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RADIAL TYPE SOLUTIONS FOR A CLASS OF THIRD ORDER EQUATIONS AND THEIR ITERATES

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ABSTRACT. We obtain solutions of radial type for a class of linear partial differential equations with singular coefficients and of third order. We also obtain similar type of solutions for the iterated forms of the same equations. Here the essential operators include Laplacian.

1. Introduction

Many physical and engineering problems are spherically symmetric and they may be solved by using partial differential equations. Spherically symmetric solutions which depend on the radius r are important for solving this type of problem.

Several authors have studied second order partial differential equations of various types and their iterates. In [1], we obtained solutions of type r^m for a class of singular equations. The essential operators there were second order elliptic or ultra-hyperbolic. In this paper, we consider a class of linear partial differential equations with singular coefficients and of third order, and using a similar method to that given in [1] we shall obtain radial type solutions for this class of equations and their iterates.

Generally here, our equation will be of the following type

$$\left(\prod_{j=1}^{p} L_j^{q_j}\right)u = \left(L_1^{q_1} \dots L_p^{q_p}\right)u = 0 \tag{1.1}$$

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where p and $q_1, \ldots, q_p$ are positive integers and

$$L_j = \sum_{i=1}^{n} \left[\frac{\partial^2}{\partial x_i^2} + \frac{1}{x_i} \left(a_j r^2 \frac{\partial^3}{\partial x_i^3} + b_i^{(j)} \frac{\partial}{\partial x_i} \right) \right] + \frac{c_j}{r^2} . \tag{1.2}$$

The iterated operators $L_j^{q_j}$ are defined by the relations

$$L_j^{k+1}(u) = L_j[L_j^k(u)] , \qquad k = 1, \ldots, q_j - 1 .$$

In (1.2), a_j, $b_i^{(j)}$ $(i = 1, \ldots, n)$ and c_j are arbitrary real parameters and r is the Euclidean distance which is defined by

$$r = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2} . \tag{1.3}$$

Equation (1.1) includes iterated forms of some well-known classical equations such as Laplace's equation and GASPT equations. The regularity domain of solutions of radial type is a spherical domain centered at the origin in $\mathbb{R}^n$.

2. Radial type solutions

Let us first prove the following lemma.

Lemma 1. *Let $q_1, \ldots, q_p$ be arbitrary positive integers. Then*

$$\left(\prod_{j=1}^{p} L_j^{q_j} \right)(r^m) = \left\{ \prod_{j=1}^{p} \prod_{k=0}^{q_j-1} \Phi_j\Big(m - 2[Q(p) - Q(j)] - 2k\Big) \right\} r^{m-2Q(p)} \tag{2.1}$$

where $Q(j) = q_1 + \cdots + q_j$, $1 \le j \le p$, and $\Phi_j(m)$ is a third degree polynomial in m given by

$$\Phi_j(m) = a_j m^3 + [3(n-2)a_j + 1]m^2 + [(8-6n)a_j + n + b_j^* - 2]m + c_j \tag{2.2}$$

with

$$b_j^* = \sum_{i=1}^{n} b_i^{(j)} .$$

Proof. From the definitions of L_j and r, for any real parameter m, we have

$$\begin{aligned} L_j(r^m) &= [a_j m(m-2)(m-4+3n) + m(m-2) + m(n+b_j^*) + c_j] r^{m-2} \\ &= \Phi_j(m) r^{m-2} . \end{aligned} \tag{2.3}$$

Applying the operator L_j repeatedly on both sides of (2.3) we then obtain by induction that

$$L_j^q(r^m) = \left\{ \prod_{k=0}^{q-1} \Phi_j(m-2k) \right\} r^{m-2q} . \tag{2.4}$$

Now replace j and q in (2.4) by p and q_p respectively, then we have

$$L_p^{q_p}(r^m) = \left\{ \prod_{k=0}^{q_p-1} \Phi_p(m-2k) \right\} r^{m-2q_p} . \tag{2.5}$$

Applying in succession the operators

$$L_{p-1}^{q_{p-1}}, L_{p-2}^{q_{p-2}}, \dots, L_1^{q_1}$$

on both sides of (2.5), we readily obtain the formula (2.1) by induction. □

Now consider the formula (2.1) and write the algebraic polynomial equation

$$\prod_{j=1}^{p} \prod_{k=0}^{q_j-1} \Phi_j(m - 2[Q(p) - Q(j)] - 2k) = 0 \tag{2.6}$$

which is of degree $3Q(p)$. When all $a_j \neq 0$ $(j = 1, \dots, p)$, the number of roots (real or complex) of equation (2.6) is $3Q(p)$.

Using Lemma 1, we can now prove the following theorem.

