Michal Fečkan

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# HIGHER DIMENSIONAL MELNIKOV MAPPINGS

MICHAL FEČKAN

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ABSTRACT. Higher dimensional Melnikov mappings are introduced for detecting the existence of transversal homoclinic orbits of period maps of autonomous ordinary differential equations with periodic nonautonomous perturbations.

## 1. Introduction

In this note, we consider ordinary differential equations of the form

$$\dot{x} = f(x) + h(x, \mu, t) \tag{1.1}$$

with $x \in \mathbb{R}^n$, $\mu \in \mathbb{R}^m$. We make the following assumptions about (1.1):

(i) $f$ and $h$ are $C^3$ in all arguments.

(ii) $f(0) = 0$ and $h(\cdot\,, 0, \cdot) = 0$.

(iii) The eigenvalues of $Df(0)$ lie off the imaginary axis.

(iv) The unperturbed equation has a homoclinic solution. That is, there exists a nonzero differentiable function $t \mapsto \gamma(t)$ such that $\lim\limits_{t\to+\infty} \gamma(t) = \lim\limits_{t\to-\infty} \gamma(t) = 0$ and $\dot{\gamma}(t) = f\bigl(\gamma(t)\bigr)$.

(v) $h(x, \mu, t+1) = h(x, \mu, t)$ for $t \in \mathbb{R}$.

Let $\Psi_\mu$ be the period map of (1.1), i.e. $\Psi_\mu(x) = \phi_\mu(x, 1)$ where $\phi_\mu(x, t)$ is the solution of (1.1) with initial condition $\phi_\mu(x, 0) = x$.

The purpose of this paper is to find a set of parameters $\mu$ for which the periodic map $\Psi_\mu$ of (1.1) has a transversal homoclinic orbit. For this purpose, higher dimensional Melnikov mappings are introduced. Simple zero points of these mappings give wedge-shaped region for $\mu$ in $\mathbb{R}^m$ where $\Psi_\mu$ possesses

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transversal homoclinic orbits. This result is a generalization of [6] when $\gamma$ is required to be nondegenerate and $m = 1$. The results of this paper are based on [2] – [4].

Finally we note that similar problems have been studied in [1] and also in [5] but by different methods by Joseph Gruendler whom the author thanks for some valuable discussions. The main difference between this paper and [5] is that by using methods from [2] – [4] for an appropriate nonlinear equation (see (2.3) below), we not only prove an existence result for homoclinic orbits of $\Psi_\mu$, which is omitted in [3] and [4] (Theorem 2.3 below does not follow from [4; Theorem 12]), but also we simultaneously establish the transversality of those orbits. In [5], another direct approach is developed for showing this transversality. Consequently, Theorem 2.3 below predicts the existence of transversal homoclinic orbits of $\Psi_\mu$ and [4; Theorem 12] gives bounded solutions of (1.1).

## 2. Melnikov mappings

We begin by considering the unperturbed equation

$$\dot{x} = f(x)\,. \tag{2.1}$$

For (2.1) we adopt the standard notation $W^s$, $W^u$ for the stable and unstable manifolds, respectively, at the origin and $d_s = \dim(W^s)$, $d_u = \dim(W^u)$. Since $x = 0$ is a hyperbolic equilibrium, $\gamma$ must approach the origin along $W^s$ as $t \to +\infty$ and along $W^u$ as $t \to -\infty$. Thus, $\gamma$ lies on $W^s \cap W^u$.

By the variational equation along $\gamma$ we mean the linear differential equation

$$\dot{u}(t) = Df\big(\gamma(t)\big)u(t)\,. \tag{2.2}$$

The next result is proved in [3; p. 706] and [4; Theorem 2].

**THEOREM 2.1.** *There exists a fundamental solution $U$ for (2.2) together with constants $M > 0$, $K_0 > 0$ and four projections $P_{ss}$, $P_{su}$, $P_{us}$, $P_{uu}$ such that $P_{ss} + P_{su} + P_{us} + P_{uu} = I$ and the following hold:*

(i) $|U(t)(P_{ss} + P_{us})U(s)^{-1}| \le K_0 e^{2M(s-t)}$ *for* $0 \le s \le t$,

(ii) $|U(t)(P_{su} + P_{uu})U(s)^{-1}| \le K_0 e^{2M(t-s)}$ *for* $0 \le t \le s$,

(iii) $|U(t)(P_{ss} + P_{su})U(s)^{-1}| \le K_0 e^{2M(t-s)}$ *for* $t \le s \le 0$,

(iv) $|U(t)(P_{us} + P_{uu})U(s)^{-1}| \le K_0 e^{2M(s-t)}$ *for* $s \le t \le 0$.

