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THE BROOKS-JEWETT THEOREM FOR k-TRIANGULAR FUNCTIONS

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ABSTRACT. It is shown a diagonal proof of the Brooks-Jewett convergence theorem for k-triangular functions on an orthomodular poset with the subsequential completeness property.

Approaching measure theory one very soon runs into the concept of outer measure, semivariation of a vector measure etc. As everybody knows these "measures" lack additivity, nevertheless they play a non trivial role in the development of the theory.

Around 1970 some mathematicians (see, e.g., [15], [16]) started considering non-additive functions by their own sake, opening a new branch of the research in measure theory. Actually, this, let us say, theoretical investigation turned out to be useful in other lines, for example potential theory, harmonic analysis, fractal geometry, functional analysis, the theory of non linear differential equations and so forth.

In fact, very recently, the theory of non-additive functions was used and sometimes improved by people working in very different fields such as artificial intelligence, game theory, statistics, economy, sociology; for more information about this see the book of Pap [26] and its bibliography.

In this paper we consider a special class of non-additive functions, the class of k-triangular functions ([1], [17], [18], [19], [20], [24], [25]). Moreover we set them in the framework of the so-called "non commutative measure theory", namely, that part of measure theory which came out by the attempt to construct a mathematical foundation of quantum mechanics initiated by Von Neumann [31] and Birkhoff [8]. Hence, we consider k-triangular functions defined on an

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orthomodular poset (a more general structure than a Boolean algebra) and we prove the classical Brooks-Jewett convergence theorem for these functions.

The technique used in proof of the theorem can be considered a "sliding hump" or a "diagonal" method. These methods have proven to be quite effective in treating various topics and have been widely used both in functional analysis and measure theory (see [4], [5], [27] and [30]). For example, in [14] Diestel and Uhl use a lemma of Rosenthal to derive some important results for vector valued measures ([2], [14], [23]). In a somewhat similar fashion Weber in his paper [33] applies a "sliding hump" method which is very closed to the Rosenthal lemma mentioned above (see also [3]). In this paper we utilize a modification of Weber's technique.

The choice was somehow "forced" by the fact that, so far, no other methods seem to be effective for proving convergence theorems in the context we are considering (i.e., functions defined on an orthomodular poset), as it is also pointed out in [12] (see also [13]).

Basic notions

We start with a short description of the structure on which the functions are defined.

A quintuple $L = (L, \leq, ', 0, 1)$ is called an *orthomodular poset* (*OMP* for short) if

(i) $(L, \leq, 0, 1)$ is a bounded partially ordered set (with 0 and 1 as the smallest and the largest element),
(ii) $' \colon L \to L$ is a unary operation (called orthocomplementation) idempotent, decreasing and such that $a \wedge a' = 0$, $a \vee a' = 1$ hold for every $a \in L$,
(iii) if $a, b \in L$ and $a \leq b'$, then $a \vee b$ exists,
(iv) if $a, b \in L$ and $a \leq b$, then $b = a \vee (b \wedge a')$.

Two elements $a, b \in L$ are *orthogonal* (write $a \perp b$) if $a \leq b'$.

A subset M of L is said to be orthogonal if $a, b \in M$ and $a \neq b$ implies $a \perp b$.

We say that two elements $a, b \in L$ *commute* (or that the pair (a, b) is compatible), in symbols $a\,C\,b$, if $a \wedge b$ and $a \wedge b'$ exist and $a = (a \wedge b) \vee (a \wedge b')$.

Let L be an OMP and ψ be a $[0, \infty[$-valued function defined on L.

We say that ψ is *exhaustive* (or strongly bounded) if $\lim\limits_k \psi(a_k) = 0$ for any orthogonal sequence $(a_k)_{k \in \mathbb{N}}$ in L.

A sequence $(\psi_n)_{n\in\mathbb{N}}$ of functions as above is *uniformly exhaustive* (uniformly strongly bounded) if for any orthogonal sequence $(a_k)_{k\in\mathbb{N}}$

$$\lim_k \psi_n(a_k) = 0 \qquad \text{uniformly in} \quad n \in \mathbb{N}.$$

For $k \in]0,\infty[$, a function ψ from L to $[0,\infty[$ is *k-triangular* if $\psi(0) = 0$ and

$$\psi(a) - k\psi(b) \le \psi(a \vee b) \le \psi(a) + k\psi(b) \qquad \text{for all} \quad a,b \in L \ \text{with} \ a \perp b\,,$$

or, equivalently, if

$$|\psi(a \vee b) - \psi(a)| \le k\psi(b) \qquad \text{for} \quad a \perp b\,.$$

It is easy to see that if ψ is not identically zero on L, then $k \ge 1$. Hence below we will consider k-triangular functions with coefficient $k \ge 1$.

