequentially complete.*

Proof. It is easy to see that in this case $K_{w^*}(E^*, F) = W^*(E^*. F)$. Thus an appeal to [4; Proposition 3.1] completes the proof. □

THEOREM 8. *Suppose $K_{w^*}(E^*,F)$ is weakly sequentially closed subspace of $L(E^*,F)$, and F is w^*-sqcu and also weakly sequentially complete Banach space. Then $W^*(E^*,F)$ is weakly sequentially complete.*

Proof. Suppose $(h_n)_n$ is a weakly Cauchy sequence in $W^*(E^*,F)$. Then $\big(h_n(x^*)\big)_n$ is weakly Cauchy for all $x^* \in E^*$. Since F is weakly sequentially complete $h_n \to h$ (weakly) in $K_{w^*}(E^*,F)$. In order to prove $h \in W^*(E^*,F)$, suppose on the contrary $(x_n^*)_n$ is a weak*-null sequence in E^* such that

$$\forall n \in \mathbb{N} \qquad \|h(x_n^*)\| \geq \varepsilon .$$

We can assume there is a sequence $(y_n^*) \subseteq B_{F^*}$ with $y_n^* \to y^*$ (weak*) and

$$y_n^* h x_n^* \geq \varepsilon ,$$

for some $y^* \in F^*$. But $(y_n^* - y^*)h(x_n^*)$ tend to zero, which is a contradiction with $h(x_n^*) \to 0$ (weakly). □

In the following we state some of the properties of $W^*(E^*,F)$. The proofs are direct and will be omitted. For undefined notations and definitions we refer to [9].

THEOREM 9. *Suppose E and F are two w^*-sqcu and $L \subseteq W^*(E^*,F)$ then:*

(a) *L is a Dunford-Pettis set (D.P) if and only if $L^* = \{h^* : h \in L\}$ is a Dunford-Pettis set in $W^*(F^*,E)$.*
(b) *If E and F have (DPrcp) and L is a (D.P)-set, then $L(x^*) = \big\{h(x^*) : h \in L\big\}$ (resp. L^*y^*) is relatively compact in F for all $x^* \in E^*$).*
(c) *Only by assuming E is w^*-sqcu, then every $T \in W^*(E^*,F)$ is a Dunford-Pettis and limited operator.*

J. Bourgain [2] in 1979 proved $L^1(E)$ is not a dual space if E contains a copy of c_0. We state a similar result for $W^*(E^*,F)$.

THEOREM 10. *Suppose E and F are two Banach spaces, $\dim E = \infty$, and F has a copy of c_0. Then $W^*(E^*,F)$ is not a dual space.*

Proof. By the Josefson-Nissenzweig theorem there exists a normalized weak* null sequence (x_n^*) in E^*. Choose (x_n) in E such that

$$x_n^* x_n = 1 , \quad \|x_n\| \leq 2 \qquad (n \in \mathbb{N}) .$$

Suppose $S\colon c_0 \to F$ is an isomorphic embedding. By the Hahn-Banach Theorem, there is a bounded sequence (y_n^*) in F^* such that $y_n^*\big(S(e_n)\big) = 1$ $(n \in \mathbb{N})$, where $(e_n)_n$ is the standard unit vector basis of c_0. It is easy to see that $\phi_n = x_n^* \otimes y_n^* \in \big(W^*(E^*,F)\big)^*$ and $\phi_n \to 0$ (weak*). An easy argument shows that $x_n \otimes S(e_n)$ is equivalent to the basis of c_0 in $W^*(E^*,F)$. Therefore there is an isomorphic embedding $\hat{S}\colon c_0 \to W^*(E^*,F)$ such that $\hat{S}(e_n) = x_n \otimes S(e_n)$. Suppose now that $W^*(E^*,F)$ is a dual space. Then there exists an isomorphism I from $W^*(E^*,F)$ onto Z^* for a Banach space Z. So $(I\hat{S})^*$ and $(\hat{S})^*$ are weak*-norm sequentially continuous. Therefore $(\hat{S})^*(\phi_n) \to 0$ (norm). On the other hand

$$\|(\hat{S})^*(\phi_n)\| \geq (\hat{S})^*(\phi_n)(e_n) = \phi_n(\hat{S}((e_n))) = \phi_n(x_n \otimes S(e_n)) = 1 \qquad (n \in \mathbb{N}),$$

which is a contradiction. □

By the same line of proof of above theorem, one can show an analogous result for the space $K_{w^*}(E^*,F)$.

