Zyta Szylicka

Proper hypersubstitutions of some generalizations of lattices and Boolean algebras

Mathematica Slovaca, Vol. 47 (1997), No. 3, 251--266

Persistent URL: http://dml.cz/dmlcz/136704

Math. Slovaca, **47** (1997), No. 3, 251–266

Mathematica Slovaca
©1997
Mathematical Institute
Slovak Academy of Sciences

PROPER HYPERSUBSTITUTIONS OF SOME GENERALIZATIONS OF LATTICES AND BOOLEAN ALGEBRAS

ZYTA SZYLICKA

(*Communicated by Tibor Katriňák*)

ABSTRACT. The notion of a proper hypersubstitution of a variety V was introduced by J. Płonka [*Proper and inner hypersubstitutions of varieties.* In: Summer School on General Algebra and Ordered Sets 1994. Proceedings of the International Conference, Palacký University, Olomouc, 1994, pp. 106–115]. Let V be a variety of a type τ. A hypersubstitution η of type τ is called a proper hypersubstitution of V if for every identity $\varphi \approx \psi$ satisfied in V the identity $\eta(\varphi) \approx \eta(\psi)$ is satisfied in V as well. In this paper, we consider proper hypersubstitutions of the uniformation and of the biregularization of a variety V. A special role in our work is played by hypersubstitutions which are regular, full and regular. We give various sufficient conditions under which a hypersubstitution η is a proper hypersubstitution of the uniformation and of the biregularization of a variety V. We determine all proper hypersubstitutions of the uniformation and of the biregularization of the variety of lattices and of the variety of Boolean algebras.

Introduction

The idea of a hypersubstitution was introduced by W. Taylor [18]. This notion was explicitly defined by E. Graczyńska and D. Schweigert [6] (see also E. Graczyńska [4]) and it was largely used for studying hyperidentities. A hypersubstitution is in fact a kind of so called *semi-weak endomorphism* (see [7] or [3]) of an algebra of terms which assigns variables to variables and terms to terms (see Section 1). Mappings which preserve identities play a crucial role for algebraists, and therefore J. Płonka [13] considered the following problem.

Let V be a variety of a given type. Which hypersubstitution η have the following property: for every identity $\varphi \approx \psi$ from $\mathrm{Id}(V)$ the identity $\eta(\varphi) \approx$

AMS Subject Classification (1991): Primary 08B99; Secondary 08B05, 06B20, 06E99.
Key words: hypersubstitution, proper hypersubstitution, variety, lattice, Boolean algebra.

$\eta(\psi)$ belongs to $\mathrm{Id}(V)$, where $\mathrm{Id}(V)$ denotes the set of all identities satisfied in V. He called such hypersubstitution a *proper hypersubstitution of* V.

J. Płonka [13] characterized proper hypersubstitutions of the varieties of lattices, of Boolean algebras and of their regularizations. In [16], [17], proper hypersubstitutions of some other generalizations of those varieties were examined. In [15], proper hypersubstitutions of the join of independent varieties were studied.

1. Preliminaries

Let us begin with the definition of a hypersubstitution. Here we quote this concept defined by E. Graczyńska and D. Schweigert [6] (see also [4]) with a slight modification from [11]. Let $\tau\colon F \to \mathbb{N}$ be a type of algebras, where F is a set of fundamental operation symbols and $\mathbb{N}$ is the set of positive integers. For a term φ of type τ let $\mathrm{Var}(\varphi)$ denote the set of all variables occurring in φ. We denote by $F(\varphi)$ the set of all fundamental operation symbols in φ. Writing $\varphi(x_{i_0},\dots,x_{i_{m-1}})$ instead of φ we shall mean that $\mathrm{Var}(\varphi) \subseteq \{x_{i_0},\dots,x_{i_{m-1}}\}$. For $f \in F$ we call the term $f(x_0,\dots,x_{\tau(f)-1})$ a *fundamental term.* Let Φ_ω^τ denote the set of all terms of type τ on variables $x_0,\dots,x_k,\dots$ $(k < \omega)$. A mapping $\eta\colon \Phi_\omega^\tau \to \Phi_\omega^\tau$ is called a *hypersubstitution of type* τ, or briefly, a *hypersubstitution* if η satisfies the following three conditions:

(H1) It assigns to every fundamental term $f(x_0,\dots,x_{\tau(f)-1})$ a term $\varphi_{f,\tau}(x_0,\dots,x_{\tau(f)-1})$ and $\eta\big(f(x_0,\dots,x_{\tau(f)-1})\big) = \varphi_{f,\tau}(x_0,\dots,x_{\tau(f)-1})$.
(H2) $\eta(x_k) = x_k$ for every variable x_k, $0 \le k < \omega$.
(H3) If $f \in F$ and $\varphi_0,\dots,\varphi_{\tau(f)-1} \in \Phi_\omega^\tau$, then

$$\eta\big(f(\varphi_0,\dots,\varphi_{\tau(f)-1})\big) = \varphi_{f,\tau}\big(\eta(\varphi_0),\dots,\eta(\varphi_{\tau(f)-1})\big).$$

By $\mathrm{Hyp}(\tau)$, we denote the set of all hypersubstitutions of type τ.

Let V be a variety of type τ. Following [1] for an identity $\varphi \approx \psi$ of type τ, we write $V \models \varphi \approx \psi$ if $\varphi \approx \psi$ belongs to $\mathrm{Id}(V)$, and we write $V \not\models \varphi \approx \psi$ otherwise.

Recall that a hypersubstitution η of type τ is called a *proper hypersubstitution of* V if for every $\varphi \approx \psi$ from $\mathrm{Id}(V)$ we have $V \models \eta(\varphi) \approx \eta(\psi)$. A hypersubstitution η of type τ is called an *inner hypersubstitution of* V (see [13]) if for every $f \in F$, $V \models f(x_0,\dots,x_{\tau(f)-1}) \approx \eta\big(f(x_0,\dots,x_{\tau(f)-1})\big)$. For general properties of proper and of inner hypersubstitutions, we refer to [13]. We denote by $P(V)$, $P_0(V)$ the set of all proper, of all inner hypersubstitutions of V, respectively. A variety V of type τ is said to be *unary* if $\tau(F) = \{1\}$. A variety V of type τ is called *idempotent* if all fundamental operations in algebras of it are idempotent.

K. Denecke and M. Reichel [2] proved the following.

RESULT 1.1. ([2]) *Let V be a variety of type τ. Then $P_0(V) = P(V) = \mathrm{Hyp}(\tau)$ if and only if the variety V is idempotent and unary, or V is trivial (i.e., $V \models x \approx y$).*

We need some notions from [13]. Let $\varphi(x_0, \dots, x_{m-1})$ be a term of type τ. A term $\varphi(x_0, \dots, x_{m-1})$ is called *$(x_0, \dots, x_{m-1})$-symmetrical in V* if $V \models \varphi(x_{k_0}, \dots, x_{k_{m-1}}) \approx \varphi(x_0, \dots, x_{m-1})$ for every permutation $(k_0, \dots, k_{m-1})$ of indices $0, \dots, m-1$.

We need the following.

LEMMA 1.2. ([13]) *Let V be a variety of type τ, let $f \in F$ and let $\eta \in P(V)$. If $f(x_0, \dots, x_{\tau(f)-1})$ is $(x_0, \dots, x_{\tau(f)-1})$-symmetrical in V, then $\eta(f(x_0, \dots, x_{\tau(f)-1}))$ is $(x_0, \dots, x_{\tau(f)-1})$-symmetrical in V.*

Let V be a variety of type τ. A term $\varphi(x_0, \dots, x_{m-1})$ will be called *weakly idempotent in V* if $V \models \varphi(\varphi(x, \dots, x), x, \dots, x) \approx \varphi(x, \dots, x)$. From (H3) and (H2), we obtain the following.

