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ON THE DIFFERENCES OF UPPER SEMICONTINUOUS QUASI-CONTINUOUS FUNCTIONS

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ABSTRACT. Functions which can be written as the difference of upper semicontinuous quasi-continuous functions or the difference of Darboux upper semicontinuous quasi-continuous functions are characterized.

1. Introduction

In 1984, J. G. Ceder and T. L. Pearson asked which functions can be written as a difference of two Darboux upper semicontinuous functions ([1; p. 186]). As each upper semicontinuous function is the sum of two Darboux upper semicontinuous quasi-continuous functions ([6; Corollary]), one could expect that the same holds for the differences of Darboux upper semicontinuous functions. However it does not.

Since upper semicontinuous functions are Baire one, their difference is in the first class of Baire, too. In 1921 W. Sierpiński constructed a bounded Baire one function which cannot be written as a difference of two upper semicontinuous functions ([8]). As in [5; p. 132], we will denote the class of all differences of upper semicontinuous functions by $\widehat{\mathcal{B}}_1$. It is proved in [7] that the class of all Darboux functions in $\widehat{\mathcal{B}}_1$ is the right answer to the above mentioned question.

An analogous question can be asked for upper semicontinuous quasi-continuous functions and Darboux upper semicontinuous quasi-continuous functions. At the very first glance the conjecture is that we will get the whole $\widehat{\mathcal{B}}_1$ class and the class of all Darboux functions in $\widehat{\mathcal{B}}_1$, respectively. But it is not true. The correct answers are: the class of all $\widehat{\mathcal{B}}_1$ functions fulfilling the condition $(*)$ below, and the class of all Darboux quasi-continuous functions in $\widehat{\mathcal{B}}_1$, respectively.

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2. Preliminaries

The real line is denoted by $\mathbb{R}$ and the set of positive integers by $\mathbb{N}$. The word *function* always means a mapping from $\mathbb{R}$ into $\mathbb{R}$. The Euclidean metric in $\mathbb{R}$ will be denoted by ϱ. For any set $A \subset \mathbb{R}$ let $\operatorname{int} A$ be its interior. If f is a function, then we write $\|f\|$ for $\sup\{|f(t)| : t \in \mathbb{R}\}$ (f need not be bounded) and we denote by $C(f)$ (respectively $D(f)$) the set of points of continuity (discontinuity) of f.

The *oscillation of a function f on a nonempty set $A \subset \mathbb{R}$* will be denoted by $\omega(f, A)$ (i.e., $\omega(f, A) = \sup\{|f(x) - f(y)| : x, y \in A\}$). Similarly, the *oscillation of a function f at a point $x \in \mathbb{R}$* will be denoted by $\omega(f, x)$ (i.e., $\omega(f, x) = \lim_{r \to 0^+} \omega(f, [x - r, x + r])$).

We say that a function f is *quasi-continuous* at a point $x \in \mathbb{R}$ if for each $\varepsilon > 0$ and each open interval $I \ni x$ there is a nonempty open interval $J \subset I$ such that $\omega(f, \{x\} \cup J) < \varepsilon$. We say that f is *quasi-continuous* if it is quasi-continuous at each point $x \in \mathbb{R}$. It is easy to prove that a Baire one function is Darboux and quasi-continuous if and only if it is bilaterally quasi-continuous. Recall that a function f is quasi-continuous if and only if the graph of $f|_{C(f)}$ is dense in the graph of f [4].

Let f be a pointwise discontinuous function and $x \in \mathbb{R}$. For brevity we define $\underline{\mathrm{LIM}}(f, x^-) = \varliminf_{t \to x^-,\, t \in C(f)} f(t)$. The symbols like $\underline{\mathrm{LIM}}(f, x^+)$, $\underline{\mathrm{LIM}}(f, x)$, etc., are defined analogously.

