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OBSERVABILITY OF A GRAPH

Ján Černý* — Mirko Horňák** — Roman Soták**

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ABSTRACT. Observability, a graph invariant inspired by point-distinguishing chromatic index, line-distinguishing chromatic number and harmonious chromatic number, is introduced. Its value has been determined for complete graphs, paths, cycles, wheels and complete bipartite graphs. A special attention is devoted to regular graphs with optimum structure as regards observability.

1. Introduction

Let G be a finite undirected graph without loops and multiple edges (for basic notions and notations see, e.g., Harary [3]), and let $E_x(G)$ be the set of all edges of G incident with $x \in V(G)$. For integers p, q we shall use the notations

$$[p,q] := \bigcup_{i=p}^{q} \{i\}, \qquad [p,\infty) := \bigcup_{i=p}^{\infty} \{i\};$$

if $q \in [1,\infty)$, the unique integer $i \in [1,q]$ fulfilling $i \equiv p \pmod q$ will be denoted by $(p)_q$. Let φ be a k-edge-colouring of G, i.e., a map from $[1,k]^{E(G)}$. The *colour set of x induced by φ* is defined by

$$\operatorname{Im}_x(\varphi) := \bigcup_{e \in E_x(G)} \{\varphi(e)\}.$$

If φ is a proper colouring, which means that $|\operatorname{Im}_x(\varphi)| = |E_x(G)|$ for each $x \in V(G)$, edges of $E_x(G)$ are distinguished by their colours (values of φ); if, moreover, vertices of G are distinguished by their colour sets, it is natural to say G is *observable through φ* (for short *φ-observable*). Denote by $\operatorname{Obs}_k(G)$ the set of all k-edge-colourings φ of G such that G is φ-observable. Of course,

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$\mathrm{Obs}_k(G) \neq \emptyset$ implies $\mathrm{Obs}_l(G) \neq \emptyset$ for each $l \in [k, \infty)$, and we can define *observability of* G by

$$\mathrm{obs}(G) := \lim_{l \to \infty} \max\{k \in [0, l] : \ \forall j \in [0, k-1] \ \ \mathrm{Obs}_j(G) = \emptyset\}.$$

Thus $\mathrm{obs}(G)$ is either the smallest $k \in [0, \infty)$ with $\mathrm{Obs}_k(G) \neq \emptyset$, or ∞ – if $\mathrm{Obs}_k(G) = \emptyset$ for all $k \in [0, \infty)$ (this happens only if G has a component K_2 or more than one component K_1).

Note that omitting the condition for φ to be proper we obtain another invariant, point-distinguishing chromatic index of a graph – see Harary and Plantholt [4]. On the other hand, the dualization of these two characteristics of a graph – when vertices are coloured and edges are required to be distinguished by colours of their vertices – leads to the notions of line-distinguishing chromatic number of a graph (Frank, Harary and Plantholt [2]) and harmonious chromatic number of a graph (Miller and Pritikin [6]).

Let $v_d(G)$ be the number of vertices of degree d in G, and $\Delta(G)$ the maximum degree of a vertex in G.

1.1. Proposition. *If* $\mathrm{Obs}_k(G) \neq \emptyset$ *for a graph* G *and* $d \in [0, \Delta(G)]$, *then* $v_d(G) \leq \binom{k}{d}$.

Proof. There are at most $\binom{k}{d}$ possibilities to distinguish d-valent vertices of G by d-element subsets of $[1, k]$. □

1.2. Corollary. *For any graph* G

$$\mathrm{obs}(G) \geq \min\Big\{k \in [0, \infty) : \ \forall d \in [0, \Delta(G)] \ \ v_d(G) \leq \binom{k}{d}\Big\}.$$

We have also the following trivial relation between $\mathrm{obs}(G)$ and the chromatic index $\chi'(G)$ of G:

1.3. Proposition. *For any graph* G, $\mathrm{obs}(G) \geq \chi'(G)$.

Proof. If $k = \mathrm{obs}(G) < \infty$, any map from $\mathrm{Obs}_k(G)$ is a proper edge-colouring of G, hence $k \geq \chi'(G)$. □

A d-regular graph G is said to be *fully observable* if $|V(G)| = \binom{\mathrm{obs}(G)}{d}$; in such a case, all d-subsets of $[1, \mathrm{obs}(G)]$ are used to distinguish vertices of G in some edge-colouring of G.

