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ON CONVEX LINEARLY ORDERED SUBGROUPS OF AN $h\ell$-GROUP

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ABSTRACT. Giraudet and Lucas proved that if the increasing part $G\uparrow$ of a half lattice ordered group is linearly ordered, then it is abelian. We present a generalization of this result.

1. Introduction

The notion of a half lattice ordered group was introduced and studied in [3]; cf. also [1], [2], [4], [5], [6], [7].

For a half lattice ordered group (briefly: $h\ell$-group) we apply the same notation as in [3]; some definitions are recalled in Sections 2 and 3 below. In particular, the set of all increasing elements of an $h\ell$-group G is denoted by $G\uparrow$; then $G\uparrow$ is a lattice ordered group.

The system of all convex linearly ordered subgroups of an $h\ell$-group G is denoted by $\mathcal{C}(G)$; next, $\mathcal{C}_m(G)$ is the set of all maximal elements of $\mathcal{C}(G)$.

For any group G we put $E(G) = \{x \in G : x \neq x^2 = e\}$, where e is the neutral element of G.

We denote by

$\mathcal{L}$ — the class of all lattice ordered groups;
$\mathcal{H}$ — the class of all $h\ell$-groups ;
$\mathcal{H}_1 = \mathcal{H} \setminus \mathcal{L}$;
$\mathcal{L}_1$ — the class of all $H \in \mathcal{L}$ such that there exists $G \in \mathcal{H}_1$ with $G\uparrow = H$.

The following result has been proven in [3]:

(A) Let $G \in \mathcal{H}_1$ such that the set $G\uparrow$ is linearly ordered. Then the group $G\uparrow$ is abelian.

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We remark that if G satisfies the assumption of (A), then $G\uparrow \in \mathcal{C}_m(G)$ and $G\uparrow$ is a normal subgroup of G.

In the present paper we prove the following generalization of (A):

(A_1) Let $G \in \mathcal{H}_1$ and let $X \in \mathcal{C}_m(G)$. Suppose that X is normal in G. Then X is abelian.

In fact, (A_1) is a particular case of:

(B_1) Let $G \in \mathcal{H}_1$ and let $X \in \mathcal{C}_m(G)$. Suppose that there exists $a \in E(G)$ such that $aX = Xa$. Then $X \in \mathcal{L}_1$ and X is abelian.

Further, we prove:

(B_2) Let $G \in \mathcal{H}_1$ and let $X \in \mathcal{C}_m(G)$. Suppose that X is not abelian. Then for each $a \in E(G)$ the relation $aX \neq Xa$ is valid. Moreover, if $a \in E(G)$, then $Y = aXa$ belongs to $C_m(G)$ and the following conditions are satisfied:

(i) $XY = X \times Y$ is a convex ℓ-subgroup of $G\uparrow$ belonging to $\mathcal{L}_1$;

(ii) X and Y are isomorphic as lattices;

(iii) X and Y are isomorphic as groups.

2. Preliminaries

Let G be a group and suppose that G is, at the same time, a partially ordered set.

We denote by $G\uparrow$ (and $G\downarrow$) the set of all $x \in G$ such that, whenever $y, z \in G$ and $y \leqq z$, then $xy \leqq xz$ (or $xy \geqq xz$, respectively).

1.1. Definition. (cf. [3]) G is called a *half lattice ordered group* if the following conditions are satisfied:

1) the partial order on G is non-trivial;
2) if $x, y, z \in G$ and $y \leqq z$, then $yx \leqq zx$;
3) $G = G\uparrow \cup G\downarrow$;
4) $G\uparrow$ is a lattice.

The neutral element of G is denoted by e. In view of 1), $G \neq \{e\}$.

1.2. Proposition. (cf. [3]) *Let $G \in \mathcal{H}_1$. Then*

(i) *$G\uparrow$ is a subgroup of G having the index 2;*

(ii) *the partially ordered sets $G\uparrow$ and $G\downarrow$ are isomorphic;*

(iii) *if $x \in G\uparrow$ and $y \in G\downarrow$, then the elements x and y are incomparable.*

Let $G \in \mathcal{H}_1$. We denote by $\mathcal{C}(G)$ the system of all subsets X of G such that

(i) X is linearly ordered;

(ii) X is a convex subset of G;
(iii) X is a subgroup of G.

The system $\mathcal{C}(G)$ is partially ordered by set-theoretical inclusion.

If $X \in \mathcal{C}(G)$, then in view of (iii) we have $e \in X$; thus according to (i) and 1.2 we have $X \subseteq G\uparrow$.

1.3. Lemma. *Let $X, Y \in \mathcal{C}(G)$ be such that $X \cap Y \neq \{e\}$. Then either $X \subseteq Y$ or $Y \subseteq X$.*

Proof. Put $X \cap Y = Z$. Assume that Y is not a subset of X. Hence there exists $y \in Y \setminus X$. Without loss of generality we can suppose that $y > e$. We have to prove that the relation $X \subseteq Y$ is valid; by way of contradiction, assume that this relation fails to hold. Hence there exists $x \in X \setminus Y$; again, it suffices to consider the case $x > e$. Both x and y belong to $G\uparrow$, hence there exists $x \wedge y$ in $G\uparrow$. Put $x \wedge y = z$. We have $z \in Z$.

