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COMPATIBILITY AND STATISTICAL INFERENCE FOR DATA IN MEASUREMENT MODEL OF FOULIS AND RANDALL

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ABSTRACT. A graph $\mathcal{G}$ is triangulated if and only if any system of functions given on cliques of $\mathcal{G}$, compatible in a natural way, can be expressed by one (unnormed) measure on $\mathcal{G}$. This explains how to connect some experimental data. Random errors are also discussed.

1. Introduction

Let $\mathcal{H} = (X, \mathcal{E})$ be a hypergraph (i.e., $\mathcal{E}$ is a family of subsets covering a non-empty set X) describing some orthogonality relation ($x \perp y$ if and only if $x, y \in E$ for some $E \in \mathcal{E}$). The family of subsets $M = A^{\perp\perp}$ ($A^{\perp} = \{x : x \perp y \text{ for all } y \in A\}$) generated by orthogonal sets in X yields an orthologic, and conversely, any finite orthologic can be obtained in this way [2], [11]. Thus the hypergraph approach of Randall and Foulis to measurement theory (and foundations of quantum theory [10]) seems to be the most general and is frequently used [7], [13], [14]. A number of attempts were made to describe state spaces and their physical interpretations (see [6], [11], [13], [18]). If $\mathcal{H}$ is a manual (i.e., an irredundant and coherent hypergraph, cf. Section 2), then a state is just a positive function μ on X satisfying the condition

$$\sum_{x \in E} \mu(x) = 1 \qquad \text{for each} \quad E \in \mathcal{E}. \tag{1.1}$$

In this paper, we examine a function $\mu \colon X \to \mathbb{R}^+$ called a measure. A celebrated construction of Greechie shows that a state space can be empty even if the hypergraph is connected with an orthomodular logic [1], [9], [10]. Thus it

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seems reasonable also to consider unnormed functions μ which do not satisfy condition (1.1), but can express some important rankings of elements of X (see examples in Section 4). We consider when such a measure μ can be obtained from experimental data connected with distinct cliques $E \in \mathcal{E}$. Such data are described by the system of functions $(\mu_E, E \in \mathcal{E})$ (compare also Finch's concept of a physical quantity [5], [12]). The triangulability of the graph describing the orthogonality relation "$\perp$" proves to be a necessary and sufficient condition.

Our main result is proved in Section 3. In Section 2, we explain the description of a physical system by means of a manual, given in 1972 by Foulis and Randall [7], while in Section 4, we give some examples and statistical models.

2. Hypergraph- and graph-theoretic approach

Let X be a non-empty finite set, and $\mathcal{E}$ a covering of X by non-empty sets. Then we call the pair $\mathcal{H} = (X, \mathcal{E})$ a *hypergraph* ([2], [8]). Elements of X are called *outcomes*, and elements of $\mathcal{E}$ — *operations*.

Let $\mathcal{H} = (X, \mathcal{E})$ be a hypergraph. A pair $x, y \in X$ is said to be *orthogonal*, denoted by $x \perp y$, if $x \neq y$ and there exists an $E \in \mathcal{E}$ with $x, y \in E$. The relation $\perp$ is irreflexive and symmetric.

A subset $A \subseteq X$ is called *orthogonal* if the condition $x, y \in A$, $x \neq y$, implies that $x \perp y$. The family of all orthogonal subsets of X will be denoted by $\mathcal{O}(H)$, i.e.,

$$\mathcal{O}(H) = \{A \subseteq X : (x, y \in A \ \&\ x \neq y) \implies (x \perp y)\}.$$

A hypergraph $\mathcal{H} = (X, \mathcal{E})$ is called *irredundant* if $E, F \in \mathcal{E}$ and $E \subseteq F$ imply that $E = F$. A hypergraph is called *coherent* if each orthogonal set $A \in \mathcal{O}(\mathcal{H})$ is contained in an operation $E \in \mathcal{E}$. An irredundant coherent hypergraph $\mathcal{H} = (X, \mathcal{E})$ is called a *manual* ([7]).