The aim of the present paper is to find observability for graphs with simple structure – the complete graph K_n, the path P_n, the cycle C_n, the wheel W_n, i.e., the graph of $(n-1)$-sided prism (n is always the number of vertices) and the complete bipartite graph $K_{m,n}$. Moreover, all pairs (k, d) are determined such that there exists a fully observable d-regular graph whose observability is k.

2. Some basic classes of graphs

First of all we consider the case of complete graphs. We have of course $\mathrm{obs}(K_0) = \mathrm{obs}(K_1) = 0$ and $\mathrm{obs}(K_2) = \infty$.

2.1. THEOREM. *If* $m \in [2, \infty)$, *then* $\mathrm{obs}(K_{2m-1}) = 2m - 1$ *and* $\mathrm{obs}(K_{2m}) = 2m + 1$.

Proof.

1. The well-known Vizing's theorem [8] together with Proposition 1.3 and the fact that K_{2m-1} does not have a matching imply

$$\mathrm{obs}(K_{2m-1}) \geq \chi'(K_{2m-1}) = 2m - 1 .$$

Let φ be any proper $(2m-1)$-edge colouring of K_{2m-1}. Each colour $i \in [1, 2m-1]$ covers an even number of vertices and, consequently, omits at least one vertex of K_{2m-1}, hence

$$\begin{aligned}(2m-1)(2m-2) &= 2|E(K_{2m-1})| \\ &= \sum_{i=1}^{2m-1} 2|\varphi^{-1}(i)| = \sum_{i=1}^{2m-1} \left|\left\{x \in V(K_{2m-1}) : \ i \in \mathrm{Im}_x(\varphi)\right\}\right| \\ &\leq (2m-1)(2m-2) .\end{aligned}$$

Since the inequality turns into equality, any colour from $[1, 2m-1]$ omits exactly one vertex of K_{2m-1}, and $\mathrm{Im}_x(\varphi) \neq \mathrm{Im}_y(\varphi)$ whenever $x \neq y$. Thus $\varphi \in \mathrm{Obs}_{2m-1}(K_{2m-1})$ and $\mathrm{obs}(K_{2m-1}) = 2m - 1$.

2. As $v_{2m-1}(K_{2m}) = 2m > 1$ and $v_d(K_{2m}) = 0$ for $d \in [0, 2m-2]$,

$$\min\left\{k \in [0, \infty) : \ \forall d \in [0, \Delta(K_{2m})] \ \ v_d(K_{2m}) \leq \binom{k}{d}\right\} = 2m ,$$

and, by Corollary 1.2, $\mathrm{obs}(K_{2m}) \geq 2m$. Suppose $\mathrm{Obs}_{2m}(K_{2m})$ contains a map φ, take $x \in V(K_{2m})$ and consider a (unique) colour i from $[1, 2m] - \mathrm{Im}_x(\varphi)$ (missing at x). Since K_{2m} has $2m$ vertices, i omits at least one vertex $y \in V(K_{2m}) - \{x\}$; but then $\mathrm{Im}_x(\varphi) = \mathrm{Im}_y(\varphi)$, and the obtained contradiction shows that $\mathrm{obs}(K_{2m}) \geq 2m + 1$.