If $(G, |\ |)$ is a quasi-normed Abelian group, one can define ψ from L to G to be k-triangular if $\psi(0) = 0$ and for $a, b \in L$ with $a \perp b$

$$|\psi(a)| - k|\psi(b)| \le |\psi(a \vee b)| \le |\psi(a)| + k|\psi(b)|\,. \tag{1}$$

In this case it is not anymore true that condition (1) is equivalent to

$$|\psi(a \vee b) - \psi(a)| \le k|\psi(b)| \qquad \text{for} \quad a \perp b \tag{2}$$

but (2) implies (1). A function satisfying (2) is called quasi-Lipschitzian.

Examples of k-triangular functions are a finitely additive $(G, |\ |)$-valued measure (hence, a vector valued measure), a finite submeasure (hence, a finite outer measure), a k-semimeasure (see [20]).

If μ is the Lebesgue measure on $[0,2]$ and $E \subseteq [0,2]$ then the function defined by

$$\phi(E) = \begin{cases} \mu(E) & \text{if } \mu(E) < 1\,, \\ 2\mu(E) - 1 & \text{if } \mu(E) \ge 1 \end{cases}$$

is a 2-triangular function.

It can be proved (see [17]) that:

PROPOSITION 1. *Let ψ be a $[0,\infty[$-valued function defined on an OMP L such that $\psi(0) = 0$.*

Then ψ is k-triangular if and only if

$$|\psi(a) - \psi(b)| \le k\psi(a \wedge b') + k\psi(b \wedge a') \qquad \textit{with} \quad a,b \in L \ \textit{and} \ a\,C\,b\,.$$

Preliminary results

In order to prove the main theorem we need some preliminary results.

We report in detail the proof of Lemma 3, since it is the key lemma and the proof is quite different from Weber's one. The other proofs can be obtained from Weber's paper by natural modifications.

The symbol Γ will stand for the set of all strictly increasing sequences of natural numbers. If $\alpha, \beta \in \Gamma$ then $\beta < \alpha$ means that β is a subsequence of α.

Lemma 2. *Let $\varepsilon > 0$. For $n \in \mathbb{N}$ let $\eta_n \colon \mathcal{P}(\mathbb{N}) \to [0, \infty]$ be a monotone function such that, for any disjoint sequence $(A_k)_{k \in \mathbb{N}}$ in $\mathcal{P}(\mathbb{N})$, there exists $l \in \mathbb{N}$ (depending on n) such that $\eta_n(A_l) \leq \varepsilon$.*

Then there exists $\gamma \in \Gamma$ such that

$$\eta_{\gamma(n)}\big(\{\gamma(h) : \ h \in \mathbb{N}, \ \ h > n\}\big) \leq \varepsilon \qquad \text{for all} \quad n \in \mathbb{N}.$$

Lemma 3. *Let $\varepsilon > 0$ and $k \geq 1$.*

(i) *For $n \in \mathbb{N}$ let $\eta_n \colon \mathcal{P}(\mathbb{N}) \to [0, \infty]$ satisfy the hypotheses of Lemma 2. Suppose that*

(ii)

$$\forall A \subseteq \mathbb{N} \quad \forall n, h \in \mathbb{N} \qquad \eta_n(A) \leq k\eta_n\big(A - \{h\}\big) + \eta_n\big(\{h\}\big),$$

(iii)

$$\lim_h \lim_n \eta_n\big(\{h\}\big) = 0.$$

Then, for all $\delta > \varepsilon$, there exists $\gamma \in \Gamma$ such that

$$\forall n \in \mathbb{N} \qquad \eta_{\gamma(n)}\big(\{\gamma(h) : \ h \in \mathbb{N}, \ \ h \neq n\}\big) \leq \delta/k.$$

Proof. Put $p_h = \lim_n \eta_n\big(\{h\}\big)$.