Let $\mathcal{G} = \big(X, E(X)\big)$ be an *undirected graph*, i.e., the set of edges satisfies $(x, y) \in E(X)$ if and only if $(y, x) \in E(X)$. A subset $W \subseteq X$ in a graph $\mathcal{G}$ is *complete* if any two distinct vertices x, y in W are *adjacent*, i.e., $(x, y) \in E(X)$. A *clique* E is a maximal (with respect to $\subseteq$) complete subset of the vertex set X, i.e., satisfies the following two conditions:

(i) For all $x, y \in E$, either $x = y$ or $(x, y) \in E$.
(ii) If $E \subseteq D \subseteq X$ and $E \neq D$, then there are $x, y \in D$ which are not adjacent.

Denote by $\mathcal{E}(G)$ the collection of all cliques of the graph $\mathcal{G} = (X, E)$. Then the pair $\big(X, \mathcal{E}(\mathcal{G})\big)$ is a hypergraph denoted by $\mathcal{H}(\mathcal{G})$. A hypergraph which is isomorphic to a hypergraph arising in this manner is called a *clique hypergraph*.

The relationship between graphs and hypergraphs is established by the following proposition.

PROPOSITION 2.1. ([11]) *A hypergraph is a clique hypergraph if and only if it is a manual.*

In the early seventies, Foulis and Randall ([7]) laid the foundations for the so-called orthologics describing a structure of measurement outcomes when the measurement results are not necessarily compatible. Such situations are encountered, for instance, when the position and momentum of an elementary physical particle are measured. In this case, one must consider more than one physical experiment, i.e., an experiment measuring the position and an experiment measuring the momentum, and, because of the uncertainty principle, these experiments need not admit a common refinement in terms of which classical probability theory and statistics are directly applicable. Therefore the experimental propositions do not form a Boolean algebra. The works of Foulis and Randall on empirical logic have given rise to a class of logics (called orthologics) which generalize Boolean algebras for the cases of measurement results obtained from more than one physical experiment.

For finite systems, a convenient way to capture these ideas in a mathematically rigorous fashion is by means of a hypergraph [2]. Outcomes are represented by the vertices $x \in X$, and operations (questions, experiments) by the edges $E \in \mathcal{E}$ of the hypergraph $\mathcal{H} = (X, \mathcal{E})$.

A *physical operation* was described by Randall and Foulis (1972) as a set of instructions that describe a well-defined, reproducible procedure which specifies what should be observed and what can be recorded as a consequence of an execution of that procedure. In this case, one and only one symbol from a specified set can be recorded as an *outcome* of the realization of the procedure. Thus $X = \bigcup \mathcal{E}$ represents the set of all outcomes. In other words, an outcome of a realization of a physical operation is merely a symbol. This definition coming from Foulis and Randall is by no means restricted to operations involving traditional laboratory procedures. Procedures like quality control test procedures, data gathering procedures such as opinion polling and procedures involving subjective approvals or disapprovals are all admissible provided they are well-defined. Some examples are also collected in Section 4.

3. Measures on manuals

Let $(X, \perp)$ be an orthogonality space, where the set X is finite or even infinite. Let the graph $\mathcal{G} = (X, E)$ describe the orthogonality relation $\perp$ on X in such a way that $x \perp y$ if and only if $x, y \in E$. Let $\mathcal{E} = E(G)$ denote the collection of all the cliques in the graph $\mathcal{G}$. Then the pair $\big(X, \mathcal{E}(G)\big)$ is a hypergraph, called a cliques hypergraph, existing by Proposition 2.1, and denoted as $\mathcal{H} = H(G)$. By Proposition 2.1, the hypergraph $\mathcal{H}$ is a manual.