If $V(K_{2m}) = [1, 2m]$, K_{2m} has a matching $M = \left\{\{2i-1, 2i\} : \ i \in \left[1, \left\lceil \frac{m}{2} \right\rceil\right]\right\}$, and we can define a $(2m+1)$-edge-colouring φ of K_{2m} by

$$\begin{aligned}\varphi(e) &= 2m + 1 && \text{for } e \in M , \\ \varphi\{i, j\} &= (i + j)_{2m} && \text{otherwise} .\end{aligned}$$

It is easy to see that φ is proper and that

$$\mathrm{Im}_i(\varphi) = [1, 2m+1] - \left\{(2i)_{2m}, n_i\right\} ,$$

where

$$\begin{aligned} n_i &= \bigl(2i - (-1)^i\bigr)_{2m} && \text{for } i \in [1,m]\,, \\ n_{m+1} &= 1 + m\bigl[1 + (-1)^m\bigr]\,, \\ n_i &= 2m + 1 && \text{for } i \in [m{+}2, 2m]\,. \end{aligned}$$

As $(2k)_{2m}$ is even and n_k is odd for each $k \in [1, 2m]$, the assumption $\mathrm{Im}_i(\varphi) = \mathrm{Im}_j(\varphi)$ with $i, j \in [1, 2m]$, $i < j$, implies $2i \equiv 2j \pmod{2m}$ and $j = i + m$. However, $n_k \le 2m - 1 < 2m + 1 = n_k + m$ for $k \in [2, m]$ and $n_1 = 3 \ne n_{m+1} \in \{1, 2m{+}1\}$, hence colour sets induced by φ are pairwise different, φ is in $\mathrm{Obs}_{2m+1}(K_{2m})$, and $\mathrm{obs}(K_{2m}) = 2m + 1$. □

The analysis concerning paths (with $n \ge 3$ vertices, otherwise P_n is isomorphic to K_n) and cycles involves an idea quite frequent in colouring techniques – see, e.g., [4], Bories and Jolivet [1] or Hell and Miller [5].

2.2. Theorem. *If $n \in [3, \infty)$, then*

$$\mathrm{obs}(P_n) = \min\left\{2\left\lceil \frac{\sqrt{8n-7}-1}{4} \right\rceil + 1,\ 2\left\lceil \frac{\sqrt{2n-5}+1}{2} \right\rceil \right\}.$$

Proof. Suppose $V(P_n) = [1, n]$ and $E(P_n) = \bigl\{\{i, i{+}1\} : i \in [1, n{-}1]\bigr\}$. Of course, $\mathrm{obs}(P_n) \ge 2$ by Corollary 1.2. If $\mathrm{Obs}_k(P_n) \ne \emptyset$ for some $k \in [2, \infty)$ and $\varphi \in \mathrm{Obs}_k(P_n)$, then $\bigl(\varphi\{1,2\}, \dots, \varphi\{n-1, n\}\bigr)$ is the sequence of vertices of a non-closed trail of length $n - 2$ in the complete graph on vertices $[1, k]$:

$$\begin{aligned} &\mathrm{Im}_1(\varphi) = \varphi\{1,2\} \ne \varphi\{n-1, n\} = \mathrm{Im}_n(\varphi)\,, \\ &\mathrm{Im}_i(\varphi) = \bigl\{\varphi\{i-1, i\}, \varphi\{i, i+1\}\bigr\} \ne \bigl\{\varphi\{j-1, j\}, \varphi\{j, j+1\}\bigr\} = \mathrm{Im}_j(\varphi) \end{aligned}$$

whenever $i, j \in [2, n{-}1]$, $i \ne j$. Conversely, if the complete graph K_k has a non-closed trail of length $n - 2$, it can be used to find a map in $\mathrm{Obs}_k(P_n)$.

Let I_k be the set of lengths of non-closed trails in K_k; evidently, $I_k \subseteq \left[1, \binom{k}{2}\right]$. Suppose first that k is odd. K_k has a closed Eulerian trail T_k, and all its trails of length $\binom{k}{2}$ are closed, hence $I_k \subseteq \left[1, \binom{k}{2} - 1\right]$. Denote as $T_k(i, j)$ the subtrail of T_k beginning in ith and ending in jth edge of this trail. If $T_k(1, l)$ is non-closed for some $l \in \left[1, \binom{k}{2} - 1\right]$, then directly $l \in I_k$. On the other hand, if $T_k(1, l)$ is closed, then necessarily $T_k(2, l + 1)$ is non-closed and $l \in I_k$ again. This shows that $I_k = \left[1, \binom{k}{2} - 1\right]$ for odd k.