By hypothesis $\lim_h p_h = 0$. Then there exists $\alpha \in \Gamma$ such that

$$\sum_{h=1}^{\infty} k^{h-1} p_{\alpha(h)} < (\delta - \varepsilon)/k.$$

Indeed, let σ be positive and such that $\sigma < (\delta - \varepsilon)/k$; then

$$\begin{aligned}
&k^0 \lim_h p_h = 0 && \Longrightarrow \ \exists\, n_1 > 0 \quad \forall n \geq n_1 \qquad p_n < \sigma/2, \\
&k \lim_h p_h = 0 = \lim_h (kp_h) && \Longrightarrow \ \exists\, n_2 > n_1 \quad \forall n \geq n_2 \qquad kp_n < \sigma/2^2, \\
&k^2 \lim_h p_h = 0 = \lim_h (k^2 p_h) && \Longrightarrow \ \exists\, n_3 > n_2 \quad \forall n \geq n_3 \qquad k^2 p_n < \sigma/2^3,
\end{aligned}$$

and so on

$$k^r \lim_h p_h = 0 = \lim_h (k^r p_h) \implies \exists\, n_{r+1} > n_r \quad \forall n \geq n_{r+1} \qquad k^r p_n < \sigma/2^{r+1}\,.$$

Put now $\alpha(h) = n_h$. Then $\alpha \in \Gamma$ and

$$k^{h-1} p_{\alpha(h)} = k^{h-1} p_{n_h} < \sigma/2^h$$

hence

$$\sum_{h=1}^{\infty} k^{h-1} p_{\alpha(h)} \leq \sum_{h=1}^{\infty} \sigma/2^h = \sigma < (\delta - \varepsilon)/k\,.$$

Now, applying iteratively the preceeding Lemma 2 to the sequence of functions $(\eta_{\alpha(n)})_{n\in\mathbb{N}}$ we obtain a sequence $\beta < \alpha$ such that

$$\eta_{\beta(n)}\big(\{\beta(h):\ h > n\}\big) \leq \varepsilon/k^{n+1}\,.$$

Indeed, putting for simplicity $\eta_n = \eta_{\alpha(n)}$ we get that

$$\exists\, \gamma_1 \in \Gamma \quad \forall n \in \mathbb{N} \qquad \eta_{\gamma_1(n)}\big(\{\gamma_1(h):\ h > n\}\big) \leq \varepsilon/k^2\,.$$

Now applying again Lemma 2 to the sequence $(\eta_{\gamma_1(n)})_{n\in\mathbb{N}}$, we can say that

$$\exists\, \gamma_2 < \gamma_1 \quad \forall n \in \mathbb{N} \qquad \eta_{\gamma_2(n)}\big(\{\gamma_2(h):\ h > n\}\big) \leq \varepsilon/k^3$$

and, iteratively,

$$\exists\, \gamma_r < \gamma_{r-1} \quad \forall n \in \mathbb{N} \qquad \eta_{\gamma_r(n)}\big(\{\gamma_r(h):\ h > n\}\big) \leq \varepsilon/k^{r+1}\,.$$

Hence

$$\alpha > \gamma_1 > \gamma_2 > \cdots > \gamma_r > \ldots\,.$$

Let us consider now the diagonal sequence defined by

$$\beta(n) = \gamma_n(n+1)\,.$$

If $n \in \mathbb{N}$ then

$$\eta_{\beta(n)}\big(\{\beta(h):\ h > n\}\big) = \eta_{\gamma_n(n+1)}\big(\{\gamma_{n+1}(n+2), \ldots, \gamma_{n+i}(n+i+1), \ldots\}\big)\,.$$

Now, for $i \geq 1$, $\gamma_{n+i} < \gamma_n$ holds, hence there exists $s_i \geq n+i+1$ such that $\gamma_{n+i}(n+i+1) = \gamma_n(s_i)$. Consequently

$$\begin{aligned}\{\beta(h):\ h > n\} &= \{\gamma_n(s_i):\ s_i \geq n+i+1,\ i \geq 1\}\\ &\subseteq \{\gamma_n(r):\ r > n+1\}\end{aligned}$$

and finally

$$\eta_{\gamma_n(n+1)}\big(\{\beta(h):\ h > n\}\big) \leq \eta_{\gamma_n(n+1)}\big(\{\gamma_n(r):\ r > n+1\}\big) \leq \varepsilon/k^{n+1}\,.$$