DEFINITION 3.1. A *measure* (or *unnormed state*) on a manual $\mathcal{H}$ is a real-valued function $\mu\colon X \to R^+$.

DEFINITION 3.2. We shall call the family of positive functions $\{\mu_E : E \to \mathbb{R}^+, E \in \mathcal{E}\}$ *compatible* on $(X, \mathcal{E})$ if, for all $E, F \in \mathcal{E}$, there exists $\alpha \in \mathbb{R}$ such that $\mu_E - \alpha\mu_F|_{E \cap F} \equiv 0$. The support of a function μ_E is defined as

$$\operatorname{supp} \mu_E = \big\{x \in E : \ \mu_E(x) > 0\big\} .$$

THEOREM 3.3. *The following conditions are equivalent:*

(1) *Each elementary cycle in $\mathcal{G}$ (i.e., not encountering the same vertex twice) of order $n \geq 4$ possesses a chord (see Appendix).*

(2) *If $\{\mu_E : \ E \in \mathcal{E}\}$ is a compatible family of positive functions on $\mathcal{H}$, there exists a measure μ on X such that for each $E \in \mathcal{E}$, there exists $\alpha \in \mathbb{R}$ such that $\mu_E = \alpha\mu\big|_E$.*

Proof.

(1) $\Longrightarrow$ (2). First of all, we remark that

$$\mu_E(x) = 0 \iff \mu_F(x) = 0 \qquad \text{for each} \quad x \in E \cap F .$$

Denote

$$\tilde{X} = \bigcup_{E \in \mathcal{E}} \operatorname{supp} \mu_E$$

and put $\mu\big|_{X \setminus \tilde{X}} \equiv 0$.

Define the subgraph induced by $\tilde{X}$ to be $\mathcal{G}_{\tilde{X}} = (\tilde{X}, E_{\tilde{X}})$, $E_{\tilde{X}} = E \cap \tilde{X}$ (cf. Appendix).

Let the relation $\sim \subseteq \tilde{X} \times \tilde{X}$ in $\tilde{X}$ be defined as follows: $x \sim y$ if and only if

(i) $x = y$,

or

(ii) $x \neq y$, and there exists a chain in $\mathcal{G}$ connecting x and y.

The above relation is an equivalence relation, i.e., is reflexive, symmetric and transitive. The classes of this equivalence relation partition $\tilde{X}$ into connected subgraphs of $\mathcal{G}_{\tilde{X}}$ called *connected components.* Let $Y \subseteq \tilde{X}$ be an equivalence class containing an element $a \in \tilde{X}$, i.e., $Y = [a]_\sim$. Our aim is to construct a measure μ on Y.

For all $x, y \in [a]_\sim$ such that $x \perp y$, we put

$$\beta_{(x,y)} = \frac{\mu_E(y)}{\mu_E(x)} \tag{3.1}$$

for any operation $E \in \mathcal{E}$ containing x and y. The number $\beta_{(x,y)}$ is well defined in virtue of Definition 3.2.

We shall prove that, for each elementary cycle $x_0, x_1, \ldots, x_n, x_0$ in $\mathcal{G}$, the equality

$$\beta_{(x_0,x_1)} \cdot \ldots \cdot \beta_{(x_{n-1},x_n)} \cdot \beta_{(x_n,x_0)} = 1 \tag{3.2}$$

holds. This fact will be proved by induction. For $n = 1$, equality (3.2) is obvious. For any cycle x_0, x_1, x_2, x_0, there exists $E \in \mathcal{E}$ containing x_0, x_1, x_2 because $(X, \mathcal{E})$ is a manual. Thus we have (3.2) for $n = 2$. Assume that, for a cycle $x_0, \ldots, x_k, x_0$, the equality

$$\beta_{(x_0,x_1)} \cdot \ldots \cdot \beta_{(x_k,x_0)} = 1 \tag{3.3}$$

holds for $k \leq n$, $n \geq 2$. For a cycle $x_0, \ldots, x_n, x_{n+1}, x_0$, we shall show the equality

$$\beta_{(x_0,x_1)} \cdot \ldots \cdot \beta_{(x_n,x_{n+1})} \beta_{(x_{n+1},x_0)} = 1 . \tag{3.4}$$