Since $Z \neq \{e\}$ there exists $z_1 \in Z$ with $z_1 \neq e$. From the fact that Z is a subgroup of G we infer that without loss of generality we can assume that $z_1 > e$. Thus for each $e < x_1 \in X \setminus Z$ and each $e < y_1 \in Y \setminus Z$ we must have $x_1 > z_1$ and $y_1 > z_1$. Hence $x_1 \wedge y_1 \geqq z_1$.

In view of the convexity of X and Y we have $x(x \wedge y)^{-1} \in X$ and $y(x \wedge y)^{-1} \in Y$. If $x(x \wedge y)^{-1} \in Z$, then $x \in Z$, which is impossible. Hence $e < x(x \wedge y)^{-1} \in X \setminus Z$. Similarly, $e < y(x \wedge y)^{-1} \in Y \setminus Z$. Then

$$\left(x(x \wedge y)^{-1}\right) \wedge \left(y(x \wedge y)^{-1}\right) = (x \wedge y)(x \wedge y)^{-1} = e .$$

On the other hand, if we put $x_1 = x(x \wedge y)^{-1}$, $y_1 = y(x \wedge y)^{-1}$, then we get $x_1 \wedge y_1 \geqq z_1$. Thus $e \geqq z_1$, which is a contradiction. □

Let $\mathcal{C}_m(G)$ be as in the Introduction.

1.4. Lemma. *Let $X \in \mathcal{C}(G)$, $X \neq \{e\}$. Then there exists an element X^0 of $\mathcal{C}_m(G)$ such that $X^0 \supseteq X$.*

Proof. Put $A = \{Y \in \mathcal{C}(G) : Y \supseteq X\}$. If $Y_1, Y_2 \in A$, then in view of 1.3 we have either $Y_1 \subseteq Y_2$ or $Y_2 \subseteq Y_1$. Put

$$X^0 = \bigcup_{Y_i \in A} Y_i .$$

Then $X^0 \in \mathcal{C}(G)$ and $X^0 \in A$. It is clear that X^0 is a maximal element of $\mathcal{C}(G)$ and that $X^0 \supseteq X$. □

1.5. Lemma. *Let $X, Y \in \mathcal{C}_m(G)$, $X \neq Y$. Then $X \cap Y = \{e\}$.*

Proof. By way of contradiction, assume that $X \cap Y = Z \neq \{e\}$. Then in view of 1.3, X and Y are comparable. This is impossible, since both X and Y are maximal. □

1.6. LEMMA. *Let X and Y be as in* 1.5. *Then*

(i) $xy = yx$ *for each* $x \in X$ *and each* $y \in Y$;
(ii) $XY = X \times Y$.

Proof. From 1.5 and from the convexity of X and Y we conclude that $x \wedge y = e$ whenever $x \in X^+$ and $y \in Y^+$; hence $xy = x \vee y = yx$. Thus (i) holds. Then by a simple calculation we obtain that (ii) is valid. □

1.7. LEMMA. *Let X and Y be as in* 1.5. *Then XY is a convex ℓ-subgroup of $G\uparrow$.*

Proof. Let $u, v \in XY$, $z \in G\uparrow$, $u \leqq z \leqq v$. Then

$$e \leqq zu^{-1} \leqq vu^{-1}$$

and in view of 1.6, $vu^{-1} \in XY$. Thus there are $x \in X^+$, $y \in Y^+$, with $vu^{-1} = xy$. According to Riesz theorem there are $x_1, y_1 \in G\uparrow$ with $e \leqq x_1 \leqq x$, $e \leqq y_1 \leqq y$, $zu^{-1} = x_1 y_1$. We have $x_1 \in X$, $y_1 \in Y$, whence $x_1 y_1 \in XY$ and so $z \in XY$. □

2. Proofs of (B_1) and (B_2)

Let H be a lattice ordered group with the neutral element e, $H \neq \{e\}$. We denote by $I(H)$ the system of all mappings $F\colon H \to H$ such that

(i) F is a group automorphism of the group $(H, \cdot)$;
(ii) F is a dual automorphism of the lattice $(H; \leqq)$;
(iii) $F(F(x)) = x$ for each $x \in H$.