LEMMA 1.3. *Let V be a variety of type τ, let $f \in F$, and let $\eta \in P(V)$. If $f(x_0, \dots, x_{\tau(f)-1})$ is weakly idempotent in V, then $\eta(f(x_0, \dots, x_{\tau(f)-1}))$ is weakly idempotent in V.*

Let p be a positive integer, let $F_p(\tau)$ denote the set of all fundamental terms $f(x_0, \dots, x_{p-1})$ with $\tau(f) = p$, and let $S_p(V)$ denote the set of all terms $\psi(x_0, \dots, x_{p-1})$ which are $(x_0, \dots, x_{p-1})$-symmetrical and weakly idempotent in V. Combining Lemmas 1.2 with 1.3 we have the following.

PROPOSITION 1.4. *Let V be a variety of type τ. If $p > 0$, $\eta \in P(V)$, every term from $F_p(\tau)$ is $(x_0, \dots, x_{p-1})$-symmetrical and weakly idempotent in V, then for every term $f(x_0, \dots, x_{p-1})$ from $F_p(\tau)$ the term $\eta(f(x_0, \dots, x_{p-1}))$ belongs to $S_p(V)$.*

Let V be a variety of type τ. Two terms φ and ψ of type τ are called *V-equivalent* if $V \models \varphi \approx \psi$. Two hypersubstitutions η_1 and η_2 are called *V-equivalent* if for every $f \in F$, $V \models \eta_1(f(x_0, \dots, x_{\tau(f)-1})) \approx \eta_2(f(x_0, \dots, x_{\tau(f)-1}))$. Clearly, if η_1 and η_2 are *V-equivalent*, then $V \models \eta_1(\varphi) \approx \eta_2(\varphi)$ for every term φ of type τ. It is known from [13] that if η_1 and η_2 are V-equivalent, then $\eta_1 \in P(V)$ if and only if $\eta_2 \in P(V)$, and so, to find all proper hypersubstitutions of V, it is enough to choose one hypersubstitution from each equivalence class of the relation "to be V-equivalent" and check if it belongs to $P(V)$ or not (see [13; Remark 1.1]).

Let V be a variety of type τ. Let ρ denote the relation "to be V-equivalent" defined on the set $\mathrm{Hyp}(\tau)$ of all hypersubstitutions of type τ, and $\rho_V = \rho \cap (P(V))^2$, where $(P(V))^2 = P(V) \times P(V)$, i.e., ρ_V is the restriction of ρ to $P(V)$. We put $p(V) = |P(V)/\rho_V|$. We shall say that two hypersubstitutions η_1 and η_2 are *essentially different* if they are not V-equivalent. So $p(V)$ is equal to the maximal number of proper hypersubstitutions of V such that every of them are essentially different.

Throughout the paper, τ_0 denotes the type such that $\tau_0\colon \{+\,,\cdot\} \to \mathbb{N}$, where $\tau_0(+) = \tau_0(\cdot) = 2$ and τ_1 the type such that $\tau_1\colon \{+\,,\cdot\,,'\} \to \mathbb{N}$, where $\tau_1(+) = \tau_1(\cdot) = 2$, $\tau_1(') = 1$.

In this paper, we shall use the following convention. Let V be a variety of type τ_0. For $\eta \in \mathrm{Hyp}(\tau_0)$ we will write (V,η,α,β) instead of η is V-equivalent to $\sigma \in \mathrm{Hyp}(\tau_0)$ defined by $\sigma(x_0 + x_1) = \alpha$, $\sigma(x_0 \cdot x_1) = \beta$ for some terms α, β of type τ_0. Let V be a variety of type τ_1. For $\eta \in \mathrm{Hyp}(\tau_1)$ we will write $(V,\eta,\alpha,\beta,\gamma)$ instead of η is V-equivalent to $\sigma \in \mathrm{Hyp}(\tau_1)$ defined by $\sigma(x_0 + x_1) = \alpha$, $\sigma(x_0 \cdot x_1) = \beta$, $\sigma(x_0') = \gamma$ for some terms α, β, γ of type τ_1.

We will say that an identity $\varphi \approx \psi$ *excludes* η *from* $P(V)$, or briefly *excludes* η if $V \models \varphi \approx \psi$ and $V \not\models \eta(\varphi) \approx \eta(\psi)$.

For a set Σ of identities of type τ we denote by $\mathrm{Mod}(\Sigma)$ the variety of type τ defined by the set Σ. An identity $\varphi \approx \psi$ of type τ is called *regular* (see [8]) if $\mathrm{Var}(\varphi) = \mathrm{Var}(\psi)$. For a variety V of type τ let $R(V)$ denote the set of all regular identities from $\mathrm{Id}(V)$. We put $V_R = \mathrm{Mod}(R(V))$. The variety V_R is called the *regularization of* V (see [14], cf. *regular part of* V of [5]).

Throughout the paper, L denotes the variety of lattices of type τ_0, and B denotes the variety of Boolean algebras of type τ_1.

The following result will be often used in the paper.

Result 1.5. ([13])

(i) *$\eta \in P(L)$ if and only if $(L,\eta,x_0 + x_1, x_0 \cdot x_1)$ or $(L,\eta,x_0 \cdot x_1, x_0 + x_1)$.*

(ii) *$\eta \in P(L_R)$ if and only if $(L_R,\eta,x_0+x_1,x_0\cdot x_1)$ or $(L_R,\eta,x_0\cdot x_1,x_0+x_1)$ or $(L_R,\eta,x_0 \cdot x_1,x_0 \cdot x_1)$ or $(L_R,\eta,x_0 + x_1,x_0 + x_1)$.*

(iii) *$\eta \in P(B)$ if and only if $(B,\eta,x_0 + x_1,x_0 \cdot x_1,x_0')$ or $(B,\eta,x_0 \cdot x_1, x_0 + x_1,x_0')$.*

(iv) *$\eta \in P(B_R)$ if and only if $(B_R,\eta,x_0 + x_1,x_0 \cdot x_1,x_0')$ or $(B_R,\eta,x_0 \cdot x_1, x_0+x_1,x_0')$ or $(B_R,\eta,x_0\cdot x_1,x_0\cdot x_1,x_0)$ or $(B_R,\eta,x_0+x_1,x_0+x_1,x_0)$.*

2. Some hypersubstitutions

A hypersubstitution η of type τ is called a *regular hypersubstitution* (see [11]) if for every $f \in F$, $\mathrm{Var}(\eta(f(x_0,\dots,x_{\tau(f)-1}))) = \{x_0,\dots,x_{\tau(f)-1}\}$. We denote by $\mathrm{RegHyp}(\tau)$ the set of all regular hypersubstitutions of type τ. In the sequel, we need the following.