For even k we have $I_k = \left[1, \frac{k(k-2)}{2} + 1\right]$:

Impossibility of lengths from $\left[\frac{k(k-2)}{2} + 2, \binom{k}{2}\right]$ follows from the fact that any subgraph G of K_k with at least $\frac{k(k-2)}{2} + 2$ edges has more than two vertices of degree $k-1$ – the opposite assumption yields

$$|E(G)| = \frac{1}{2}\sum_{i=1}^{k-1} i v_i(G) \le \frac{1}{2}\left[2(k-1) + (k-2)(k-2)\right] = \frac{k(k-2)}{2} + 1\,;$$

thus, clearly, G does not have an Eulerian trail.

$I_2 = \{1\}$ is trivial. For even $k \in [4, \infty)$ consider a factor F of K_k consisting of two components K_1 and $\frac{k-2}{2}$ components K_2. The connected graph $K_k - E(F)$ has $\frac{k(k-2)}{2} + 1$ edges and exactly two vertices of odd degree, hence it has a non-closed Eulerian trail T connecting these two vertices, and the trail induced by first $l \in \left[1, \frac{k(k-2)}{2} + 1\right]$ edges of T shows $l \in I_k$.

If $\mathrm{Obs}_k(P_n) \neq \emptyset$ with k odd, then $n - 2 \le \binom{k}{2} - 1$. The resulting quadratic inequality in k is for $k \in [3, \infty)$ equivalent to

$$k \ge \frac{\sqrt{8n-7}+1}{2}\,, \quad \frac{k-1}{2} \ge \left\lceil \frac{\sqrt{8n-7}-1}{4} \right\rceil \quad \text{and} \quad k \ge 2\left\lceil \frac{\sqrt{8n-7}-1}{4} \right\rceil + 1\,.$$

From the assumption $\mathrm{Obs}_k(P_n) \neq \emptyset$ with k even, we have $n - 2 \le \frac{k(k-2)}{2} + 1$, which is for $k \in [2, \infty)$ equivalent to inequalities

$$k \ge \sqrt{2n-5} + 1\,, \quad \frac{k}{2} \ge \left\lceil \frac{\sqrt{2n-5}+1}{2} \right\rceil \quad \text{and} \quad k \ge 2\left\lceil \frac{\sqrt{2n-5}+1}{2} \right\rceil.$$

Combining both systems of inequalities we get

$$\begin{aligned} \mathrm{obs}(P_n) &= \min\{k \in [3, \infty) :\ n \in I_k\} \\ &= \min\left\{2\left\lceil \frac{\sqrt{8n-7}-1}{4} \right\rceil + 1,\ 2\left\lceil \frac{\sqrt{2n-5}+1}{2} \right\rceil\right\}. \end{aligned}$$

□

2.3. THEOREM. *If $k, n \in [3, \infty)$, then* $\mathrm{obs}(C_n) = k$ *if and only if either*

(1) *k is odd and* $n \in \left[\frac{k^2-4k+5}{2}, \frac{k^2-k-6}{2}\right] \cup \left\{\frac{k^2-k}{2}\right\}$,

or

(2) *k is even and* $n \in \left[\frac{k^2-3k-2}{2}, \frac{k^2-3k}{2}\right] \cup \left[\frac{k^2-3k+4}{2}, \frac{k^2-2k}{2}\right]$.

P r o o f . Analogously as in the proof of Theorem 2.2 our task is to examine the structure of the set $J_k \subseteq \left[3, \binom{k}{2}\right]$ of all lengths of closed trails in K_k – maps from $\mathrm{Obs}_k(C_n)$ are in $1-1$ correspondence with closed trails of length n in K_k.