3. Main results

Proposition 3.1. *Suppose f_1 and f_2 are upper semicontinuous quasi-continuous functions. Then for each $x \in \mathbb{R}$*

$$\underline{\mathrm{LIM}}(f_1 - f_2, x) \le (f_1 - f_2)(x) \le \overline{\mathrm{LIM}}(f_1 - f_2, x). \tag{$*$}$$

Proof. Let $x \in \mathbb{R}$. Since f_2 is quasi-continuous, there are $x_1, x_2, \ldots \in C(f_1) \cap C(f_2)$ with $x_n \to x$ and $f_2(x_n) \to f_2(x)$. So

$$\begin{aligned} \underline{\mathrm{LIM}}(f_1 - f_2, x) &\le \liminf_{n \to \infty} (f_1 - f_2)(x_n) \\ &= \liminf_{n \to \infty} f_1(x_n) - \lim_{n \to \infty} f_2(x_n) \le (f_1 - f_2)(x). \end{aligned}$$

Similarly we can prove the other inequality. □

LEMMA 3.2. *Let f_1 and f_2 be upper semicontinuous, let A and B, $B \subset A$, be nowhere dense closed sets, suppose $(*)$ holds for $x \in A \setminus B$, and $\overline{\mathrm{LIM}}(f_i, x) = f_i(x)$ for $x \in B$. There is a nonnegative lower semicontinuous function α such that $f_1 + \alpha$ and $f_2 + \alpha$ are upper semicontinuous, $D(\alpha) \subset A$, $\|\alpha\| \le \sup\{\omega(f_i, x) : x \in A \setminus B,\ i \in \{1,2\}\}$, $\overline{\mathrm{LIM}}(f_i + \alpha, x) = (f_i + \alpha)(x)$ for $x \in A$ and $i \in \{1,2\}$, and $\alpha = 0$ on A.*

Proof. Set $S = \sup\{\omega(f_i, x) : x \in A \setminus B,\ i \in \{1,2\}\}$. For $i \in \{1,2\}$ choose a countable set $C_i \subset A$ such that the graph of $f_i|_{C_i}$ is dense in the graph of $f_i|_A$ (we use the fact that $\mathbb{R}^2$ is hereditarily separable). Set $C = C_1 \cup C_2$ and arrange the elements of C in a sequence, $\{c_k : k \in \mathbb{N}\}$.

I. First we will show that for each k there is a nonnegative function α_k such that:

(a) $f_i + \alpha_k \le \max\{f_i(c_k) + 2/k, f_i\}$ on $\mathbb{R}$ for $i \in \{1,2\}$,
(b) $D(\alpha_k) \subset \{c_k\}$ and $\alpha_k(x) = 0$ whenever $|x - c_k| > 2/k$ or $x \in A$ (so α_k is lower semicontinuous),
(c) for $i \in \{1,2\}$ we have $\overline{\mathrm{LIM}}(f_i + \alpha_k, c_k) = \varlimsup_{t \to c_k} (f_i + \alpha_k)(t) = (f_i + \alpha_k)(c_k)$ (so $f_i + \alpha_k$ is upper semicontinuous),
(d) $\|\alpha_k\| \le S$.

If $c_k \in B$, then we set $\alpha_k = 0$.

Otherwise for $i \in \{1,2\}$ find a strictly monotone sequence of points of continuity of both f_1 and f_2, (c_{kn}^i), such that $\lim_{n\to\infty} (f_{3-i} - f_i)(c_{kn}^i) = \underline{\mathrm{LIM}}(f_{3-i} - f_i, c_k)$, $c_{kn}^i \to c_k$, and $|c_{k1}^i - c_k| < 1/k$. We may assume that these two sequences are disjoint from A, that $(f_{3-i} - f_i)(c_{kn}^i) \le \underline{\mathrm{LIM}}(f_{3-i} - f_i, c_k) + 1/k$ for each n and i, and that $c_{km}^2 \ne c_{kn}^1$ for each $m, n \in \mathbb{N}$. There is a family, $\{I_{kn}^i : n \in \mathbb{N},\ i \in \{1,2\}\}$, consisting of pairwise disjoint open intervals of length less than $1/k$, such that for each n and i, $c_{kn}^i \in I_{kn}^i \subset \mathbb{R} \setminus A$ and