LEMMA 2.1. ([13]) *Let φ be a term of type τ. If $\eta \in \mathrm{RegHyp}(\tau)$, then*

$$F(\eta(\varphi)) = \bigcup_{f\in F(\varphi)} F(\eta(f(x_0,\dots,x_{\tau(f)-1}))) .$$

A hypersubstitution η of type τ will be called a *full hypersubstitution* if $\bigcup_{f\in F} F(\eta(f(x_0,\dots,x_{\tau(f)-1}))) = F$. By $\mathrm{FullHyp}(\tau)$ we denote the set of all full hypersubstitutions of type τ. Let η_{id} be the hypersubstitution of type τ defined by $\eta_{\mathrm{id}}(f(x_0,\dots,x_{\tau(f)-1})) = f(x_0,\dots,x_{\tau(f)-1})$ for every $f \in F$. K. Denecke and M. Reichel [2] proved that $\underline{\mathrm{Hyp}(\tau)} = (\mathrm{Hyp}(\tau),\circ,\eta_{\mathrm{id}})$ is a monoid, where $\circ$ denotes the superposition. Let us put $\mathrm{FRHyp}(\tau) = \mathrm{FullHyp}(\tau) \cap \mathrm{RegHyp}(\tau)$. We have:

PROPOSITION 2.2. *$\underline{\mathrm{FRHyp}(\tau)}$ is a submonoid of $\underline{\mathrm{Hyp}(\tau)}$.*

Proof. Obviously, η_{id} is a full and regular hypersubstitution. Let $\eta_1, \eta_2 \in \mathrm{FRHyp}(\tau)$. Then $\eta_1 \circ \eta_2 \in \mathrm{RegHyp}(\tau)$ because it is known from [11] that $\underline{\mathrm{RegHyp}(\tau)}$ is a submonoid of $\underline{\mathrm{Hyp}(\tau)}$. In view of Lemma 2.1, we obtain

$$\bigcup_{f\in F} F(\eta_1(\eta_2(f(x_0,\dots,x_{\tau(f)-1})))) = F .$$

Thus $\eta_1 \circ \eta_2 \in \mathrm{FullHyp}(\tau)$. Similarly, we obtain that $\eta_2 \circ \eta_1 \in \mathrm{FullHyp}(\tau)$. Consequently, $\mathrm{FRHyp}(\tau)$ is closed under $\circ$. □

Remark. $\underline{\mathrm{FullHyp}(\tau)}$ need not be a submonoid of $\underline{\mathrm{Hyp}(\tau)}$. In fact, let us consider the type τ_0, and let $\eta_1 \in \mathrm{FullHyp}(\tau_0)$ be defined by $\eta_1(x_0 + x_1) = x_0$, $\eta_1(x_0 \cdot x_1) = x_0 + (x_1 \cdot x_0)$. We see that $\eta_1(\eta_1(x_0 + x_1)) = \eta_1(x_0) = x_0$, $\eta_1(\eta_1(x_0 \cdot x_1)) = \eta_1(x_0 + (x_1 \cdot x_0)) = x_0$. Thus, $\eta_1 \circ \eta_1 \notin \mathrm{FullHyp}(\tau_0)$.

3. Uniformations of varieties

An identity $\varphi \approx \psi$ of type τ is called *uniform* (see [9]) if $F(\varphi) = F(\psi) = F$ or $F(\varphi) = F(\psi) \neq F$ and $\mathrm{Var}(\varphi) = \mathrm{Var}(\psi)$. For example, the identity $x_0 + x_0 \cdot x_1 \approx x_0 + x_0 \cdot x_0$ is uniform in L, however, it is not regular and it is not uniform in B. For a variety V of type τ we denote by $U(V)$ the set of all uniform identities from $\mathrm{Id}(V)$. We put $V_U = \mathrm{Mod}\big(U(V)\big)$. The variety V_U will be called the *uniformation of* V.

We need the following lemmas.

Lemma 3.1. ([13]) *If $\varphi \approx \psi$ is a regular identity of type τ and $\eta \in \mathrm{RegHyp}(\tau)$, then $\eta(\varphi) \approx \eta(\psi)$ is a regular identity of type τ and* $\mathrm{Var}\big(\eta(\varphi)\big) = \mathrm{Var}(\varphi) = \mathrm{Var}(\psi) = \mathrm{Var}\big(\eta(\psi)\big)$.

Lemma 3.2. *If $\varphi \approx \psi$ is a uniform identity of type τ and $\eta \in \mathrm{FRHyp}(\tau)$, then $\eta(\varphi) \approx \eta(\psi)$ is a uniform identity of type τ.*

Proof. Let $F(\varphi) = F(\psi) \neq F$ and $\mathrm{Var}(\varphi) = \mathrm{Var}(\psi)$. Then, since $\eta \in \mathrm{RegHyp}(\tau)$, by Lemma 2.1, we get

$$\begin{aligned} F\big(\eta(\varphi)\big) &= \bigcup_{f \in F(\varphi)} F\Big(\eta\big(f(x_0, \dots, x_{\tau(f)-1})\big)\Big) \\ &= \bigcup_{f \in F(\psi)} F\Big(\eta\big(f(x_0, \dots, x_{\tau(f)-1})\big)\Big) = F\big(\eta(\psi)\big). \end{aligned}$$

Moreover, according to Lemma 3.1, we have $\mathrm{Var}\big(\eta(\varphi)\big) = \mathrm{Var}\big(\eta(\psi)\big)$. Now, assume that $F(\varphi) = F(\psi) = F$. Note that $\eta \in \mathrm{FRHyp}(\tau)$. Hence, using Lemma 2.1, we obtain

$$\begin{aligned} F\big(\eta(\varphi)\big) &= \bigcup_{f \in F(\varphi)} F\Big(\eta\big(f(x_0, \dots, x_{\tau(f)-1})\big)\Big) = \bigcup_{f \in F} F\Big(\eta\big(f(x_0, \dots, x_{\tau(f)-1})\big)\Big) = F \\ &= \bigcup_{f \in F} F\Big(\eta\big(f(x_0, \dots, x_{\tau(f)-1})\big)\Big) = \bigcup_{f \in F(\psi)} F\Big(\eta\big(f(x_0, \dots, x_{\tau(f)-1})\big)\Big) \\ &= F\big(\eta(\psi)\big). \end{aligned}$$

This completes the proof. □

Proposition 3.3. *Let V be a variety of type τ. If $\eta \in \mathrm{FRHyp}(\tau)$ and $\eta \in P(V)$, then $\eta \in P(V_U)$.*

Proof. Let $\varphi \approx \psi$ belong to $U(V)$. Then $\varphi \approx \psi$ belongs to $\mathrm{Id}(V)$, and since $\eta \in P(V)$, we get $V \models \eta(\varphi) \approx \eta(\psi)$. However, the assumptions of Lemma 3.2 are satisfied, and hence, the identity $\eta(\varphi) \approx \eta(\psi)$ is uniform. Consequently, $V_U \models \eta(\varphi) \approx \eta(\psi)$, and so the proof is completed. □

Consider the following 12 terms:

$$x_0,\ x_0+x_0,\ x_0\cdot x_0,\ x_0+x_0\cdot x_1,\ x_1,\ x_1+x_1,\ x_1\cdot x_1,\ x_1+x_0\cdot x_1,\ x_0+x_1,\ (x_0+x_1)\cdot(x_0+x_1),\ x_0\cdot x_1,\ x_0\cdot x_1+x_0\cdot x_1\,. \tag{3.1}$$

Let us put

$$L_1=\{x_0+x_1,\ (x_0+x_1)\cdot(x_0+x_1)\}\,,$$
$$L_2=\{x_0\cdot x_1,\ x_0\cdot x_1+x_0\cdot x_1\}\,.$$

THEOREM 3.4. *Let L_U be the uniformation of the variety L of lattices. Then $\eta\in P(L_U)$ if and only if (L_U,η,α,β), where $(\alpha,\beta)\in(L_1\times L_2)\cup(L_2\times L_1)$.*

Proof.