THEOREM 1. *Let the algebraic polynomial equation* (2.6) *have distinct real roots* $\gamma_1, \dots, \gamma_M$ *each having multiplicity* $\lambda_1, \dots, \lambda_M$*, respectively, and distinct complex roots* $\alpha_1 \pm \mathrm{i}\beta_1, \dots, \alpha_N \pm \mathrm{i}\beta_N$*, each having multiplicity* $\mu_1, \dots, \mu_N$*, respectively. Then the solutions of type* $u = r^m$ *of equation* (1.1) *are given by the formula*

$$u = \sum_{v=1}^{M} \sum_{j=0}^{\lambda_v-1} A_{vj} r^{\gamma_v} (\ln r)^j + \sum_{s=1}^{N} \sum_{k=0}^{\mu_s-1} r^{\alpha_s} (\ln r)^k \left[B_{sk} \cos(\beta_s \ln r) + C_{sk} \sin(\beta_s \ln r) \right] \tag{2.7}$$

where A_{vj}, B_{sk} *and* C_{sk} *are arbitrary constants.*

Proof. By the hypothesis concerning the real and complex roots of (2.6), we have the following two factors for this algebraic equation

$$\prod_{v=1}^{M} (m - \gamma_v)^{\lambda_v} \qquad \text{and} \qquad \prod_{s=1}^{N} \left(m^2 - 2\alpha_s m + \alpha_s^2 + \beta_s^2\right)^{\mu_s} .$$

Therefore, (2.1) can be written as

$$\left(\prod_{j=1}^{p} L_j^{q_j}\right)(r^m) = \left\{K \prod_{v=1}^{M}(m-\gamma_v)^{\lambda_v} \prod_{s=1}^{N}\left(m^2 - 2\alpha_s m + \alpha_s^2 + \beta_s^2\right)^{\mu_s}\right\} r^{m-2Q(p)} \tag{2.8}$$

where $K = \prod_{j=1}^{p} a_j^{q_j}$, $\lambda_v \in \{1,2,\dots,3Q(p)\}$, $\mu_s \in \{0,1,\dots,Q(p)\}$ and $\sum_{v=1}^{M} \lambda_v + 2\sum_{s=1}^{N} \mu_s = 3Q(p)$ is the order of equation (1.1).

On the other hand, the following equalities are well known

$$\begin{aligned}\frac{\partial^\ell}{\partial m^\ell}\left(\prod_{j=1}^{p} L_j^{q_j}\right)(r^m) &= \left(\prod_{j=1}^{p} L_j^{q_j}\right)\left(\frac{\partial^\ell r^m}{\partial m^\ell}\right)\\ &= \left(\prod_{j=1}^{p} L_1^{q_j}\right)\left[r^m(\ln r)^\ell\right]\ ; \qquad \ell \in \mathbb{N},\end{aligned} \tag{2.9}$$

$$r^{\alpha_s \pm \mathrm{i}\beta_s} = r^{\alpha_s}\left[\cos(\beta_s \ln r) \pm \mathrm{i}\sin(\beta_s \ln r)\right]. \tag{2.10}$$

Now again consider (2.8). It is obvious that the right hand side of (2.8) has the factor $(m-\gamma_v)^{\lambda_v}$ which vanishes for $m = \gamma_v$ together with its derivatives (with respect to m)

$$\frac{\mathrm{d}^j}{\mathrm{d}m^j}(m-\gamma_v)^{\lambda_v}\ ; \qquad j = 1,\dots,\lambda_v - 1;\ \ v = 1,\dots,M.$$

Thus the function r^{γ_v}, and by (2.9), each of the functions

$$\frac{\mathrm{d}^j}{\mathrm{d}m^j}(r^m)\big|_{m=\gamma_v} = r^{\gamma_v}(\ln r)^j\ ; \qquad j = 1,\dots,\lambda_v - 1;\ \ v = 1,\dots,M$$

satisfy (1.1). Since this equation is linear, by the superposition principle the sum

$$\sum_{v=1}^{M} \sum_{j=0}^{\lambda_v - 1} A_{vj} r^{\gamma_v}(\ln r)^j \tag{2.11}$$

also satisfies (1.1). Similarly, the factors of (2.8)

$$\left(m^2 - 2\alpha_s m + \alpha_s^2 + \beta_s^2\right)^{\mu_s} = \left[m - (\alpha_s + \mathrm{i}\beta_s)\right]^{\mu_s}\left[m - (\alpha_s - \mathrm{i}\beta_s)\right]^{\mu_s}$$

and the expressions

$$\frac{\mathrm{d}^k}{\mathrm{d}m^k}\left[m - (\alpha_s \pm \mathrm{i}\beta_s)\right]^{\mu_s}\ ; \qquad k = 1,\dots,\mu_s - 1;\ \ s = 1,\dots,M$$

are zero for $m = \alpha_s \pm \mathrm{i}\,\beta_s$. Hence by (2.9) and (2.10), for $k = 0, 1, \ldots, \mu_s - 1$, $s = 1, \ldots, N$, each of the functions

$$\begin{aligned} \frac{\mathrm{d}^k}{\mathrm{d}m^k}(r^m)\big|_{m=\alpha_s \pm \mathrm{i}\,\beta_s} &= r^{\alpha_s \pm \mathrm{i}\,\beta_s}(\ln \tilde{r})^k \\ &= r^{\alpha_s}(\ln r)^k\big[\cos(\beta_s \ln r) \pm \mathrm{i}\sin(\beta_s \ln r)\big] \end{aligned}$$

and their superposition

$$\sum_{s=1}^{N}\sum_{k=0}^{\mu_s - 1} r^{\alpha_s}(\ln r)^k\big[B_{sk}\cos(\beta_s \ln r) + C_{sk}\sin(\beta_s \ln r)\big] \tag{2.12}$$

satisfy (1.1). Therefore, the sum of (2.11) and (2.12) gives (2.7). Thus the theorem is proved. □

Since the solution (2.7) depends only on the radial distance r, we have called this a radial type solution.

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