*Also, there exists an integer $d$ with* $\operatorname{rank} P_{ss} = \operatorname{rank} P_{uu} = d$.

In the language of exponential dichotomies ([6]), we see that Theorem 2.1 provides a two-sided exponential dichotomy. For $t \to -\infty$ an exponential dichotomy is given by the fundamental solution $U$ and the projection $P_{us} + P_{uu}$ while for $t \to +\infty$ a similar exponential dichotomy is given by $U$ and $P_{ss} + P_{us}$.

Let $u_j$ denote the $j$th column of $U$ and assume these are numbered so that

$$P_{uu} = \begin{pmatrix} I_d & 0_d & 0 \\ 0_d & 0_d & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad P_{ss} = \begin{pmatrix} 0_d & 0_d & 0 \\ 0_d & I_d & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here, $I_d$ denotes the $d\times d$ identity matrix and $0_d$ denotes the $d\times d$ zero matrix.

For each $i = 1, \dots, n$ we define $u_i^\perp(t)$ by $\langle u_i^\perp(t), u_j(t)\rangle = \delta_{ij}$, where $\langle \cdot\,,\cdot\rangle$ is the scalar product on $\mathbb{R}^n$. The vectors $u_i^\perp$ can be computed from the formula $U^{\perp *} = U^{-1}$ where $U^\perp$ denotes the matrix with $u_j^\perp$ as column $j$. Differentiating $UU^{\perp *} = I$ we obtain $\dot{U}U^{\perp *} + U\dot{U}^{\perp *} = 0$ so that $\dot{U}^\perp = -(U^{-1}\dot{U}U^{\perp *})^* = -Df(\gamma)^* U^\perp$. Thus, $U^\perp$ is the adjoint of $U$.

The function $\dot\gamma$ is always a solution to the variational equation (2.2) and we may assume that $u_{2d} = \dot\gamma$, since $\dot\gamma$ is a linear combination of the columns $u_{d+1}$ to $u_{2d}$ of $U$ and a linear transformation of these columns preserves the projections.

Now we define the following Banach spaces

$$Z = \left\{ z \in C^0\big((-\infty,\infty), \mathbb{R}^n\big) \mid \sup_{t\in\mathbb{R}} |z(t)| < \infty \right\},$$
$$Y = \left\{ z \in C^1\big((-\infty,\infty), \mathbb{R}^n\big) \mid z, \dot z \in Z \right\}.$$

Without loss of generality, we can suppose that $f$ and $h$ as well as all their partial derivatives up to order 3 are uniformly bounded over the whole spaces of definition.

We study the equation

$$F_{\mu,\varepsilon,y}(x) = \dot x - f(x) - h(x,\mu,t) - \varepsilon|\mu| L(x-y) = 0, \tag{2.3}$$
$$F_{\mu,\varepsilon,y}\colon Y \to Z,$$

where $L\colon Y \to Z$ is a linear continuous mapping such that $\|L\| \le 1$, $y \in Y$ and $\varepsilon \in \mathbb{R}$ is small. It is clear that solutions of (2.3) near $\gamma$ with $\varepsilon = 0$ are homoclinic solutions of (1.1).

We make the change of variable in (2.3)

$$x(t) = \gamma(t-\alpha) + w(t), \qquad \langle w(0), \dot\gamma^\perp(-\alpha)\rangle = 0, \tag{2.4}$$

where $\alpha \in \mathcal{I} \subset \mathbb{R}$ and $\mathcal{I}$ is a given bounded open interval. We note that (2.4) defines a tubular neighbourhood of the manifold $\{\gamma(t-\alpha)\}_{\alpha\in\mathcal{I}}$ in $Y$ when $w$ is sufficiently small. Hence (2.3) has the form

$$\begin{aligned} G_{\alpha,\mu,\varepsilon,y}(w) = \dot w - f\big(\gamma(t-\alpha)+w\big) + f\big(\gamma(t-\alpha)\big) \\ - h\big(\gamma(t-\alpha)+w,\mu,t\big) - \varepsilon|\mu| L\big(w+\gamma(t-\alpha)-y\big) = 0 \end{aligned}$$
$$G_{\alpha,\mu,\varepsilon,y}\colon Y \to Z.$$