$$\max\{f_i(x) - f_i(c_{kn}^i),\ f_{3-i}(x) - f_{3-i}(c_{kn}^i)\} \le 1/(k+n)$$

whenever $x \in I_{kn}^i$. For each n and i construct a nonnegative continuous function α_{kn}^i such that $\alpha_{kn}^i(x) = 0$ whenever $x \notin I_{kn}^i$ and

$$\|\alpha_{kn}^i\| = \alpha_{kn}^i(c_{kn}^i) = \min\Big\{\max\{f_i(c_k) - f_i(c_{kn}^i), 0\}, S\Big\};$$

observe that for each $x \in \mathbb{R}$: if $\alpha_{kn}^i(x) > 0$, then $x \in I_{kn}^i$ and $f_i(c_k) > f_i(c_{kn}^i)$, so

$$(f_i + \alpha_{kn}^i)(x) \le f_i(c_{kn}^i) + 1/(k+n) + \|\alpha_{kn}^i\| \le f_i(c_k) + 1/(k+n),$$

and by $(*)$,

$$\begin{aligned}(f_{3-i} + \alpha_{kn}^i)(x) &= f_{3-i}(x) - f_{3-i}(c_{kn}^i) + (f_{3-i} - f_i)(c_{kn}^i) + f_i(c_{kn}^i) + \alpha_{kn}^i(x) \\ &\le 1/(k+n) + \underline{\mathrm{LIM}}(f_{3-i} - f_i, c_k) + 1/k + f_i(c_k) \\ &< f_{3-i}(c_k) + 2/k.\end{aligned}$$

Now define $\alpha_k = \sum_{n\in\mathbb{N}} \sum_{i=1}^{2} \alpha_{kn}^i$. It is easy to show that conditions (a), (b), and (d) are fulfilled.

To prove (c) fix an $s \in \{1,2\}$. First note that

$$\begin{aligned}\overline{\mathrm{LIM}}(f_s + \alpha_k, c_k) &\ge \limsup_{n\to\infty}(f_s + \alpha_k)(c_{kn}^s) = \limsup_{n\to\infty}(f_s + \alpha_{kn}^s)(c_{kn}^s)\\ &\ge \limsup_{n\to\infty} \min\{f_s(c_k), f_s(c_{kn}^s) + S\} = f_s(c_k)\,.\end{aligned}$$

Hence we need only to show that $f_s + \alpha_k$ is upper semicontinuous at c_k. Take a sequence $x_m \to c_k$ with $\lim_{m\to\infty}(f_s + \alpha_k)(x_m) = \overline{\lim}_{t\to c_k}(f_s + \alpha_k)(t)$. If $\alpha_k(x_m) = 0$ for infinitely many m, then

$$\lim_{m\to\infty}(f_s + \alpha_k)(x_m) = \lim_{m\to\infty} f_s(x_m) \le f_s(c_k) = (f_s + \alpha_k)(c_k)\,.$$

So assume that $\alpha_k(x_m) > 0$ for each m. For each m there are n_m and i_m with $\alpha_k(x_m) = \alpha_{kn_m}^{i_m}(x_m)$. We may assume that the sequence (i_m) is constant, and we will write i instead of i_m. We have $n_m \to \infty$, $x_m \in I_{kn_m}^i$, and $f_i(c_k) > f_i(c_{kn_m}^i)$ for each m. Now:

- If $s = i$, then by the above,

$$\begin{aligned}\lim_{m\to\infty}(f_s + \alpha_k)(x_m) &= \lim_{m\to\infty}(f_i + \alpha_{kn_m}^i)(x_m)\\ &\le \limsup_{m\to\infty}\big(f_i(c_k) + 1/(k + n_m)\big)\\ &= f_i(c_k) = (f_i + \alpha_k)(c_k)\,.\end{aligned}$$