COROLLARY 11. *If F contains a copy of c_0, then $K_{w^*}(E^*,F)$ is not a dual space.*

A bounded set $B \subseteq E$ is called a *limited set* if $\lim_n \sup_{x\in B} |x_n^*(x)| = 0$, for every weak* null sequence (x_n^*) in E^*.

E is said to be a *Gelfand-Phillips space* if every limited set in E is relatively compact ([7]).

THEOREM 12. *Suppose E is w^*-sqcu. Then F is a Gelfand-Phillips space if and only if $W^*(E^*,F)$ is a Gelfand-Phillips space.*

Proof. It is well known that on the assumption $K_{w^*}(E^*,F)$ is a Gelfand-Phillips space, and the Gelfand-Phillips property is inherited by closed subspaces ([8]), which completes the proof. □

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REFERENCES

[1] BOMBAL, F.: *On (V^*) sets and Pełczyński's property (V^*)*, Glasgow Math. J. **32** (1990), 109–120.

[2] BOURGAIN, J.: *A note on the Lebesgue spaces of vector valued functions*, Bull. Soc. Math. Belg. Sér. B **31** (1979), 45–47.

[3] BOURGAIN, J.—DIESTEL, J.: *Limited operators and strictly cosingularity*, Math. Nachr. **119** (1984), 55–58.

[4] COLLINS, H. S.—RUESS, W.: *Weak compactness in spaces of compact operators and of vector valued functions*, Pacific J. Math. **106** (1983), 45–71.

[5] DIESTEL, J.: *Sequences and Series in Banach Spaces*. Grad. Texts in Math., Springer-Verlag, New York, 1984.

[6] DIESTEL, J.—UHL Jr., J. J.: *Vector Masures*. Math. Surveys Monographs 15, Amer. Math. Soc., Providence, RI, 1977.

[7] DREWNOWSKI, L.: *On Banach spaces with the Gelfand-Phillips property*, Math. Z. **193** (1986), 405–411.

[8] DREWNOWSKI, L.—EMMANUELE, G.: *On Banach spaces with the Gelfand-Phillips property II*, Rend. Circ. Mat. Palermo (2) **38** (1989), 377–391.

[9] EMMANUELE, G.: *Banach spaces in which Dunford-Pettis sets are relatively compact*, Arch. Math. (Basel) **58** (1992), 477–485.

[10] EMMANUELE, G.: *On complemented copies of c_0 in spaces of operators II*, Comment. Math. Univ. Carolin. **35** (1994), 259–261.

[11] EMMANUELE, G.: *On the reciprocal, Dunford-Pettis property in projective tensor products*, Math. Proc. Cambridge Philos. Soc. **109** (1991), 161–166.

[12] EMMANUELE, G.: *Remarks on the uncomplemented subspace $W(E,F)$*, J. Funct. Anal. **99** (1991), 125–130.

[13] FEDER, M.: *On the non-existence of a projection onto the space of compact operators*, Canad. Math. Bull. **25** (1962), 78–81.

[14] FERRANDO, J. C.: *Copies of c_0 in certain vector valued function Banach spaces*, Math. Scand. **77** (1995), 148–152.

[15] JOHNSON, J.: *Remarks on Banach spaces of compact operators*, J. Funct. Anal. **32** (1979), 304–311.

[16] KALTON, N. J.: *Spaces of compact operators*, Math. Ann. **208** (1974), 267–278.

[17] MENDOZA, J.: *Copies of classical sequence spaces in vector valued function Banach spaces.* In: Lecture Notes in Pure and Appl. Math. 172, Dekker, New York, 1996, pp. 311–320.

[18] PEŁCZYŃSKI, A.: *Banach spaces on which every unconditionally converging operator is weakly compact*, Bull. Acad. Pol. Sci., Ser. Sci. Math. Astron. Phys. **10** (1962), 641–648.

[19] ROSENTHAL, H. P.: *On relatively disjoint families of measures, with some application to Banach space theory*, Studia Math. **37** (1979), 13–36.

[20] RUESS, W.—STEGALL, C. P.: *Extreme points in duals of operator spaces*, Math. Ann. **261** (1980), 535–546.

[21] RYAN, R. A.: *Complemented copies of c_0 in the space of compact operators*, Proc. Roy. Irish Acad. Sect. A **91A** (1991), 239–241.

[22] ZAFARANI, J.: *Grothendieck space of compact operators*, Math. Nachr. **174** (1995), 317–322.

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Department of Mathematics
Faculty of Basic Sciences
University of Mazandran
Babolsar
IRAN

E-mail: amohsen@umcc.ac.ir