We have

$$D_w G_{\alpha,0,0,y}(0)u = \dot{u} - Df\big(\gamma(t-\alpha)\big)u\,.$$

By putting

$$U_\alpha(t) = U(t-\alpha)\,, \qquad U_\alpha^\perp(t) = U^\perp(t-\alpha)\,,$$

Theorem 2.1 holds when $U$ is replaced by $U_\alpha$ and (2.2) by

$$\dot{u} = Df\big(\gamma(t-\alpha)\big)u\,,$$

respectively, but $K_0 > 0$ should be enlarged. We note that $\alpha \in \mathcal{I} \subset \mathbb{R}$ and $\mathcal{I}$ is a given bounded open interval. Moreover, we put

$$\gamma_\alpha(t) = \gamma(t-\alpha)\,, \qquad u_{j,\alpha} = u_j(t-\alpha)\,, \qquad u_{j,\alpha}^\perp = u_j^\perp(t-\alpha)\,.$$

Consequently, by putting

$$Q = \Big\{ y \in Y \mid \sup_{t\in\mathbb{R}} \big(|y(t)| + |\dot{y}(t)|\big) < \sup_{t\in\mathbb{R}} \big(|\gamma(t)| + |\dot{\gamma}(t)|\big) + 1 \Big\}$$

and by using the same approach as in [3; p. 709] and [4], there are open small neighborhoods $0 \in O \subset \mathbb{R}^{d-1}$, $0 \in V \subset \mathbb{R}$, $0 \in W \subset \mathbb{R}^m$ and a mapping

$$G \in C^3(Y \times O \times \mathcal{I} \times W \times V \times Q, Z)$$

such that any solution of (2.3) near $\gamma_\alpha$ for $\mu \in W$, $\varepsilon \in V$, $y \in Q$ is determined by the equation $G(z,\beta,\alpha,\mu,\varepsilon,y) = 0$ and this solution has the form

$$x = \gamma_\alpha + z\,, \qquad P_{ss} U_\alpha^{-1}(0)\Big(z(0) - \sum_{j=1}^{d-1} \beta_j u_{j+d,\alpha}(0)\Big) = 0\,, \tag{2.5}$$

where $\beta = (\beta_1,\dots,\beta_{d-1})$. We remark that $\{u_{j,\alpha}(0)\}_{j=1}^n$ are linearly independent, $u_{2d,\alpha}(0) = \dot{\gamma}_\alpha(0) = \dot{\gamma}(-\alpha)$; also

$$\big\{ v \in \mathbb{R}^n \mid \langle v, \dot{\gamma}^\perp(-\alpha)\rangle = 0 \big\} = \operatorname{span}\big\{ \{u_{j,\alpha}(0)\}_{j=1}^n \setminus \{u_{2d,\alpha}(0)\} \big\}\,,$$

and

$$0 = P_{ss} U_\alpha^{-1}(0) w = P_{ss} U_\alpha^{\perp *}(0) w \iff \forall\, 1 \le i \le d \quad \langle u_{j+d,\alpha}^\perp(0), w\rangle = 0\,.$$

Hence (2.4) and (2.5) provide a suitable decomposition of any $x$ in (2.3) near the manifold $\{\gamma(t-\alpha)\}_{\alpha\in\mathcal{I}}$.

Now by using the Lyapunov-Schmidt procedure (see again [3; p. 709] and [4; Theorem 8]), the study of the equation $G(z,\beta,\alpha,\mu,\varepsilon,y) = 0$ can be expressed in the following theorem for $z$, $\mu$, $\varepsilon$, $\beta$ small, $y \in Q$ and $\alpha \in \mathcal{I}$.

**THEOREM 2.2.** *Let $U$ and $d$ be as in Theorem 2.1. Then there exist small neighborhoods $0 \in O_1 \subset \mathbb{R}^{d-1}$, $0 \in W_1 \subset \mathbb{R}^m$, $0 \in V_1 \subset \mathbb{R}$ and a $C^3$ function $H: Q \times O_1 \times \mathcal{I} \times W_1 \times V_1 \to \mathbb{R}^d$ denoted $(y, \beta, \alpha, \mu, \varepsilon) \mapsto H(y, \beta, \alpha, \mu, \varepsilon)$ with the following properties:*

(i) *The equation $H(y, \beta, \alpha, \mu, \varepsilon) = 0$ holds if and only if (2.3) has a solution near $\gamma_\alpha$ and moreover, each such $(y, \beta, \alpha, \mu, \varepsilon)$ determines a unique solution of (2.3),*