- If $s = 3 - i$, then by $(*)$,

$$\begin{aligned}&\lim_{m\to\infty}(f_s + \alpha_k)(x_m)\\ &= \lim_{m\to\infty}(f_{3-i} + \alpha_{kn_m}^i)(x_m)\\ &\le \limsup_{m\to\infty}\big(f_{3-i}(x_m) - f_{3-i}(c_{kn_m}^i)\big) + \lim_{m\to\infty}(f_{3-i} - f_i)(c_{kn_m}^i)\\ &\qquad\qquad\qquad + \limsup_{m\to\infty}\big(f_i(c_{kn_m}^i) + \alpha_{kn_m}^i(x_m)\big)\\ &\le \limsup_{m\to\infty} 1/(k + n_m) + \underline{\mathrm{LIM}}(f_{3-i} - f_i, c_k) + f_i(c_k)\\ &\le f_{3-i}(c_k) = (f_{3-i} + \alpha_k)(c_k)\,.\end{aligned}$$

II. Define $\alpha = \sup\{\alpha_k : k \in \mathbb{N}\}$. It is clear that α is lower semicontinuous, $D(\alpha) \subset A$, $\alpha = 0$ on A, and $0 \le \alpha \le S$ on $\mathbb{R}$. To complete the proof fix an $i \in \{1,2\}$ and an $x \in A$. (Since $D(\alpha) \subset A$, so $f_i + \alpha$ is upper semicontinuous on $\mathbb{R} \setminus A$.)

II.a) First we will prove that $f_i + \alpha$ is upper semicontinuous at x. Take a sequence $x_n \to x$ such that $\lim_{n\to\infty}(f_i+\alpha)(x_n) = \overline{\lim_{t\to x}}(f_i+\alpha)(t)$. We will consider several cases.

If $\alpha(x_n) = 0$ for infinitely many n, then

$$\lim_{n\to\infty}(f_i+\alpha)(x_n) = \lim_{n\to\infty} f_i(x_n) \le f_i(x) = (f_i+\alpha)(x).$$

So we may assume that $\alpha(x_n) > 0$ for each n. Since by (b), $\alpha_k(x_n) = 0$ whenever $k > 2/\varrho(x_n, A)$, there is a $k_n \in \mathbb{N}$ with $\alpha(x_n) = \alpha_{k_n}(x_n)$.

If k_n does not tend to infinity, then there is a $k \in \mathbb{N}$ such that $\alpha(x_n) = \alpha_k(x_n)$ for infinitely many n. By (c),

$$\lim_{n\to\infty}(f_i+\alpha)(x_n) = \lim_{n\to\infty}(f_i+\alpha_k)(x_n) \le (f_i+\alpha_k)(x) = (f_i+\alpha)(x).$$

Finally let $k_n \to \infty$. Then $|x_n - c_{k_n}| \le 2/k_n \to 0$ and by (a),

$$\begin{aligned}\lim_{n\to\infty}(f_i+\alpha)(x_n) &= \lim_{n\to\infty}(f_i+\alpha_{k_n})(x_n)\\ &\le \limsup_{n\to\infty}\big(f_i(c_{k_n}) + 2/k_n\big) \le f_i(x) = (f_i+\alpha)(x).\end{aligned}$$

II.b) By the property of the set C, there are $k_1, k_2, \dots$ such that $c_{k_n} \to x$ and $f_i(c_{k_n}) \to f_i(x)$. By (c), for each $n \in \mathbb{N}$ there exists an $x_n \in C(f_i)$ such that $|x_n - c_{k_n}| < 1/n$ and $|(f_i+\alpha_{k_n})(x_n) - f_i(c_{k_n})| \le 1/n$. Then

$$\begin{aligned}\overline{\mathrm{LIM}}(f_i+\alpha, x) &\ge \limsup_{n\to\infty}(f_i+\alpha)(x_n) \ge \limsup_{n\to\infty}(f_i+\alpha_{k_n})(x_n)\\ &\ge \lim_{n\to\infty} f_i(c_{k_n}) = f_i(x) = (f_i+\alpha)(x).\end{aligned}$$