2.1. LEMMA. (cf. [3; III.3]) *For each $F \in I(H)$ there exists $G = G_{H,F} \in \mathcal{H}_1$ such that*

(i) $G\uparrow = H$;
(ii) *there is $a \in E(G)$ with $F(x) = axa$ for each $x \in H$;*
(iii) *G is uniquely determined up to H-isomorphism.*

2.2. PROPOSITION. (cf. [3; I.3.1]) *If $G \in \mathcal{H}_1$, then $E(G) \neq \emptyset$.*

2.3. LEMMA. *Let $G \in \mathcal{H}_1$, $a \in E(G)$. Put $F(x) = axa$ for each $x \in G\uparrow$. Then $F \in I(G\uparrow)$.*

Proof. This is an immediate consequence of the definition of $E(G)$. □

In what follows we suppose that G is an element of $\mathcal{H}_1$, $H = G\uparrow$. Let $X \in \mathcal{C}_m(G)$. Further let a and F be as in 2.3.

Put $F(X) = Y$. Then Y is an element of $\mathcal{C}_m(G)$. We distinguish two cases:

(a) $Y = X$;

(b) $Y \neq X$.

First suppose that (a) is valid. Put $T = X \cup aX$.

2.4. LEMMA. *T is a subgroup of the group G.*

Proof. If $x_1, x_2 \in X$, then clearly $x_1 x_2 \in T$. Further, let $x \in X$, $y \in aX$. Hence $y = ax_1$ for some $x_1 \in X$. Also, $x = F(x')$ for some $x' \in X$. Then

$$xy = ax'aax_1 = ax'x_1 \in aX\,,$$

since $x'x_1 \in X$. Further

$$yx = ax_1x \in aX\,.$$

If y_1 is another element of aX, i.e., $y_1 = ax_2$ for some $x_2 \in X$, then

$$yy_1 = ax_1ax_2 = x_1'x_2 \in X\,,$$

where $x_1' = F(x_1)$.

Thus the set T is closed under the group operation.

Let $t \in T$. If $t \in X$, then $t^{-1} \in X$. Suppose that $t \in aX$, thus $t = ax$ for some $x \in X$. Hence $t^{-1} = x^{-1}a^{-1} = x^{-1}a$. From (a) we conclude that $Xa = aX$, therefore $t^{-1} \in aX$. □

2.5. LEMMA. *T is a half lattice ordered group; moreover, $T \in \mathcal{H}_1$.*

Proof. We have $X \subseteq H = G\uparrow$ and $aX \subseteq G\downarrow$. This yields that $X = T\uparrow$ and $aX = T\downarrow$. Then in view of 1.1 we infer that T is a half lattice ordered group. Further, since $aX \neq \emptyset$, we get $T \in \mathcal{H}_1$. □

2.6. PROPOSITION. *If* (a) *is valid, then the group X is abelian.*

Proof. This is a consequence of (A) and 2.5. □

Now, (B_1) is a corollary of 2.3 and 2.6. Also, (A_1) follows immediately from (B_1).

Suppose that the condition (b) holds. Put

$$Z = XY\,, \qquad T_1 = Z \cup aZ\,.$$

According to the results of Section 1, Z is a convex ℓ-subgroup of $G\uparrow$ and $Z = X \times Y$.

2.7. LEMMA. *T_1 is a subgroup of G.*

Proof. From the definition of Y we obtain

$$aX = Ya, \qquad Xa = aY, \qquad aZ = Za.$$

Let $t_1, t_2 \in T_1$. We have to show that $t_1 t_2$ belongs to T_1.

If $t_1, t_2 \in Z$, then $t_1 t_2 \in Z \subseteq T_1$. Suppose that $t_1 \in Z$, $t_2 \in aZ$. Hence there exist x_i, y_i $(i = 1, 2)$ such that $x_i \in X$, $y_i \in Y$ and

$$t_1 = x_1 y_1, \qquad t_2 = a x_2 y_2.$$

Thus

$$t_1 t_2 = x_1 y_1 a x_2 y_2 = a x_1' y_1' x_2 y_2$$

for some $x_1' \in X$, $y_1' \in Y$. Hence $t_1 t_2 \in T_1$. Similarly, $t_2 t_1 \in T_1$. Next, let $t_2^0 \in aZ$, $t_2^0 = a x_0 y_0$. Then

$$t_2 t_2^0 = a x_2 y_2 a x_0 y_0 = a a x_2' y_2' x_0 y_0 = x_2' y_2' x_0 y_0$$

for some $x_2' \in X$ and $y_2' \in Y$. Hence $t_2 t_2^0 \in T_1$. Therefore T_1 is closed under the group operation. If $z \in Z$, then $z^{-1} \in Z$. Next, $(az)^{-1} = z^{-1} a^{-1} = z^{-1} a \in Za = aZ$, which completes the proof. □

The proof of the following lemma is analogous to that of 2.5.

2.8. LEMMA. *T_1 is a half lattice ordered group; moreover, $T_1 \in \mathcal{H}_1$.*

Proof of (B_2). Let the assumptions of (B_2) be valid. Hence, in particular, X is not abelian. Then in view of 2.6, for each $a \in E(G)$ the relation $aX \neq Xa$ is valid. Now it suffices to apply 2.8, 1.6, 1.7 and the properties of the mappings $F \in I(H)$. □

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