($\Longrightarrow$) First note that every binary term $q(x_0,x_1)$ of type τ_0 is L_U-equivalent to one of the terms (3.1). According to Proposition 1.4, if $\eta\in P(L_U)$, then (L_U,η,α,β), where $\alpha,\beta\in L_1\cup L_2$. This follows from the fact that among the terms (3.1) only the terms from $L_1\cup L_2$ are (x_0,x_1)-symmetrical and weakly idempotent in L_U. Further, if $(\alpha,\beta)\in L_1^2\cup L_2^2$ (where $L_i^2=L_i\times L_i$, $i=1,2$), then $\eta\notin P(L_U)$. In fact, it is enough to observe that the identity $x_0+x_0\cdot x_1\approx x_0+x_0\cdot x_0$ excludes η from $P(L_U)$. For example, let us take $\alpha=x_0+x_1$, $\beta=(x_0+x_1)\cdot(x_0+x_1)$. We have $L_U\models\eta(x_0+x_0\cdot x_1)\approx x_0+(x_0+x_1)\cdot(x_0+x_1)$ and $L_U\models\eta(x_0+x_0\cdot x_0)\approx x_0+(x_0+x_0)\cdot(x_0+x_0)$. But $L\not\models x_0+(x_0+x_1)\cdot(x_0+x_1)\approx x_0+(x_0+x_0)\cdot(x_0+x_0)$ and consequently, $L_U\not\models x_0+(x_0+x_1)\cdot(x_0+x_1)\approx x_0+(x_0+x_0)\cdot(x_0+x_0)$.

($\Longleftarrow$) Let (L_U,η,α,β), where $(\alpha,\beta)\in(L_1\times L_2)\cup(L_2\times L_1)$. Then, in view of Result 1.5 (i), we conclude that $\eta\in P(L)$. Further, we see that $\eta\in\mathrm{FRHyp}(\tau)$. Hence, by Proposition 3.3, $\eta\in P(L_U)$ as required. □

COROLLARY 3.5. $P(L_U)=P(L)$.

Proof. Since $P(L)\subseteq\mathrm{FRHyp}(\tau_0)$, we see that $P(L)\subseteq P(L_U)$ follows from Proposition 3.3. On the other hand, assume that $L\models\varphi\approx\psi$. Combining Result 1.5 (i) and Theorem 3.4 we conclude that, if $\eta\in P(L_U)$, then there exists $\eta^*\in P(L)$ such that η and η^* are L-equivalent. Hence, we have $L\models\eta(\varphi)\approx\eta^*(\varphi)\approx\eta^*(\psi)\approx\eta(\psi)$. Finally, $\eta\in P(L)$, and so we proved the statement. □

To prove the next theorem, we need a simple but technical lemma.

LEMMA 3.6. *Let φ be a term of type τ_1 with $\mathrm{Var}(\varphi)=\{x_0,\dots,x_{m-1}\}$. Then we have:*

(i) *If $(B,\eta,\alpha,\alpha,\gamma)$, where $(\alpha,\gamma)\in\{(x_0+x_1,x_0+x_0'),(x_0\cdot x_1,x_0\cdot x_0')\}$, then*

$$B\models\eta(\varphi)\approx\begin{cases}\gamma & \text{if } {}'\in F(\varphi)\,,\\ x_0+\dots+x_{m-1} & \text{if } \alpha=x_0+x_1\,,\ {}'\notin F(\varphi)\,,\\ x_0\cdot\dots\cdot x_{m-1} & \text{if } \alpha=x_0\cdot x_1\,,\ {}'\notin F(\varphi)\,.\end{cases}$$

(ii) *If* $(B,\eta,\alpha,\beta,x_0)$, *where* $(\alpha,\beta) \in \{(x_0+x_1, x_0+x_0'), (x_0+x_0', x_0+x_1), (x_0\cdot x_1, x_0\cdot x_0'), (x_0\cdot x_0', x_0\cdot x_1)\}$, *then*

$$B \models \eta(\varphi) \approx \begin{cases} \alpha & \text{if } + \in F(\varphi),\ \alpha \in \{x_0+x_0', x_0\cdot x_0'\}; \\ \beta & \text{if } \cdot \in F(\varphi),\ \beta \in \{x_0+x_0', x_0\cdot x_0'\}; \\ x_0+\cdots+x_{m-1} & \text{if } \alpha = x_0+x_1,\ \cdot \notin F(\varphi) \text{ or } \beta = x_0+x_1,\ + \notin F(\varphi); \\ x_0\cdot\ldots\cdot x_{m-1} & \text{if } \alpha = x_0\cdot x_1,\ \cdot \notin F(\varphi) \text{ or } \beta = x_0\cdot x_1,\ + \notin F(\varphi). \end{cases}$$

(iii) *If* $(B,\eta,\alpha,\gamma,\gamma)$, *where* $(\alpha,\gamma) \in \{(x_0+x_1, x_0+x_0'), (x_0\cdot x_1, x_0\cdot x_0')\}$, *then*

$$B \models \eta(\varphi) \approx \begin{cases} \gamma & \text{if } \cdot \in F(\varphi) \text{ or } {}' \in F(\varphi), \\ x_0+\cdots+x_{m-1} & \text{if } \alpha = x_0+x_1,\ F(\varphi) \subseteq \{+\}, \\ x_0\cdot\ldots\cdot x_{m-1} & \text{if } \alpha = x_0\cdot x_1 F(\varphi) \subseteq \{+\}. \end{cases}$$

(iv) *If* $(B,\eta,\gamma,\beta,\gamma)$, *where* $(\beta,\gamma) \in \{(x_0+x_1, x_0+x_0'), (x_0\cdot x_1, x_0\cdot x_0')\}$, *then*

$$B \models \eta(\varphi) \approx \begin{cases} \gamma & \text{if } + \in F(\varphi) \text{ or } {}' \in F(\varphi), \\ x_0+\cdots+x_{m-1} & \text{if } \beta = x_0+x_1,\ F(\varphi) \subseteq \{\cdot\}, \\ x_0\cdot\ldots\cdot x_{m-1} & \text{if } \beta = x_0\cdot x_1,\ F(\varphi) \subseteq \{\cdot\}. \end{cases}$$

(v) *If* $(B,\eta,\alpha,\alpha,x_0)$, *where* $\alpha \in \{x_0+x_0', x_0\cdot x_0'\}$, *then* $B \models \eta(\varphi) \approx x_k$ *for every* φ *such that* $F(\varphi) \subseteq \{'\}$, $\mathrm{Var}(\varphi) = \{x_k\}$, *and* $B \models \eta(\varphi) \approx \alpha$ *otherwise.*

(vi) *If* $(B,\eta,\alpha,\alpha,\alpha)$, *where* $\alpha \in \{x_0+x_0', x_0\cdot x_0'\}$, *then* $B \models \eta(\varphi) \approx \alpha$ *for every* φ *with* $F(\varphi) \neq \emptyset$.

Proof. We use induction with respect to the complexity of φ. If φ is a fundamental term, then the statement is clear for each of the conditions (i) – (iv).