The proof is complete. □

THEOREM 3.3. *Let f_1 and f_2 be upper semicontinuous functions such that $(*)$ holds for each $x \in \mathbb{R}$. For each $\eta > 0$ there is a nonnegative lower semicontinuous function α such that $f_1 + \alpha$ and $f_2 + \alpha$ are upper semicontinuous and quasi-continuous, $D(\alpha) \subset D(f_1) \cup D(f_2)$, and $\|\alpha\| \le \sup\{\omega(f_i, x) : x \in \mathbb{R}, i \in \{1,2\}\} + \eta$.*

Proof. We proceed by induction. Let $S_0 = \sup\{\omega(f_i,x) : x \in \mathbb{R}, i \in \{1,2\}\}$, $A_0 = \emptyset$ and $S_n = \eta/2^n$, $A_n = \{x \in \mathbb{R} : \omega(f_1,x) \ge S_n$ or $\omega(f_2,x) \ge S_n\}$ for $n \in \mathbb{N}$. Let $\alpha_0 = 0$. For each $n \in \mathbb{N}$ use Lemma 3.2 to construct a lower semicontinuous function α_n such that the followings conditions hold:

(i) $f_1 + \alpha_n$ and $f_2 + \alpha_n$ are upper semicontinuous;
(ii) $D(\alpha_n - \alpha_{n-1}) \subset A_n$;
(iii) $0 \le \alpha_n - \alpha_{n-1} \le S_{n-1}$ on $\mathbb{R}$;
(iv) $\overline{\mathrm{LIM}}(f_i + \alpha_n, x) = (f_i + \alpha_n)(x)$ for $x \in A_n$ and $i \in \{1,2\}$;
(v) $\alpha_n - \alpha_{n-1} = 0$ on A_n.

Define $\alpha = \lim_{n\to\infty} \alpha_n$. By (iii), this sequence is uniformly convergent (so α is lower semicontinuous and by (i), $f_1 + \alpha$ and $f_2 + \alpha$ are upper semicontinuous), and $\|\alpha\| \le \sum_{n=0}^{\infty} S_n = S_0 + \eta$. Moreover by (ii), $D(\alpha) \subset \bigcup_{n\in\mathbb{N}} A_n = D(f_1) \cup D(f_2)$. To complete the proof we have to show that $f_1 + \alpha$ and $f_2 + \alpha$ are quasi-continuous.

Fix an $i \in \{1,2\}$ and an $x \in \mathbb{R}$. If $x \notin \bigcup_{n\in\mathbb{N}} A_n$, then clearly $f_i + \alpha$ is continuous at x. So assume the opposite case. There is an $n \in \mathbb{N}$ with $x \in A_n$. By (iii), (iv), and (v), we get

$$(f_i + \alpha)(x) \ge \overline{\mathrm{LIM}}(f_i + \alpha, x) \ge \overline{\mathrm{LIM}}(f_i + \alpha_n, x) = (f_i + \alpha_n)(x) = (f_i + \alpha)(x)\,.$$

It follows that $f_i + \alpha$ is quasi-continuous and the proof is complete. $\square$

Remark 3.1. It is clear that not every upper semicontinuous function fulfills the condition $(*)$. (Consider, e.g., the characteristic function of a singleton.) On the other hand, there are functions in $\widehat{\mathcal{B}}_1$ fulfilling $(*)$ which are neither quasi-continuous, nor Darboux (e.g., the "sign" function).

Now we will consider Darboux upper semicontinuous quasi-continuous functions. We start with a lemma.