1. For odd $k \in [3, \infty)$ we shall show $J_k = \left[3, \binom{k}{2}-3\right] \cup \left\{\binom{k}{2}\right\}$. Clearly, $\binom{k}{2} - i \notin J_k$ for $i = 1, 2$, since deleting i edges from K_k leads to a graph with at least two vertices of odd degree which is non-Eulerian.

For the rest proceed by induction on k. $J_3 = \{3\}$ being trivial, suppose $J_k = \left[3, \binom{k}{2}-3\right] \cup \left\{\binom{k}{2}\right\}$ for some $k \in [3, \infty)$. Let $V(K_{k+2}) = [1, k+2]$, choose a closed trail T of length $\binom{k}{2}$ in the complete subgraph of K_{k+2} on vertices $[3, k+2]$, delete from it the edge $\{3, 4\}$ and close the obtained trail T' by another trail T'' determined by its sequence of vertices:

$$(3,1,4), \qquad (3,1,2,4), \qquad (3,1,5,2,4), \qquad (3,1,2,5,1,4).$$

Thus we have $\binom{k}{2} + i \in J_{k+2}$, $i = 1, 2, 3, 4$.

Closed trails of length 4 induced by the sequence $(1, 2j, 2, 2j+1, 1)$ for $j \in \left[3, \frac{k+1}{2}\right]$ can be attached to the trails above to prove $\left[\binom{k}{2}+5, \binom{k+2}{2}-3\right] \subseteq J_{k+2}$.

For $k \in [5, \infty)$ a closed trail of length $\binom{k}{2} - 3$ on vertices $[3, k+2]$ serves similarly as a base for $\left[\binom{k}{2}-2, \binom{k}{2}-1\right] \subseteq J_{k+2}$.

Finally, $\binom{k+2}{2} \in J_{k+2}$ since K_{k+2} is an Eulerian graph.

2. For even $k \in [4, \infty)$ we are going to prove $J_k = \left[3, \frac{k(k-2)}{2}\right]$:

A subgraph of K_k with at least $\frac{k^2-2k+2}{2}$ edges has its average degree at least $k - 2 + \frac{2}{k}$, consequently it must contain vertices of odd degree $k-1$ and cannot be Eulerian.

The rest is easy for $k = 4$, so we can suppose $k \in [6, \infty)$. With respect to the first part of the proof, $J_{k-1} = \left[3, \binom{k-1}{2}-3\right] \cup \left\{\binom{k-1}{2}\right\}$, and it suffices to show that $\left[\binom{k-1}{2}-2, \binom{k-1}{2}-1\right]$ and $\left[\binom{k-1}{2}+1, \frac{k(k-2)}{2}\right]$ are subsets of J_k.

Denote by M the perfect matching $\left\{\{2i-1, 2i\} : i \in \left[1, \frac{k}{2}\right]\right\}$ of K_k. Then the graph $K_k - M$ has a closed Eulerian trail. The same is true for $F_l = K_k - (M \cup E(G_l))$, where G_l is the cycle of length $l \in \left[3, \frac{k}{2}+1\right]$ with $E(G_l) \cap M = \emptyset$ on vertices $1, 4, 5$ (for $l = 3$) or successively $1, 3, \ldots, 2l-3, 4$. (It is easy to see that F_l is connected.) Thus $\binom{k}{2} - \left(\frac{k}{2} + l\right) \in J_k$ for $l \in \{0\} \cup \left[3, \frac{k}{2}+1\right]$.

The graphs induced by the edges of $M - \{\{1,2\}\} \cup \{\{1,3\}, \{2,3\}\}$ or $M - \{\{1,2\}\} \cup \{\{1,3\}, \{2,5\}, \{3,5\}\}$ have all their vertices of odd degree, their complements in K_k are connected graphs with only vertices of even degree, and we can claim $\binom{k}{2} - \left[\left(\frac{k}{2} - 1\right) + l\right] \in J_k$ for $l = 2, 3$.