(i) We prove only this for $(\alpha,\gamma) = (x_0+x_1, x_0+x_0')$ because the proof for $(\alpha,\gamma) = (x_0\cdot x_1, x_0\cdot x_0')$ is analogous. Let $\varphi = \varphi_0+\varphi_1$ for some terms φ_0, φ_1 of type τ_1. If $' \in F(\varphi)$, then without loss of generality, we can assume that $' \in F(\varphi_0)$. Then, by the inductive assumption, we have

$$B \models \eta(\varphi) \approx \eta(\varphi_0+\varphi_1) \approx \eta(\varphi_0)+\eta(\varphi_1) \approx x_0+x_0'+\eta(\varphi_1) \approx x_0+x_0'. \quad (3.2)$$

If $' \notin F(\varphi)$, then we can assume that $\mathrm{Var}(\varphi_0) = \{x_{i_0},\ldots,x_{i_{s-1}}\}$, $\mathrm{Var}(\varphi_1) = \{x_{j_0},\ldots,x_{j_{t-1}}\}$, where $i_0,\ldots,i_{s-1},j_0,\ldots,j_{t-1} \in \{0,\ldots,m-1\}$. According to the inductive assumption, we conclude that

$$\begin{aligned} B \models \eta(\varphi) &\approx \eta(\varphi_0+\varphi_1) \approx \eta(\varphi_0)+\eta(\varphi_1) \\ &\approx x_{i_0}+\cdots+x_{i_{s-1}}+x_{j_0}+\cdots+x_{j_{t-1}} \approx x_0+\cdots+x_{m-1}. \end{aligned} \quad (3.3)$$

Similarly, we deal with $\varphi = \varphi_0 \cdot \varphi_1$. Further, let $\varphi = (\varphi_0)'$. Then $B \models \eta(\varphi) \approx \eta((\varphi_0)') \approx \eta(\varphi_0) + (\eta(\varphi_0))' \approx x_0 + x_0'$.

(ii) Let $(B, \eta, \alpha, \beta, x_0)$. First, let $(\alpha, \beta) = (x_0 + x_1, x_0 + x_0')$. If $\varphi = \varphi_0 \cdot \varphi_1$ for some terms φ_0, φ_1 of type τ_1, then, by the inductive assumption, $B \models \eta(\varphi) \approx \eta(\varphi_0 \cdot \varphi_1) \approx \eta(\varphi_0) + (\eta(\varphi_0))' \approx x_0 + x_0'$. Let $\varphi = \varphi_0 + \varphi_1$. If $\cdot \in F(\varphi)$, then without loss of generality, we can assume that $\cdot \in F(\varphi_0)$. By the inductive assumption, we get (3.2) as required. If $\cdot \notin F(\varphi)$, then we can assume that $\mathrm{Var}(\varphi_0) = \{x_{i_0}, \dots, x_{i_{s-1}}\}$, $\mathrm{Var}(\varphi_1) = \{x_{j_0}, \dots, x_{j_{t-1}}\}$, where $i_0, \dots, i_{s-1}, j_0, \dots, j_{t-1} \in \{0, \dots, m-1\}$. Just as in (i), we obtain (3.3). Let $\varphi = (\varphi_0)'$. Then we use the inductive assumption. If $\cdot \in F(\varphi_0)$, then $B \models \eta(\varphi) \approx \eta((\varphi_0)') \approx \eta(\varphi_0) \approx x_0 + x_0'$. If $\cdot \notin F(\varphi_0)$, then $B \models \eta(\varphi) \approx \eta((\varphi_0)') \approx \eta(\varphi_0) \approx x_0 + \dots + x_{m-1}$ because *$Var(\varphi_0) = Var(\varphi)$. The remaining* $(\alpha, \beta) \in \{(x_0 \cdot x_1, x_0 \cdot x_0'), (x_0 + x_0', x_0 + x_1), (x_0 \cdot x_0', x_0 \cdot x_1)\}$ are handled in the same way.

Proofs of (iii) and (iv) are similar to that of (ii).

(v) We prove only this for $\alpha = x_0 + x_0'$ because the proof for $\alpha = x_0 \cdot x_0'$ is similar. Assume that $\varphi = (\varphi_0)'$. Then, by the inductive assumption, we have

$$B \models \eta(\varphi) \approx \eta((\varphi_0)') \approx \eta(\varphi_0) \approx \begin{cases} x_k & \text{if } F(\varphi_0) \subseteq \{'\}, \\ x_0 + x_0' & \text{otherwise.} \end{cases}$$

Now, let $\varphi = \varphi_0 + \varphi_1$. We see that $B \models \eta(\varphi) \approx \eta(\varphi_0 + \varphi_1) \approx \eta(\varphi_0) + (\eta(\varphi_0))' \approx x_0 + x_0'$. The term $\varphi = \varphi_0 \cdot \varphi_1$ is handled in the same way. Thus the proof of (v) is completed.

(vi) Trivial. □

Let us put

$$U_1 = L_1 \cup \{(x_0 + x_1) \cdot (x_0 + x_0'),\ ((x_0 + x_1)')',\ (x_0' \cdot x_1')'\},$$
$$U_2 = L_2 \cup \{(x_0 \cdot x_1) \cdot (x_0 + x_0'),\ ((x_0 \cdot x_1)')',\ (x_0' + x_1')'\},$$
$$U_3 = \{(x_0 + x_0') + (x_1 + x_1'),\ (x_0 \cdot x_0')' \cdot (x_1 \cdot x_1')',\ (x_0 + x_0') \cdot (x_1 + x_1')\},$$
$$U_4 = \{(x_0 \cdot x_0') \cdot (x_1 \cdot x_1'),\ (x_0 + x_0')' + (x_1 + x_1')',\ x_0 \cdot x_0' + x_1 \cdot x_1'\},$$
$$U_5 = \{x_0',\ x_0' + x_0',\ x_0' \cdot x_0',\ x_0' + x_0' \cdot x_0'\},$$
$$U_6 = \{x_0,\ x_0 + x_0,\ x_0 \cdot x_0,\ (x_0')',\ x_0 + x_0 \cdot x_0,\ x_0 + (x_0')',\ x_0 \cdot (x_0')',\ x_0 + x_0 \cdot x_0'\},$$
$$U_7 = \{x_0 + x_0',\ (x_0 \cdot x_0')',\ (x_0 + x_0') \cdot (x_0 + x_0')\},$$
$$U_8 = \{x_0 \cdot x_0',\ (x_0 + x_0')',\ (x_0 \cdot x_0') + (x_0 \cdot x_0')\}.$$

THEOREM 3.7. *Let B_U be the uniformation of the variety B of Boolean algebras. Then $\eta \in P(B_U)$ if and only if η is a full hypersubstitution with $(B_U, \eta, \alpha, \beta, \gamma)$, where $(\alpha, \beta, \gamma) \in ((U_1 \times U_2) \cup (U_2 \times U_1)) \times U_5$ or $(\alpha, \beta, \gamma) \in ((U_1 \times U_3) \cup (U_3 \times U_1) \cup (U_2 \times U_4) \cup (U_4 \times U_2) \cup U_3^2 \cup U_4^2) \times U_6$ or $(\alpha, \beta, \gamma) \in$*

$\big((U_1 \times U_3) \cup (U_3 \times U_1) \cup U_1^2 \cup U_3^2\big) \times U_7$ *or* $(\alpha,\beta,\gamma) \in \big((U_2 \times U_4) \cup (U_4 \times U_2) \cup U_2^2 \cup U_4^2\big) \times U_8$.

Proof.