Let $(x_i, x_j)_{i<j}$ be a chord in the cycle $x_0, \ldots, x_{n+1}, x_0$. By assumption (3.3),

$$\beta_{(x_0,x_1)} \cdot \ldots \cdot \beta_{(x_{i-1},x_i)} \cdot \beta_{(x_i,x_j)} \cdot \beta_{(x_j,x_{j+1})} \cdot \ldots \cdot \beta_{(x_{n+1},x_0)} = 1 \tag{3.5}$$

and

$$\beta_{(x_i,x_{i+1})} \cdot \ldots \cdot \beta_{(x_{j-1},x_j)} \cdot \beta_{(x_j,x_i)} = 1 . \tag{3.6}$$

Now, by (3.1),

$$\beta_{(x_i,x_j)} \cdot \beta_{(x_j,x_i)} = 1 . \tag{3.7}$$

Multiplying (3.5) and (3.6) by sides and taking account of (3.7), we obtain (3.4).

We also show, by induction, that (3.2) is true if a cycle $x_0, \ldots, x_n, x_0$ is not necessarily elementary. For $n = 1$, equality (3.2) is obvious. Assume that, for a cycle $x_0, \ldots, x_k, x_0$, $k \leq n$, equality (3.3) is true. We shall show that, for a cycle $x_0, \ldots, x_{n+1}, x_0$, the equality

$$\beta_{(x_0,x_1)} \cdot \ldots \cdot \beta_{(x_{n+1},x_0)} = 1 \tag{3.8}$$

holds. If the cycle $x_0, \ldots, x_{n+1}, x_0$ is simple, then equality (3.8) is proved; if not, let $i < j$ be indices such that $x_i = x_j$, and that the difference $j - i$ is minimal. Then the cycle $x_i, \ldots, x_j, x_i$ is elementary, and the equality

$$\beta_{(x_i,x_{i+1})} \cdot \ldots \cdot \beta_{(x_{j-1},x_j)} = 1 \tag{3.9}$$

is true. Take another chain of the form $x_0, x_1, \dots, x_i, x_{j+1}, \dots, x_{n+1}, x_0$. By assumption (3.3), we have

$$\beta_{(x_0,x_1)} \cdot \dots \cdot \beta_{(x_{i'-1},x_{i'})} \beta_{(x_j,x_{j+1})} \cdot \dots \cdot \beta_{(x_{n+1},x_0)} = 1 . \tag{3.10}$$

Multiplying (3.9) by (3.10), we obtain the inductive assertion (3.8).

Define, for any chain $x_0, x_1, \dots, x_n$,

$$\beta_{(x_0,\dots,x_n)} = \beta_{(x_0,x_1)} \cdot \dots \cdot \beta_{(x_{n-1},x_n)} . \tag{3.11}$$

Let us observe that if $x_0 = x'_0$, $x_n = x'_{n'}$, for another chain $x'_0, \dots, x'_{n'}$ we have

$$\beta_{(x_0,\dots,x_n)} = \beta_{(x'_0,\dots,x'_{n'})} . \tag{3.12}$$

In fact, the chain $x_0, x_1, \dots, x_n = x'_{n'}, x'_{n'-1}, \dots, x'_0 = x_0$ is a cycle, and therefore, by (3.2), the following equality

$$\beta_{(x_0,x_1)} \cdot \dots \cdot \beta_{(x_{n-1},x_n)} \cdot \beta_{(x'_{n'},x'_{n'-1})} \cdot \dots \cdot \beta_{(x'_1,x'_0)} = 1$$

should be true. Using (3.1), we have

$$\beta_{(x'_0,\dots,x'_{n'})} = \frac{1}{\beta_{(x_1,x_0)} \cdot \dots \cdot \beta_{(x'_{n'},x'_{n'-1})}} = \beta_{(x_0,x_1)} \cdot \dots \cdot \beta_{(x_{n-1},x_n)} .$$

Hence (3.12) is true.