Observe now that if F is a finite subset of $\beta(\mathbb{N})$, we have that

$$\begin{aligned}\lim_n \sum_{h\in F} k^{h-1}\eta_{\beta(n)}(\{h\}) &= \sum_{h\in F} k^{h-1} \lim_n \eta_{\beta(n)}(\{h\})\\ &= \sum_{h\in F} k^{h-1} p_h < (\delta-\varepsilon)/k\,.\end{aligned}$$

Using this observation and the transfinite induction it is possible to construct a sequence $(l_n)_{n\in\mathbb{N}}$ with elements in $\beta(\mathbb{N})$ having the following properties:

$$l_n \in \beta(\mathbb{N})\,, \quad l_n > l_{n-1}\,, \quad l_0 = 0$$

and

$$\sum_{h=1}^{n-1} k^{h-1}\eta_{l_n}(\{l_h\}) < (\delta-\varepsilon)/k\,.$$

Let us show how.

Put $l_1 = \min\beta(\mathbb{N}) \subseteq \mathbb{N}$, $l_1 > 0$. Then $l_1 > l_0$.

Suppose there exists $l_n \in \beta(\mathbb{N})$ such that

$$l_n > l_{n-1} \qquad \text{and} \qquad \sum_{h=1}^{n-1} k^{h-1}\eta_{l_n}(\{l_h\}) < (\delta-\varepsilon)/k\,,$$

we have to show that there exists $l_{n+1} \in \beta(\mathbb{N})$ such that

$$l_{n+1} > l_n \qquad \text{and} \qquad \sum_{h=1}^{n} \eta_{l_{n+1}}(\{l_h\}) < (\delta-\varepsilon)/k\,.$$

Let $F = \{l_1, l_2, \ldots, l_n\} \subseteq \beta(\mathbb{N})$, then,

$$\rho_n = \lim_m \sum_{h=1}^{n} k^{h-1}\eta_{\beta(m)}(\{l_h\}) < (\delta-\varepsilon)/k$$

hence, taken a positive σ_n such that $\sigma_n < \big((\delta-\varepsilon)/k\big) - \rho_n$, it is possible to find $m_n \in \mathbb{N}$ such that

$$\forall\, m > m_n \qquad \left|\sum_{h=1}^{n} k^{h-1}\eta_{\beta(m)}(\{l_h\}) - \rho_n\right| < \sigma_n$$

hence, for $m > m_n$

$$\sum_{h=1}^{n} k^{h-1}\eta_{\beta(m)}(\{l_h\}) < \sigma_n + \rho_n < (\delta-\varepsilon)/k\,.$$

Finally choose

$$l_{n+1} = \min\{l \in \beta(\mathbb{N}) : \ l > l_n \ \text{and} \ l = \beta(m) \ \text{with} \ m > m_n\} .$$

We now conclude the proof.

Let us define $\gamma = (l_n)_{n \in \mathbb{N}}$, then $\gamma < \beta$.

Let $n, m \in \mathbb{N}$ be such that $\beta(m) = \gamma(n)$. Recursively applying the hypothesis (ii) you get

$$\begin{aligned}
\eta_{\gamma(n)}\big(\{\gamma(h) : \ h \neq n\}\big) &\leq k^{n-1}\eta_{\gamma(n)}\big(\{\gamma(h) : \ h > n\}\big) + \sum_{h=1}^{n-1} k^{h-1}\eta_{\gamma(n)}\big(\{\gamma(h)\}\big) \\
&\leq k^{m}\eta_{\beta(m)}\big(\{\beta(h) : \ h > m\}\big) + \sum_{h=1}^{n-1} k^{h-1}\eta_{l_n}\big(\{l_h\}\big) \\
&\leq k^{m}(\varepsilon/k^{m+1}) + (\delta - \varepsilon)/k = \delta/k .
\end{aligned}$$

□

LEMMA 4. *Let $\varepsilon > 0$ and $k \geq 1$. Assume that*

(i) *$\mathcal{R}$ is a ring of subsets of $\mathbb{N}$ such that $\{h\} \in \mathcal{R}$ for all $h \in \mathbb{N}$,*

(ii) *for any $n \in \mathbb{N}$, $\phi_n \colon \mathcal{R} \to [0, \infty]$ is a function satisfying the following properties:*

  (1) *$\big|\phi_n(A) - \phi_n(\{h\})\big| \leq k\phi_n\big(A \setminus \{h\}\big)$ for $h \in A \in \mathcal{R}$,*

  (2) *for all (A_k), disjoint sequence in $\mathcal{R}$, there exists $l \in \mathbb{N}$ such that $\phi_n(A_l) \leq \varepsilon$,*

(iii) *$\lim\limits_{h}\lim\limits_{n}\phi_n\big(\{h\}\big) = 0$,*

(iv) *$\inf\{\phi_m(A) : \ m \in A \subseteq M , \ A \in \mathcal{R}\} \leq \varepsilon$ for any infinite set $M \subseteq \mathbb{N}$.*

Then $\limsup \phi_n\big(\{n\}\big) \leq 2\varepsilon$.