$(\implies)$: In view of the arguments used in the first part of the proof of Theorem 3.4, we can infer that if $\eta \in P(B_U)$, then $(B_U, \eta, \alpha, \beta, \gamma)$, where $\alpha, \beta \in \bigcup_{i=1}^{4} U_i$. Thus $\eta \in \mathrm{RegHyp}(\tau_1)$. First we exclude all hypersubstitutions which are not full. Let $\eta \notin \mathrm{FullHyp}(\tau_1)$, i.e., if $(B_U, \eta, \alpha, \beta, \gamma)$, then $F(\alpha) \cup F(\beta) \cup F(\gamma) \neq F$. Let us take the following identity

$$x_0 + x_0 \cdot x_1' \approx x_0 + x_0 \cdot x_0'. \tag{3.4}$$

Clearly, this identity is uniform in B, but is not regular. By Lemma 2.1, we have $F\big(\eta(x_0 + x_0 \cdot x_1')\big) = F(\alpha) \cup F(\beta) \cup F(\gamma) \neq F$ and $F\big(\eta(x_0 + x_0 \cdot x_0')\big) = F(\alpha) \cup F(\beta) \cup F(\gamma) \neq F$. In view of Lemma 3.1, $\mathrm{Var}\big(\eta(x_0 + x_0 \cdot x_1')\big) = \{x_0, x_1\}$, $\mathrm{Var}\big(\eta(x_0 + x_0 \cdot x_0')\big) = \{x_0\}$. Hence, the identity $\eta(x_0 + x_0 \cdot x_1') \approx \eta(x_0 + x_0 \cdot x_0')$ is not uniform in B and consequently, $\eta \notin P(B_U)$. Assume that $\eta \in \mathrm{FullHyp}(\tau_1)$ and $(B_U, \eta, \alpha, \beta, \gamma)$. If $(\alpha, \beta) \in \big((U_1 \cup U_3) \times U_4\big) \cup \big((U_2 \cup U_4) \times U_3\big) \cup (U_4 \times U_1) \cup (U_3 \times U_2)$, then the identity

$$x_0 + x_0 \cdot x_0 \approx x_0 \cdot (x_0 + x_0) \tag{3.5}$$

excludes η from $P(B_U)$. Further, it is not difficult to observe that every unary term $q(x_0)$ of type τ_1 must be B_U-equivalent to one of terms from $\bigcup_{i=5}^{8} U_i$. If $(\alpha, \beta) \in (U_1 \times U_2) \cup (U_2 \times U_1)$, then we have to exclude three possibilities: $\gamma \in U_6$, $\gamma \in U_7$, $\gamma \in U_8$. If $\gamma \in U_6$, then the "biregular" de Morgan law (the term "biregular" de Morgan law is justified by the next section), namely,

$$(x_0 + x_1)' \cdot (x_0 + x_1)' \approx x_0' \cdot x_1' + x_0' \cdot x_1'$$

excludes η. If $\gamma \in U_7 \cup U_8$, then to prove that $\eta \notin P(B_U)$ it is enough to take the identity

$$x_0 + x_0 \cdot x_0' \approx (x_0')' + x_0 \cdot x_0'.$$

Let $(\alpha, \beta) \in U_1^2$. Then we have to deal with three possibilities: $\gamma \in U_5$, $\gamma \in U_6$, $\gamma \in U_8$. First, note that the identity

$$(x_0 + x_0 \cdot x_0)' \approx x_0' + x_0 \cdot x_0' \tag{3.6}$$

excludes η if $\gamma \in U_5 \cup U_8$. If $\gamma \in U_6$, then it is not difficult to check that the identity

$$(x_0 + x_0 \cdot x_1)' \approx x_0' + x_0 \cdot x_0' \tag{3.7}$$

excludes η. Similarly, we deal with $(\alpha,\beta,\gamma) \in U_2^2 \times (U_5 \cup U_6 \cup U_7)$. Further, let $(\alpha,\beta) \in U_1 \times U_3$. Then we have to consider two cases: if $\gamma \in U_5$, then the identity (3.6) excludes η, and if $\gamma \in U_8$, then the identity

$$(x_0 \cdot x_0)' \approx x_0' \cdot x_0' \tag{3.8}$$

excludes η. For $(\alpha,\beta) \in U_3 \times U_1$, it is enough to take the identity $\big(x_0 \cdot (x_0 + x_0)\big)' \approx x_0' \cdot (x_0 + x_0')$ for $\gamma \in U_5$, and the identity

$$(x_0 + x_0)' \approx x_0' + x_0' \tag{3.9}$$

for $\gamma \in U_8$. If $(\alpha,\beta) \in (U_2 \times U_4) \cup (U_4 \times U_2)$, then we have again two possibilities: $\gamma \in U_5$, $\gamma \in U_7$, for which we proceed in the same way as above for $\gamma \in U_5$, $\gamma \in U_8$, respectively. We complete the proof by noting that if $(\alpha,\beta,\gamma) \in U_3^2 \times (U_5 \cup U_8)$ or $(\alpha,\beta,\gamma) \in U_4^2 \times (U_5 \cup U_7)$, then the identity (3.9) excludes η.

($\Longleftarrow$) : Assume that $\eta \in \mathrm{FullHyp}(\tau_1)$ and $(B_U, \eta, \alpha, \beta, \gamma)$, where (α,β,γ) is identical as in the statement. Clearly, $\eta \in \mathrm{RegHyp}(\tau_1)$. If $(\alpha,\beta,\gamma) \in \big((U_1 \times U_2) \cup (U_2 \times U_1)\big) \times U_5$, then combining Result 1.5(iii) with Proposition 3.3, we get that $\eta \in P(B_U)$. Let $(\alpha,\beta,\gamma) \in (U_1^2 \times U_7) \cup (U_2^2 \times U_8)$, and let $B_U \models \varphi \approx \psi$. First, note that $F(\varphi) = F(\psi)$. If $' \in F(\varphi)$, then $B \models \eta(\varphi) \approx \gamma \approx \eta(\psi)$ by Lemma 3.6(i). If $' \notin F(\varphi)$, then we can assume that $\mathrm{Var}(\varphi) = \{x_0, \dots, x_{m-1}\} = \mathrm{Var}(\psi)$. Again applying Lemma 3.6(i), we obtain $B \models \eta(\varphi) \approx x_0 + \dots + x_{m-1} \approx \eta(\psi)$ if $(\alpha,\beta,\gamma) \in U_1^2 \times U_7$, and $B \models \eta(\varphi) \approx x_0 \cdot \ldots \cdot x_{m-1} \approx \eta(\psi)$ if $(\alpha,\beta,\gamma) \in U_2^2 \times U_8$. Since $\eta \in \mathrm{FRHyp}(\tau_1)$, we use Lemma 3.2 to both cases and we get that an identity $\eta(\varphi) \approx \eta(\psi)$ is uniform in B. Thus $B_U \models \eta(\varphi) \approx \eta(\psi)$. Consequently, $\eta \in P(B_U)$. Similarly, if $(\alpha,\beta,\gamma) \in \big((U_1 \times U_3) \cup (U_3 \times U_1) \cup (U_2 \times U_4) \cup (U_4 \times U_2) \cup U_3^2 \cup U_4^2\big) \times U_6$ or $(\alpha,\beta,\gamma) \in \big((U_1 \times U_3) \cup (U_3 \times U_1) \cup U_3^2\big) \times U_7$ or $(\alpha,\beta,\gamma) \in \big((U_2 \times U_4) \cup (U_4 \times U_2) \cup U_4^2\big) \times U_8$, then applying Lemmas 3.6 and 3.2 we conclude that $\eta \in P(B_U)$. Thus the proof is completed. □

Remarks.

1. One can find simpler excluding identities, but here we use these above since they are convenient for further considerations, e.g., in the proof of Theorem 4.5.

2. Theorem 3.7 shows that it can happen that $P_0(V) \not\subseteq P(V_U)$, and consequently, $P(V) \not\subseteq P(V_U)$ because of $P_0(V) \subseteq P(V)$ (see [13; (1.iv)]). In fact, let us take $\eta \in \mathrm{Hyp}(\tau_1)$ defined by $\eta(x_0 + x_1) = x_0 + x_1$, $\eta(x_0 \cdot x_1) = (x_0' + x_1')'$, $\eta(x_0') = x_0'$. Clearly, $\eta \in P_0(B)$, and thus $\eta \in P(B)$ ($\eta \in P(B)$ also follows from Result 1.5(iii)). Let us consider the identity (3.7). Then $\eta\big((x_0 + x_0 \cdot x_1)'\big) = \big(x_0 + (x_0' + x_1')'\big)'$ and $\eta(x_0' + x_0 \cdot x_0') = x_0' + \big(x_0' + (x_0')'\big)'$. Hence, the identity $\eta\big((x_0 + x_0 \cdot x_1)'\big) \approx \eta(x_0' + x_0 \cdot x_0')$ is not uniform in B, and consequently, $\eta \notin P(B_U)$.