(ii) $H(y, 0, \alpha, 0, 0) = 0$,

(iii)

$$\frac{\partial H_i}{\partial \mu_j}(y, 0, \alpha, 0, 0) = -\int_{-\infty}^{\infty} \left\langle u_{i,\alpha}^{\perp}(t), \frac{\partial h}{\partial \mu_j}(\gamma_\alpha(t), 0, t)\right\rangle dt$$
$$= -\int_{-\infty}^{\infty} \left\langle u_i^{\perp}(t), \frac{\partial h}{\partial \mu_j}(\gamma(t), 0, t+\alpha)\right\rangle dt,$$

(iv) $\frac{\partial H_i}{\partial \beta_j}(y, 0, \alpha, 0, 0) = 0$,

(v)

$$\frac{\partial^2 H_i}{\partial \beta_k \partial \beta_j}(y, 0, \alpha, 0, 0) = -\int_{-\infty}^{\infty} \left\langle u_{i,\alpha}^{\perp}, D^2 f(\gamma_\alpha) u_{d+j,\alpha} u_{d+k,\alpha}\right\rangle dt$$
$$= -\int_{-\infty}^{\infty} \left\langle u_i^{\perp}, D^2 f(\gamma) u_{d+j} u_{d+k}\right\rangle dt.$$

We introduce the following notations.

$$a_{ij}(\alpha) = -\int_{-\infty}^{\infty} \left\langle u_i^{\perp}(t), \frac{\partial h}{\partial \mu_j}(\gamma(t), 0, t+\alpha)\right\rangle dt,$$

$$b_{ijk} = -\int_{-\infty}^{\infty} \left\langle u_i^{\perp}, D^2 f(\gamma) u_{d+j} u_{d+k}\right\rangle dt.$$

Finally, we take the mapping $M_\mu: \mathbb{R}^d \to \mathbb{R}^d$ defined by

$$\left(M_\mu(\alpha, \beta)\right)_i = \sum_{j=1}^{m} a_{ij}(\alpha)\mu_j + \frac{1}{2}\sum_{j,k=1}^{d-1} b_{ijk}\beta_j\beta_k.$$

Now we can state the main result of this note.

**THEOREM 2.3.** *Let $d > 1$. If $M_{\mu_0}$ has a simple zero point $(\alpha_0, \beta_0)$, i.e., $(\alpha_0, \beta_0)$ satisfies $M_{\mu_0}(\alpha_0, \beta_0) = 0$ and $D_{(\alpha,\beta)} M_{\mu_0}(\alpha_0, \beta_0)$ is a regular matrix, then there is a wedge-shaped region in $\mathbb{R}^m$ for $\mu$ of the form*

$$\mathcal{R} = \{ s^2 \tilde{\mu} \mid s \text{ is from a small open neighborhood of } 0 \in \mathbb{R} \text{ and } \tilde{\mu} \text{ is from a small open neighborhood of } \mu_0 \in \mathbb{R}^m \text{ satisfying } |\tilde{\mu}| = |\mu_0| \}$$

*such that for any $\mu \in \mathcal{R} \setminus \{0\}$, the period map $\Psi_\mu$ of the equation (1.1) possesses a transversal homoclinic orbit.*

Proof. Let us take $\mathcal{I} = (\alpha_0 - 1, \alpha_0 + 1)$ and let us consider the mapping defined by

$$\Phi(y, \tilde{\beta}, \alpha, \tilde{\mu}, \tilde{\varepsilon}, s) = \begin{cases} \frac{1}{s^2} H(y, s\tilde{\beta}, \alpha, s^2\tilde{\mu}, s^3\tilde{\varepsilon}) & \text{for } s \neq 0, \\ M_{\tilde{\mu}}(\alpha, \tilde{\beta}) & \text{for } s = 0. \end{cases}$$

According to (ii) – (v) of Theorem 2.2, the mapping $\Phi$ is $C^1$ smooth near

$$(y, \tilde{\beta}, \alpha, \tilde{\mu}, \tilde{\varepsilon}, s) = (y, \beta_0, \alpha_0, \mu_0, 0, 0), \qquad y \in Q$$

with respect to the variables $\tilde{\beta}$, $\alpha$. Since

$$M_{\mu_0}(\alpha_0, \beta_0) = 0 \qquad \text{and} \qquad D_{(\alpha,\beta)} M_{\mu_0}(\alpha_0, \beta_0) \quad \text{is a regular matrix},$$

we can apply the implicit function theorem to find a local and unique solution of the equation $\Phi = 0$ in the variables $\tilde{\beta}$, $\alpha$, where $\tilde{\mu}$ is near $\tilde{\mu}_0$ satisfying $|\tilde{\mu}| = |\mu_0|$. By (i) of Theorem 2.2, this gives for $\varepsilon = 0$ the existence of $\mathcal{R}$ on which $\Psi_\mu$ has a homoclinic orbit. Moreover, we may suppose that the corresponding solutions of (2.3) lie in $Q$.