LEMMA 3.4. *For a Baire one function f the following are equivalent:*

(i) *f is Darboux and quasi-continuous;*

(ii) *for each $x \in \mathbb{R}$*

$$\max\{\underline{\mathrm{LIM}}(f, x^-), \underline{\mathrm{LIM}}(f, x^+)\} \le f(x) \le \min\{\overline{\mathrm{LIM}}(f, x^-), \overline{\mathrm{LIM}}(f, x^+)\}\,;$$

(iii) *for each $x \in \mathbb{R}$ there exist sequences $y_1, y_2, \dots \in C(f)$ and $z_1, z_2, \dots \in C(f)$ such that $y_n \nearrow x$, $z_n \searrow x$, and $\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} f(z_n) = f(x)$.*

P r o o f. The equivalence (i) $\iff$ (iii) coincides with Lemma of [6], and the implication (iii) $\implies$ (ii) is evident. Let f satisfy the condition (ii) and suppose that (iii) is not fulfilled, i.e., either

$$\underline{\mathrm{LIM}}(f, x^-) < f(x) < \overline{\mathrm{LIM}}(f, x^-)$$

or

$$\underline{\mathrm{LIM}}(f, x^+) < f(x) < \overline{\mathrm{LIM}}(f, x^+)$$

for some $x \in \mathbb{R}$. Assume that, e.g., the first case holds. (The other case is analogous.) There is an $\eta > 0$ such that $|f(t) - f(x)| > 2\eta$ for each $t \in [x - \eta, x] \cap C(f)$. Set

$$U_1 = \operatorname{int}\{t \in [x - \eta, x] :\ f(t) < f(x) - \eta\}$$

and

$$U_2 = \operatorname{int}\{t \in [x-\eta, x] : \ f(t) > f(x) + \eta\}.$$

The sets U_1 and U_2 are disjoint, open, and nonempty. Denote by A_i the family of the end points of the components of U_i ($i \in \{1,2\}$). Observe that both A_1 and A_2 are dense in the boundary of U_1 (= the boundary of U_2), i.e., in the set $A = [x-\eta, x] \setminus (U_1 \cup U_2)$. But by (ii), $f(t) \le f(x) - 2\eta$ for $t \in A_1$, and $f(t) \ge f(x) + 2\eta$ for $t \in A_2$. So $f|_A$ has no points of continuity and f is not Baire one, an impossibility. □

The proofs of Proposition 3.5 and Theorem 3.6 are actually repetitions of the proofs of Proposition 3.1 and Theorem 3.3, respectively. (We just use the condition (iii) of Lemma 3.4 instead of $(*)$.) Therefore we omit them.

PROPOSITION 3.5. *Suppose f_1 and f_2 are Darboux upper semicontinuous quasi-continuous functions. Then $f_1 - f_2$ is Darboux and quasi-continuous.*

THEOREM 3.6. *Suppose f_1 and f_2 are upper semicontinuous functions such that $f_1 - f_2$ is Darboux and quasi-continuous. Then for each $\eta > 0$ there exists a nonnegative lower semicontinuous function α such that $f_1 + \alpha$ and $f_2 + \alpha$ are Darboux, upper semicontinuous, and quasi-continuous, $D(\alpha) \subset D(f_1) \cup D(f_2)$, and $\|\alpha\| \le \sup\{\omega(f_i, x) : \ x \in \mathbb{R}, \ i \in \{1,2\}\} + \eta$.*

Remark 3.2. The class of Darboux quasi-continuous functions in $\widehat{\mathcal{B}}_1$ is essentially smaller than both the class of Darboux $\widehat{\mathcal{B}}_1$ functions and the class of $\widehat{\mathcal{B}}_1$ functions fulfilling $(*)$. (Consider, e.g., a Darboux upper semicontinuous function which vanishes on a dense set but not everywhere ([2]), and the already mentioned "sign" function.)

Remark 3.3. It was already mentioned that the difference of Darboux upper semicontinuous functions is Darboux ([7]). By Proposition 3.5, the difference of Darboux upper semicontinuous quasi-continuous functions is both Darboux and quasi-continuous. So it may seem strange that the difference of upper semicontinuous quasi-continuous functions need not be quasi-continuous. However, condition $(*)$ is equivalent to simultaneous upper and lower quasi-continuity of the difference $f_1 - f_2$. (See [3] for the definitions of the latter two notions.) On the other hand, one can easily show that an upper semicontinuous function is quasi-continuous if and only if it is both upper and lower quasi-continuous.

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