COROLLARY 3.8. $p(L_U) = 8$, $p(B_U) = 964$.

4. Biregularizations of varieties

An identity $\varphi \approx \psi$ of type τ is called *biregular* (see [9]) if $F(\varphi) = F(\psi)$ and $\mathrm{Var}(\varphi) = \mathrm{Var}(\psi)$. For example, the identity $x_0 + x_0 \cdot x_1' \approx x_0 + x_1 \cdot x_1'$ is biregular in B, however, the identity $x_0 + x_0 \cdot x_1 \approx x_0 + x_1 \cdot x_1'$ is regular but not biregular. For a variety V of type τ we denote by $B(V)$ the set of all biregular identities from $\mathrm{Id}(V)$. We put $V_B = \mathrm{Mod}(B(V))$. The variety V_B is called the *biregularization of* V (see [12]). Observe that $V_B = (V_U)_R = (V_R)_U$ (see [10]), and so it means that operators U and R commute. The proof of the next lemma is analogous to the second part of the proof of Lemma 3.2.

LEMMA 4.1. (cf. [13]) *If $\varphi \approx \psi$ is a biregular identity of type τ and $\eta \in \mathrm{RegHyp}(\tau)$, then $\eta(\varphi) \approx \eta(\psi)$ is a biregular identity of type τ.*

We need the following.

PROPOSITION 4.2. *Let V be a variety of type τ. Then we have:*

(i) *If $\eta \in \mathrm{RegHyp}(\tau)$ and $\eta \in P(V)$, then $\eta \in P(V_R)$.*
(ii) *If $\eta \in \mathrm{RegHyp}(\tau)$ and $\eta \in P(V_U)$, then $\eta \in P(V_B)$.*
(iii) *If $\eta \in \mathrm{RegHyp}(\tau)$ and $\eta \in P(V_R)$, then $\eta \in P(V_B)$.*

Proof.

(i) was proved in [13].

(ii) Substitute V by V_U and apply (i).

(iii) Similarly to the proof of Proposition 3.3, but using Lemma 4.1 we get the statement. □

Let us consider the following two terms:

$$x_0 + x_0 \cdot x_0, \qquad x_1 + x_1 \cdot x_1 \tag{4.1}$$

THEOREM 4.3. *Let L_B be the biregularization of the variety L of lattices. Then $\eta \in P(L_B)$ if and only if $(L_B, \eta, \alpha, \beta)$, where $(\alpha, \beta) \in (L_1 \cup L_2)^2$.*

Proof.

$(\Longrightarrow)$: First note that every binary term $q(x_0, x_1)$ of type τ_0 is L_B-equivalent to one of the terms (3.1) or (4.1) (see [12]). Arguing analogously as in the proof of Theorem 3.4 we get that, if $\eta \in P(L_B)$, then $(L_B, \eta, \alpha, \beta)$, where $(\alpha, \beta) \in (L_1 \cup L_2)^2$.

$(\Longleftarrow)$: Assume that $(L_B, \eta, \alpha, \beta)$, where $(\alpha, \beta) \in (L_1 \cup L_2)^2$. Then, from Result 1.5(ii), it follows that $\eta \in P(L_R)$. But $\eta \in \mathrm{RegHyp}(\tau_0)$. So, in view of Proposition 4.2(iii), we have $\eta \in P(L_B)$ as required. □

COROLLARY 4.4. $P(L_B) = P(L_R)$.

Proof. Since $P(L_R) \subseteq \mathrm{RegHyp}(\tau_0)$, from Proposition 4.2(iii), we obtain that $P(L_R) \subseteq P(L_B)$. In order to prove the converse inclusion $\supseteq$, let $L_R \models \varphi \approx \psi$, and let $\eta \in P(L_B)$. Then combining Result 1.5(ii) with Theorem 4.3 we conclude that there exists $\eta_1 \in P(L_R)$ such that η and η_1 are L_R-equivalent. So we have $L_R \models \eta(\varphi) \approx \eta_1(\varphi) \approx \eta_1(\psi) \approx \eta(\psi)$, what shows that $\eta \in P(L_R)$. □

THEOREM 4.5. *Let B_B be the biregularization of the variety B of Boolean algebras. Then $\eta \in P(B_B)$ if and only if $(B_B, \eta, \alpha, \beta, \gamma)$, where $(\alpha, \beta, \gamma) \in ((U_1 \times U_2) \cup (U_2 \times U_1)) \times U_5$ or $(\alpha, \beta, \gamma) \in ((U_1 \times U_3) \cup (U_3 \times U_1) \cup (U_2 \times U_4) \cup (U_4 \times U_2) \cup U_1^2 \cup U_2^2 \cup U_3^2 \cup U_4^2) \times U_6$ or $(\alpha, \beta, \gamma) \in ((U_1 \times U_3) \cup (U_1 \times U_3) \cup U_1^2 \cup U_3^2) \times U_7$ or $(\alpha, \beta, \gamma) \in ((U_2 \times U_4) \cup (U_4 \times U_2) \cup U_2^2 \cup U_4^2) \times U_8$.*

Proof.

($\Longrightarrow$) : Analogously as in the proof of Theorem 3.7 and using results of Joel Berman [12; Section 2, Example 3], we conclude that if $\eta \in P(B_B)$, then $(B_B, \eta, \alpha, \beta, \gamma)$, where $\alpha, \beta \in \bigcup_{i=1}^{4} U_i$. First note that $\eta \in \mathrm{RegHyp}(\tau_1)$. To complete the proof of this part, it is enough to repeat considerations from the first part of the proof of Theorem 3.7 substituting B_U by B_B. Therefore, let $(B_B, \eta, \alpha, \beta, \gamma)$. If $(\alpha, \beta) \in ((U_1 \cup U_3) \times U_4) \cup ((U_2 \cup U_4) \times U_3) \cup (U_4 \times U_1) \cup (U_3 \times U_2)$, then the identity (3.5) excludes η from $P(B_B)$. It is not difficult to verify that every unary term $q(x_0)$ of type τ_1 must be B_B-equivalent to one of the terms from $\bigcup_{i=5}^{8} U_i$. For $(\alpha, \beta) \in (U_1 \times U_2) \cup (U_2 \times U_1) \cup (U_1 \times U_3) \cup (U_3 \times U_1) \cup (U_2 \times U_4) \cup (U_4 \times U_2)$ it is enough to repeat the arguments from the proof of Theorem 3.7. If $(\alpha, \beta) \in U_1^2$, then we have to exclude only the case when $\gamma \in U_5 \cup U_8$. This case is handled just as in the proof of Theorem 3.7, so we omit this proof. Similarly, we treat η with $(\alpha, \beta, \gamma) \in U_2^2 \times (U_5 \cup U_7)$. To end the proof of this part, it is enough to observe that, if $(\alpha, \beta, \gamma) \in U_3^2 \times (U_5 \cup U_8)$ or $(\alpha, \beta, \gamma) \in U_4^2 \times (U_5 \cup U_7)$, then the identity (3.9) excludes η from $P(B_B)$.