Define a measure μ on an equivalence class $[a]_\sim$ in the form

$$\mu_a(x) = \beta_{(x_0,\dots,x_1)} , \tag{3.13}$$

where $x_0, x_1, \dots, x_n$ is any chain in $\mathcal{G}$ such that $x_0 = a$ and $x_n = x$. If $[a]_\sim = [a']_\sim$, then we have

$$\mu_{a'} = \frac{1}{\mu_a(a')} \cdot \mu_a . \tag{3.14}$$

Indeed, take a chain $a = a_0, a_1, \dots, a_n = a' = x_0, \dots, x_n = x$. Then, according to (3.13), we have

$$\begin{aligned}
\mu_{a'}(x) = \beta_{(x_0,\dots,x_n)} &= \frac{1}{\beta_{(a=a_0,\,a_1,\dots,a_n=a')}} \cdot \beta_{(a=a_0,\,a_1,\dots,a_n,x_1,\dots,x_n=x)} \\
&= \frac{1}{\mu_a(a')} \cdot \mu_a(x) .
\end{aligned}$$

By the Axiom of Choice (the space X may be infinite), we put $\tilde{X} = \bigcup_{j\in J} [a_j]_\sim$ for a family $\{a_j : j \in J\}$ of elements mutually non-equivalent with respect to the relation $\sim$.

Let $E \in \mathcal{E}$. Denote

$$\tilde{E} = \operatorname{supp} \mu_E .$$

Since E is a clique, there exists $j \in J$ such that $\tilde{E} \subseteq [a_j]_\sim$. Let $a' \in \tilde{E}$. Since E is a clique, therefore, for $x \in E$, by (3.1), we have

$$\mu_{a'}(x) = \beta_{(a',x)} = \frac{\mu_E(x)}{\mu_E(a')} .$$

Hence, using (3.14), we obtain

$$\begin{aligned} \frac{\mu_E(x)}{\mu_E(a')} = \mu_{a'}(x) &= \frac{1}{\mu_{a_j}(a')} \cdot \mu_{a_j}(x) \\ &= \frac{1}{\mu_{a_j}(a')} \cdot \mu(x) . \end{aligned}$$

Now, it suffices to put

$$\alpha_E = \frac{\mu_E(a')}{\mu_{a_j}(a')} .$$

(2) $\Longrightarrow$ (1). Suppose that (1) is not satisfied. In other words, an elementary cycle $x_0, \dots, x_n, x_0$ does not possesses a chord. Hence, for each $E \in \mathcal{E}$, there exists $0 \le i(E) \le n$ such that

$$E \cap \{x_0, x_1, \dots, x_n\} \subseteq \{x_{i(E)}, x_{i(E)+1}\} \qquad \text{with} \quad x_{n+1} = x_0 .$$

Define a measure μ_E on $E \in \mathcal{E}$ as follows:

$$\mu_E(x) = \begin{cases} 0 & \text{if } x \notin \{x_0, \dots, x_n\} , \\ 1 & \text{if } x = x_{i(E)} , \\ 2 & \text{if } x = x_{i(E)+1} . \end{cases}$$

It can easily be seen that $\{\mu_E : E \in \mathcal{E}\}$ forms a compatible measure family. It suffices to prove that

$$(\mu_E - \alpha\mu_F)|E \cap F \cap \{x_0, \dots, x_n\} \equiv 0$$

in the case when $E \cap F \cap \{x_0, \dots, x_n\} = \{x_{i(E)}, x_{i(E)+1}\}$. Then $i(E) = i(F) = i_0$, and we have $\mu_E = \mu_F|E \cap F$.