The following theorem provides a criterion for a sequence of s-bounded functions to be uniformly exhaustive. Reading the hypotheses in the light of Proposition 1, it is easy to see that this criterion is applicable to the case of a sequence of k-triangular functions.

THEOREM 5. *Let L be an OMP.*

For $n \in \mathbb{N}$, let $\psi_n \colon L \to [0, \infty]$ be an s-bounded function such that

$$|\psi_n(b) - \psi_n(a)| \leq k\psi_n(b \wedge a') \qquad \text{for} \quad a, b \in L , \ \ a \leq b .$$

The following are equivalent:

(1) *For any orthogonal sequence $(a_k)_{k \in \mathbb{N}}$ in L and any $\alpha \in \Gamma$,*

$$\inf\Big\{\psi_{\alpha(n)}\Big(\bigvee_{k \in A} a_{\alpha(k)}\Big) : \ n \in A \subseteq \mathbb{N}, \ \ \bigvee_{k \in A} a_{\alpha(k)} \ \ \textit{exist in} \ \ L\Big\} = 0$$

and α has a subsequence β such that

$$\lim_k \lim_n \psi_{\beta(n)}\big(a_{\beta(k)}\big) = 0\,.$$

(2) $\{\psi_n : n \in \mathbb{N}\}$ *is uniformly exhaustive.*

The main result

We are now ready to prove the main result of this paper.

An orthomodular poset L has the *subsequential completeness property* (SCP) if for every orthogonal sequence in L there exists a subsequence which has supremum in L (see [9], [10]).

For examples of posets having this property see [29].

THEOREM 6. *Let L be an OMP with the SCP and $k \geq 1$.*

For $n \in \mathbb{N}$, let $\psi_n \colon L \to [0, \infty[$.

Suppose ψ_n to be exhaustive and such that

$$|\psi_n(b) - \psi_n(a)| \leq k\psi_n(b \wedge a') \qquad \text{if} \quad a \leq b\,.$$

If $\psi \colon L \to [0, \infty[$ is an exhaustive function with the property that $\lim\limits_n \psi_n(a) = \psi(a)$ for all $a \in L$, then $\{\psi_n : n \in \mathbb{N}\}$ is uniformly exhaustive and ψ is k-triangular.

Proof. The proof utilizes the preceeding characterization of uniform exhaustivity.

Let $(a_k)_{k\in\mathbb{N}}$ be an orthogonal sequence and let $\alpha \in \Gamma$. It is sufficient to prove that

(1) $\inf\Big\{\psi_{\alpha(n)}\Big(\bigvee\limits_{k\in A} a_{\alpha(k)}\Big) : n \in A \subseteq \mathbb{N},\ \exists \bigvee\limits_{k\in A} a_{\alpha(k)} \in L\Big\} = 0,$

(2) $\exists\, \beta < \alpha \qquad \lim\limits_k \lim\limits_n \psi_{\beta(n)}(a_{\beta(k)}) = 0\,.$

(1) Let $\varepsilon > 0$. If $\alpha(\mathbb{N})$ is an infinite subset of $\mathbb{N}$, it contains a disjoint sequence $(A_n)_{n\in\mathbb{N}}$ of infinite subsets of $\mathbb{N}$. Using the SCP and the s-boundedness of ψ it is possible to construct an infinite subset B of $\alpha(\mathbb{N})$ such that $\psi\Big(\bigvee\limits_{k\in B} a_k\Big) \leq \varepsilon$.