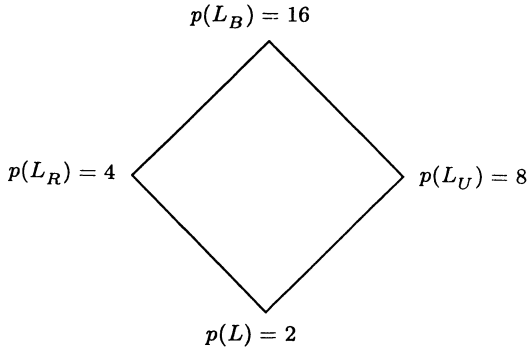
($\Longleftarrow$) : Let $\eta \in \mathrm{Hyp}(\tau_1)$ and $(B_B, \eta, \alpha, \beta, \gamma)$, where (α, β, γ) is identical as in the statement. Obviously, $\eta \in \mathrm{RegHyp}(\tau_1)$. If $(\alpha, \beta, \gamma) \in (U_1^2 \cup U_2^2) \times U_6$, then $\eta \in P(B_B)$ by Proposition 4.2(iii). Further, comparing the statements of Theorems 3.7 and 4.5 we see that they are much the same. Therefore, we need the following fact: If $\eta_1 \in P(B_B)$ and η_1, η_2 are B_R-equivalent, then $\eta_2 \in P(B_B)$. In fact, first note that $\eta_2 \in \mathrm{RegHyp}(\tau_1)$. Further, let $B_B \models \varphi \approx \psi$. Then $B_B \models \eta_1(\varphi) \approx \eta_1(\psi)$ because $\eta_1 \in P(B_B)$. Since every biregular identity is regular, we have $B_R \models \eta_1(\varphi) \approx \eta_1(\psi)$. Because η_1, η_2 are B_R-equivalent, we get $B_R \models \eta_2(\varphi) \approx \eta_1(\varphi) \approx \eta_1(\psi) \approx \eta_2(\psi)$. Moreover, in view of Lemma 4.1, we obtain that the identity $\eta_2(\varphi) \approx \eta_2(\psi)$ is biregular, and so $B_B \models \eta_2(\varphi) \approx$

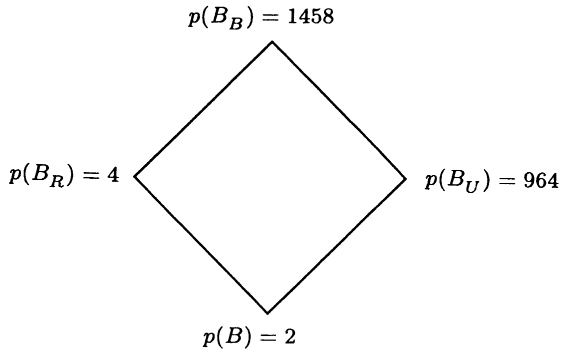
$\eta_2(\psi)$. Finally, $\eta_2 \in P(B_B)$. Combining Proposition 4.2(ii), (iii) with the fact proved above we conclude that, if $(\alpha,\beta,\gamma) \in ((U_1 \times U_2) \cup (U_2 \times U_1)) \times U_5$ or $(\alpha,\beta,\gamma) \in ((U_1 \times U_3) \cup (U_3 \times U_1) \cup (U_2 \times U_4) \cup (U_4 \times U_2) \cup U_3^2 \cup U_4^2) \times U_6$ or $(\alpha,\beta,\gamma) \in ((U_1 \times U_3) \cup (U_3 \times U_1) \cup U_1^2 \cup U_3^2) \times U_7$ or $(\alpha,\beta,\gamma) \in ((U_2 \times U_4) \cup (U_4 \times U_2) \cup U_2^2 \cup U_4^2) \times U_8$, then $\eta \in P(B_B)$. Thus the proof is completed. □

Remark. The fact from the proof of Theorem 4.5 can be generalized for arbitrary variety V of type τ.

COROLLARY 4.6. $p(L_B) = 16$, $p(B_B) = 1458$.

We collect in the figure below Corollaries 3.8 and 4.6. So we have the following diagrams, which present numbers of essentially different proper hypersubstitutions of varieties discussed in the paper.





REFERENCES

[1] BURRIS, S.—SANKAPPANAVAR, H. P.: *A Course in Universal Algebra*, Springer-Verlag, Berlin-New York, 1981.

[2] DENECKE, K.—REICHEL, M.: *Monoids of hypersubstitutions and M-solid varieties.* In: Contributions to General Algebra 9, Verlag Hölder-Pichler-Tempsky, Wien, 1995, pp. 117–126.

[3] GŁAZEK, K.: *Weak homomorphisms of general algebras and related topics*, Math. Seminar Notes **8** (1980), 1–36.

[4] GRACZYŃSKA, E.: *On normal and regular identities and hyperidentities.* In: Universal and Applied Algebra (K. Hałkowska, B. Stawski, eds.), World Sci. Publ., Singapore, 1989, pp. 107–135.

[5] GRACZYŃSKA, E.—KELLEY, D.—WINKLER, P.: *On the regular part of varieties of algebras*, Algebra Universalis **23** (1986), 77–84.

[6] GRACZYŃSKA, E.—SCHWEIGERT, D.: *Hyperidentities of a given type*, Algebra Universalis **27** (1990), 305–318.

[7] KOLIBIAR, M.: *Weak homomorphisms in some classes of algebras*, Studia Sci. Math. Hungar. **19** (1984), 413–420.

[8] PŁONKA, J.: *On method of constructions of abstract algebras*, Fund. Math. **61** (1967), 183–189.

[9] PŁONKA, J.: *On varieties of algebras defined by identities of some special forms*, Houston J. Math. **14** (1988), 253–263.

[10] PŁONKA, J.: *Biregular and uniform identities of algebras*, Czechoslovak Math. J. **40(115)** (1990), 367–387.

[11] PŁONKA, J.: *On hyperidentities of some varieties.* In: General Algebra and Discrete Mathematics (K. Denecke, O. Lüders, eds.), Heldermann Verlag, Berlin, 1995, pp. 199–213.

[12] PŁONKA, J.: *Free algebras over biregularizations of varieties.* In: Proceedings of the Conference "Algebra and Combinatorics, Interactions and Applications", Köningstein, 6–12 March 1994 (To appear).

[13] PŁONKA, J.: *Proper and inner hypersubstitutions of varieties.* In: Summer School on General Algebra and Ordered Sets 1994. Proceedings of the International Conference, Palacký University, Olomouc, 1994, pp. 106–115.

[14] PŁONKA, J.—ROMANOWSKA, A.: *Semilattice sums.* In: Universal Algebra and Quasigroup Theory (A. Romanowska, J. Smith, eds.), Heldermann Verlag, Berlin, 1992, pp. 123–158.

[15] PŁONKA, J.—SZYLICKA, Z.: *Proper hypersubstitutions of the join of independent varieties*, Disscuss. Math., Algebra and Stochastic Methods **15** (1995), 127–134.

[16] SZYLICKA, Z.: *Proper hypersubstitutions of normalizations and externalizations of varieties.* In: Summer School on General Algebra and Ordered Sets 1994. Proceedings of the International Conference, Palacký University, Olomouc, 1994, pp. 144–155.

[17] SZYLICKA, Z.: *Proper hypersubstitutions of outerizations of varieties*, Disscuss. Math., Algebra and Stochastic Methods **15** (1995), 69–80.

[18] TAYLOR, W.: *Hyperidentities and hypervarieties*, Aequationes Math. **23** (1981), 30–49.

Received October 27, 1995
Revised April 15, 1997

Institute of Mathematics
Technical University of Opole
ul. Luboszycka 3
PL–45–036 Opole
POLAND
E-mail: zszyl@ss5.po.opole.pl