Suppose that $\mu_E = \alpha(E)\mu|_E$. Let $E(i)$ be an operation such that $x_i, x_{i+1} \in E(i)$. Then we have

$$\frac{\mu(x_{i+1})}{\mu(x_i)} = \frac{\mu_{E(i)}(x_{i+1})}{\mu_{E_i}(x_i)} = 2 .$$

Hence we obtain

$$\begin{aligned} \mu(x_0) &= \frac{\mu(x_0)}{\mu(x_n)} \cdot \frac{\mu(x_n)}{\mu(x_{n-1})} \cdot \ldots \cdot \frac{\mu(x_2)}{\mu(x_1)} \cdot \mu(x_1) \\ &= 2^n \cdot \mu(x_1) = 2^{n+1} \cdot \mu(x_0) \neq 0 , \end{aligned}$$

which gives a contradiction. This ends the proof. □

4. Measures and empirical data — statistical aspects

We conclude by pointing out a number of situations where an (unnormed) measure on a hypergraph arises in a natural way from a compatible family of positive functions constructed on the basis of empirical data (cf. Section 3). We would like to propose a minimum of statistical methods which are useful when we expect some random errors. It is important that we assume almost nothing about the hypergraph (cf. Theorem 3.3). Look at the following examples.

Let $\mu(x)$ be a long-run frequency of the occurrence of the outcome $x \in X$ provided that one executes operations $E \in \mathcal{E}$ with given frequencies $P(E)$. It is natural to estimate $\mu(x)$ on the basis of the outcome for some finite series of realizations of the operations $E \in \mathcal{E}$.

Analogous problems arise when we establish a ranking $\mu(x)$ (defined in a suitable way) for football teams from a set X. The scores of some games are then the only empirical data.

The proportions of chemical roots in chemical reactions or a relative position of a point in a geodesic measurement are all established by means of analogous procedures connected with certain graphs.

The following simple model, being a trivial example of the Markov random field on a hypergraph (cf. David and Lauritzen [4]), can be helpful in a number of such situations. Let $\mathcal{G} = \big(X, E(X)\big)$ be a graph describing the orthogonality relation for a hypergraph $\mathcal{H}$. Let $\mu\colon X \to \mathbb{R}^+ \setminus 0$, and let $\xi_{(x,y)}$ be a normally distributed random variable

$$\xi_{(x,y)} \sim N\left(\ln \frac{\mu(y)}{\mu(x)}; \sigma_{(x,y)}\right), \qquad \sigma_{(x,y)} = \sigma_{(y,x)},$$

defined for any pair (x,y) if $(x,y) \in E(X)$. Moreover, we shall assume that $\xi_{(x,y)} = -\xi_{(y,x)}$, and that the random variables

$$\xi_{(x_1,y_1)}, \xi_{(x_2,y_2)}, \ldots, \xi_{(x_n,y_n)}$$

are independent when the sets $\{x_1, y_1\}, \ldots, \{x_n, y_n\}$ are distinct. Denote by $a_{(x,y)}$ any realization of $\xi_{(x,y)}$. With such realizations of $\xi_{(x,y)}$, one must find the estimators $\theta_{(x,y)}$ of $\ln \frac{\mu(y)}{\mu(x)}$, satisfying the conditions

$$\theta_{(x,y)} = -\theta_{(y,x)}, \tag{4.1}$$

$$\theta_{\left(x_0(i), x_1(i)\right)} + \cdots + \theta_{(x_{n(i)}(i), x_0(i))} = 0, \tag{4.2}$$

where $\big(x_0(i), \ldots, x_{n(i)}, x_0(i)\big)$ is any cycle in $\mathcal{G}$ for $i \in I$.