Put $D = \big\{n \in \mathbb{N} : \exists\, m \in B \quad \alpha(n) = m\big\}$. Then

$$\Big\{\psi_m\Big(\bigvee_{k\in B} a_k\Big) : m \in B\Big\} \subseteq \Big\{\psi_{\alpha(n)}\Big(\bigvee_{k\in A} a_{\alpha(k)}\Big) : n \in A \subseteq \mathbb{N},\ \exists \bigvee_{k\in A} a_{\alpha(k)} \in L\Big\}$$

hence

$$\inf\Big\{\psi_{\alpha(n)}\Big(\bigvee_{k\in A} a_{\alpha(k)}\Big) : n \in A \subseteq \mathbb{N},\ \exists \bigvee_{k\in A} a_{\alpha(k)} \in L\Big\} \leq \inf_{m\in B} \psi_m\Big(\bigvee_{k\in B} a_k\Big)\,.$$

Moreover, since $B \subseteq \alpha(\mathbb{N})$, then $\{\psi_m : m \in B\}$ can be regarded as a sequence $(\psi_{\beta(n)})_{n\in\mathbb{N}}$ with $\beta < \alpha$. Therefore

$$\lim_n \psi_{\beta(n)}\Big(\bigvee_{k\in B} a_k\Big) = \psi\Big(\bigvee_{k\in B} a_k\Big) \le \varepsilon$$

hence

$$\inf_{m\in B} \psi_m\Big(\bigvee_{k\in B} a_k\Big) \le \lim_n \psi_{\beta(n)}\Big(\bigvee_{k\in B} a_k\Big) \le \varepsilon .$$

(2) Let $\beta = \alpha$, then

$$\lim_k \lim_n \psi_{\alpha(n)}\big(a_{\alpha(k)}\big) = \lim_k \psi\big(a_{\alpha(k)}\big) = 0$$

in view of the exhaustivity of ψ.

Finally, it is easy to see that ψ is k-triangular. □

Generalization

Let S be a commutative semigroup with a neutral element 0. Suppose S is endowed with a function $f\colon S \to [0,\infty[$ satisfying $f(0) = 0$ and

$$|f(x+y) - f(x)| \le f(y) \qquad \text{for all} \quad x, y \in S .$$

Then (S, f) is called a *triangular semigroup.*

If $(x_n)_{n\in\mathbb{N}}$ is a sequence in S, we say that $(x_n)_{n\in\mathbb{N}}$ converges to 0 ($\lim_n x_n = 0$) if $\lim_n f(x_n) = 0$.

A function $\phi\colon L \to S$ (L OMP) is *k-triangular* ($k \ge 1$) if the composite function $f \circ \phi\colon L \to [0,\infty[$ is k-triangular.

The function ϕ is *exhaustive* (s-bounded) if $\lim_k \phi(a_k) = 0$ for every orthogonal sequence $(a_k)_{k\in\mathbb{N}}$ in L.

A sequence $(\phi_n)_{n\in\mathbb{N}}$ of functions from L to S is *uniformly exhaustive* (uniformly s-bounded) if $\lim_k \phi_n(a_k) = 0$ uniformly in $\mathbb{N}$.

The following generalization of Theorem 6 holds:

THEOREM 7. *Let L be an OMP with the SCP, $k \ge 1$ and let (S, f) be a triangular semigroup.*

For $n \in \mathbb{N}$, let $\phi_n\colon L \to S$ be k-triangular and exhaustive.

If ϕ is an exhaustive function with the property that $\lim_n f\big(\phi_n(a)\big) = \phi(a)$ for all $a \in L$, then $\{\phi_n : n \in \mathbb{N}\}$ is uniformly exhaustive and ϕ is k-triangular.

P r o o f. Apply Theorem 6 to the sequence of functions $(f \circ \phi_n)_{n\in\mathbb{N}}$. □

Remarks. Recently a similar result was published for a sequence of k-triangular functions defined on a difference poset with values in a triangular semigroup ([21]). Its proof is essentially based on [17; Theorem 3.3], which is not comparable with Theorem 6 of this paper, in the sense that, while it considers a less general structure than an orthomodular poset, namely an orthomodular lattice, it gives it a weaker property than SCP, namely the subsequential interpolation property.

The case where L is an orthomodular lattice seems so far to be quite different from the case of an OMP. Indeed, the technique utilized in [17] fails for an OMP even if we assign to it the stronger SCP property.

A very general result of the type Brooks-Jewett is also contained in [11]. Here the functions take values in a uniform space Y and a concept of exhaustivity at a point $x_0 \in Y$ is introduced through the set of all uniformly continuous pseudometric defined on Y.

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