To prove the transversality of these homoclinic orbits, we fix $\mu \in \mathcal{R} \setminus \{0\}$ and take

$$y = \tilde{\gamma},$$

where $\tilde{\gamma}$ is the solution of (2.3) for which the transversality of the corresponding homoclinic orbit of $\Psi_\mu$ must be proved. Then we vary $\varepsilon = s^3\tilde{\varepsilon}$ sufficiently small. Note that $s \neq 0$ is also fixed because $\mu = s^2\tilde{\mu}$ and also $|\tilde{\mu}| = |\mu_0|$. Since the local uniqueness of solutions of (2.3) close to $\tilde{\gamma}$ is satisfied for any $\tilde{\varepsilon}$ sufficiently small according to the above application of the implicit function theorem, such equation (2.3) (*with fixed $\mu \in \mathcal{R} \setminus \{0\}$, $\varepsilon = s^3\tilde{\varepsilon}$ where $s \neq 0$ is also fixed and special $y = \tilde{\gamma}$*) has the unique solution $x = \tilde{\gamma}$ near $\tilde{\gamma}$ for any $\tilde{\varepsilon}$ sufficiently small.

Hence [2; Theorem] gives the invertibility of $DF_{\mu,0,\tilde{\gamma}}(\tilde{\gamma})$ and so the only bounded solution on $\mathbb{R}$ of the equation

$$\dot{v} = Df(\tilde{\gamma})v + D_x h(\tilde{\gamma}, \mu, t)v$$

is $v = 0$. Then [6; Corollary 3.6] implies the transversality of these homoclinic orbits of $\Psi_\mu$ for $\mu \in \mathcal{R} \setminus \{0\}$. □

**Remark 2.4.**

a) If $M_{\mu_0}$ has a simple zero point $(\alpha_0, \beta_0)$, then $M_{r^2\mu_0}$ also has a simple zero point at $(\alpha_0, r\beta_0)$ for any $r \in \mathbb{R} \setminus \{0\}$.

b) If $d = 1$, then we take the function $M_\mu(\alpha) = \sum_{j=1}^{m} a_{1j}(\alpha)\mu_j$, which is the usual Melnikov function. So for any simple zero $\alpha_0$ of $M_{\mu_0}(\alpha) = 0$, when $\mu_0$ is fixed, there is a two-sided wedge-shaped region in $\mathbb{R}^m$ for $\mu$ of the form

$$\mathcal{R} = \{ s\tilde{\mu} \mid s \text{ is from a small open neighborhood of } 0 \in \mathbb{R} \text{ and } \tilde{\mu} \text{ is from a small open neighborhood of } \mu_0 \in \mathbb{R}^m \text{ satisfying } |\tilde{\mu}| = |\mu_0| \}$$

such that for any $\mu \in \mathcal{R} \setminus \{0\}$, the period map $\Psi_\mu$ of the equation (1.1) possesses a transversal homoclinic orbit.

## 3. An example

We complete this note with the following example. Consider the equation

$$\begin{aligned} \ddot{x} &= x - 2xz^2 + \dot{x}^2 + \mu_1 \cos\omega t - \mu_2 z, \\ \ddot{y} &= y - 2yz^2 + \dot{x}\dot{y}, \\ \ddot{z} &= z - 2z^3 + y\dot{y} + \mu_1 \cos\omega t + (\mu_2 - \mu_1)\dot{z}. \end{aligned} \tag{3.1}$$

This equation is studied in Example 1 of [4]. In the space $(x, \dot{x}, y, \dot{y}, z, \dot{z})$, the eigenvalues of $Df(0)$ are $\lambda_1 = \lambda_2 = \lambda_3 = -1$, $\lambda_4 = \lambda_5 = \lambda_6 = 1$. When $\mu = 0$ a homoclinic solution is given by $x = 0$, $y = 0$, $z = r$, i.e. $\gamma = (0, 0, 0, 0, r, \dot{r})$ where $r(t) = \operatorname{sech} t$. Note $\ddot{r} = r - r^3$ and $\ddot{z} = z - 2z^3$ is the familiar Duffing's equation.