By the commonly used maximum likelihood method (see Cramer [3]), the function

$$\ln\left(\prod_{(x,y)\in E(X)} \exp\left[-\frac{\big(\theta_{(x,y)} - a_{(x,y)}\big)^2}{2\sigma^2_{(x,y)}}\right]\right) = -\sum_{(x,y)\in E(X)} \frac{\big(\theta_{(x,y)} - a_{(x,y)}\big)^2}{2\sigma^2_{(x,y)}}$$

should take its maximum provided that (4.1) and (4.2) hold. By the standard Lagrange multipliers method, one obtains a system of linear equations: (4.1), (4.2) and

$$\frac{\theta_{(x,y)} - a_{(x,y)}}{\sigma_{(x,y)}} + \sum_{\substack{i \in I \,\&\, (x,y) \in \\ \{(x_0(i), x_1(i)), \dots, (x_n(i), x_0(i))\}}} \lambda_i = 0 \qquad \text{for} \quad (x,y) \in E(X),$$

which uniquely determine the numbers $\theta_{(x,y)}$.

Appendix

Elements of graph theory.

Given a subset A of X, we shall use $A^\perp$ to denote the set $\{x \in X : x \perp a \text{ for all } a \in A\}$, i.e., $A^\perp$ is a set of vertices in $X \setminus A$ because the relation $\perp$ is antireflexive. The set $A^\perp$ is said to be the *boundary* of a subset A of X. A set $\{x\}^\perp = \{y \in x : (x,y) \in E\}$ will then denote the *adjacency set* of the vertex $x \in X$. For simplicity, we shall use $x^\perp$ instead of $\{x\}^\perp$.

Given a subset $A \subseteq X$ of the vertex set of a graph $\mathcal{G}$, we define the *subgraph induced* by A to be $\mathcal{G}_A = (A, E_A)$, where the edge set

$$E_A = \{(x,y) \in E : \ x \in A \text{ and } y \in A\}$$

is obtained from $\mathcal{G}$ by keeping the edges with both endpoints in A. In other words, for two distinct points $a, b \in \mathcal{G}_A$, the relation $a \perp b$ is true in $\mathcal{G}_A$ whenever it is true in $\mathcal{G}$.

By a *path* of length $n \geq 0$ between two vertices $x, y \in X$, we mean a sequence of distinct vertices $x_0, \dots, x_n \in X$ such that $\{x_{i-1}, x_i\} \in E$ in the sense that the terminal point of the edge $\{x_{i-1}, x_i\}$ is the initial point of the edge $\{x_i, x_{i+1}\}$ for all $i = 1, \dots, n-1$ with $x_1 = x$ and $x_n = y$.

A *chain* is a sequence of edges of $\mathcal{G}$ such that each edge in the sequence has one endpoint in common with its predecessor in the sequence, and its other endpoint in common with its successor in the sequence. A chain that does not encounter the same vertex twice is called *elementary.* A chain that does not use the same edge twice is called *simple.*

A graph $\mathcal{G}$ is *connected* if, between any two vertices, there exists a chain in $\mathcal{G}$ joining them.

The relation $\sim$ defined as $x \sim y$ if and only if

(i) $x = y$,

or

(ii) $x \neq y$, and there exists a chain in $\mathcal{G}$ connecting x and y

is an equivalence relation. The classes of this equivalence relation partition X into connected subgraphs of $\mathcal{G}$ called *connected components.*

An n*-cycle* is a path of length n with the modification that $x = y$, i.e., it begins and ends at the same point. An n-cycle $x_1, x_2, \dots, x_n, x_1$ is called an *irreducible cycle* if $x_i \neq x_j$ for $i \neq j$ and $x_i \perp x_j$ if $j = i \pm 1$ or $j = i \pm (n-1)$.

A graph $\mathcal{G}$ is said to be *acyclic* if it does not contain an irreducible cycle with more than three points. A graph is called *triangulated* if each cycle of length at least four possesses a chord that is an edge joining two non-consecutive vertices of the cycle. Triangulated graphs are also called chordal graphs. They are simply acyclic graphs ([8], [14], [15]).

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