In Example 5 of [3], $d = 3$ and

$$\begin{aligned} u_1^\perp &= (-\dot{r}, r, 0, 0, 0, 0), \\ u_2^\perp &= (0, 0, -\dot{r}, r, 0, 0,), \\ u_3^\perp &= (0, 0, 0, 0, -\ddot{r}, \dot{r}). \end{aligned}$$

Using these results, we easily get

$$M_\mu(\alpha, \beta_1, \beta_2) = \begin{cases} a_{11}(\alpha)\mu_1 + 2\mu_2 - \frac{\pi}{8}\beta_1^2, \\ -\frac{\pi}{8}\beta_1\beta_2, \\ a_{31}(\alpha)\mu_1 - \frac{2}{3}\mu_2 - \frac{\pi}{8}\beta_2^2, \end{cases}$$

where

$$a_{11}(\alpha) = -\pi \cos \omega\alpha \operatorname{sech} \frac{\pi\omega}{2}, \qquad a_{31}(\alpha) = \frac{2}{3} - \pi\omega \sin \omega\alpha \operatorname{sech} \frac{\pi\omega}{2}.$$

There are the following solutions of $M_\mu(\alpha, \beta) = 0$ (see Remark 2.4a)

$$\beta(\alpha) = \left(\sqrt{\frac{8}{\pi}(a_{11} + 3a_{31})}, 0\right), \qquad \mu(\alpha) = \left(1, \tfrac{3}{2}a_{31}\right), \tag{i}$$

$$\beta(\alpha) = \left(0, \sqrt{\frac{8}{3\pi}(a_{11} + 3a_{31})}\right), \qquad \mu(\alpha) = \left(1, -\tfrac{1}{2}a_{11}\right). \tag{ii}$$

The linearization $D_{(\alpha,\beta)}M_\mu(\alpha, \beta)$ at the points (i) is

$$\begin{pmatrix} a'_{11} & -\frac{\pi}{4}\sqrt{\frac{8}{\pi}(a_{11} + 3a_{31})} & 0 \\ 0 & 0 & -\frac{\pi}{8}\sqrt{\frac{8}{\pi}(a_{11} + 3a_{31})} \\ a'_{31} & 0 & 0 \end{pmatrix},$$

and at all points (ii) has the form

$$\begin{pmatrix} a'_{11} & 0 & 0 \\ 0 & -\frac{\pi}{8}\sqrt{\frac{8}{3\pi}(a_{11} + 3a_{31})} & 0 \\ a'_{31} & 0 & -\frac{\pi}{4}\sqrt{\frac{8}{3\pi}(a_{11} + 3a_{31})} \end{pmatrix}.$$

Since $\lim_{\omega\to\infty} \left(a_{11}(\alpha) + 3a_{31}(\alpha)\right) = 2$, we see for $\omega$ sufficiently large that points (i), respectively (ii), are simple zero points when $\alpha \neq \frac{\pi(2k+1)}{2\omega}$, $k = \{0, 1, \dots\}$, respectively $\alpha \neq \frac{\pi k}{\omega}$, $k = \{0, 1, \dots\}$.

Hence for $\omega$ sufficiently large, there are two small open wedge-shaped regions in the $\mu_1$–$\mu_2$ plane with the limit slopes given by

$$1 \pm \frac{3}{2}\pi\omega \operatorname{sech} \frac{\pi\omega}{2} \qquad \text{and} \qquad \pm \frac{\pi}{2} \operatorname{sech} \frac{\pi\omega}{2}$$

containing parameters for which the period map of (3.1) possesses a transversal homoclinic orbit near $\gamma$. Since according to the above results the limit slopes correspond to nonsimple zero points of $M_\mu(\alpha, \beta)$, the intersection of the closures of these wedge-shaped regions with their limit slopes is by Theorem 2.3 the set $\{(0,0)\}$. On the other hand, for any slope included between these limit slopes, there is a curve tangent to this slope on which by [4; Example 1] the period map of (3.1) has a homoclinic orbit.

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*Department of Mathematical Analysis*
*Comenius University*
*Mlynská dolina*
*SK–842 15 Bratislava*
*SLOVAKIA*
*E-mail*: Michal.Feckan@fmph.uniba.sk

*Mathematical Institute*
*Slovak Academy of Sciences*
*Štefánikova 49*
*SK–814 73 Bratislava*
*SLOVAKIA*