The proof of our theorem now follows from the structure of the sets J_k, $k \in [3, \infty)$. □

The wheel on 4 vertices is isomorphic to K_4, hence $\operatorname{obs}(W_4) = 5$ by Theorem 2.1.

2.4. THEOREM. *If* $n \in [5, \infty)$, *then* $\operatorname{obs}(W_n) = n - 1$.

P r o o f. From Corollary 1.2, it is evident that $\operatorname{obs}(W_n) \geq n - 1$. W_n has a vertex of degree $n - 1$. On the other hand, if

$$V(W_n) = [1, n], \qquad E(W_n) = \left\{\{i, n\}, \{i, (i+1)_{n-1}\} : i \in [1, n-1]\right\},$$

then the map φ defined by

$$\begin{aligned} \varphi\{i, n\} &= i, \\ \varphi\{i, (i+1)_{n-1}\} &= (i+2)_{n-1} \qquad \text{for } i \in [1, n-1] \end{aligned}$$

belongs to $\operatorname{Obs}_{n-1}(W_n)$, and we obtain $\operatorname{obs}(W_n) = n - 1$. □

We end this paragraph with turning our attention to complete bipartite graphs $K_{m,n}$. For the graph $K_{1,1}$ isomorphic to K_2 we have $\operatorname{obs}(K_{1,1}) = \infty$.

2.5. THEOREM. *If* $n \in [2, \infty)$ *and* $m \in [2, n-1]$, *then* $\operatorname{obs}(K_{1,n}) = n$, $\operatorname{obs}(K_{m,n}) = n + 1$ *and* $\operatorname{obs}(K_{n,n}) = n + 2$.

Proof.

1. The star $K_{1,n}$ on $n+1$ vertices has observability at least n by Corollary 1.2, and the result follows since each proper n-edge-colouring of $K_{1,n}$ distinguishes vertices of $K_{1,n}$ by its induced colour sets.

2. As the graph $K_{m,n}$ has $m \in [2, n-1]$ vertices of degree n, Corollary 1.2 yields $\operatorname{obs}(K_{m,n}) \geq n+1$. The desired result can be obtained provided

$$V(K_{m,n}) = [1, m+n], \qquad E(K_{m,n}) = \{\{i,j\} : i \in [1,m],\ j \in [m+1, m+n]\},$$

by considering the map $\varphi \in \mathrm{Obs}_{n+1}(K_{m,n})$ defined by

$$\varphi\{i,j\} = (i+j)_{n+1} \qquad \text{for} \quad i \in [1,m] \text{ and } j \in [m+1, m+n].$$

3. Since

$$v_n(K_{n,n}) = 2n > n+1 = \binom{n+1}{n},$$

$\operatorname{obs}(K_{n,n}) \geq n+2$ by Corollary 1.2. A map $\psi \in \mathrm{Obs}_{n+2}(K_{n,n})$ can be defined by

$$\psi\{i,j\} = (i+j)_{n+2},$$
$$\psi\{n,j\} = (j-1)_{n+2} \qquad \text{for} \quad i \in [1, n-1],\ j \in [n+1, 2n].$$

□

Note that, according to Horňák and Soták [7], $\operatorname{obs}(K(p\times q)) = (p-1)q+2$ for $p \in [3, \infty)$ and $q \in [2, \infty)$, where $K(p \times q)$ is the complete p-partite graph with all parts of cardinality q.

3. Fully observable graphs

When searching for observability of a d-regular graph G with n vertices, according to Corollary 1.2, the first candidate is the minimum k such that $\binom{k}{d} \geq n$. It is quite natural to expect problems in an attempt to find a map in $\mathrm{Obs}_k(G)$ if the difference $\binom{k}{d} - n$ is small – it represents a "degree of freedom" for such a map. G is fully observable just if its observability corresponds to a map with "degree of freedom" equal to 0. An example of such a graph for $d = 3$, $n = 20$ and $k = 6$ is the graph D of dodecahedron – see Fig. 1, where values of a map from $\mathrm{Obs}_6(D)$ are indicated.

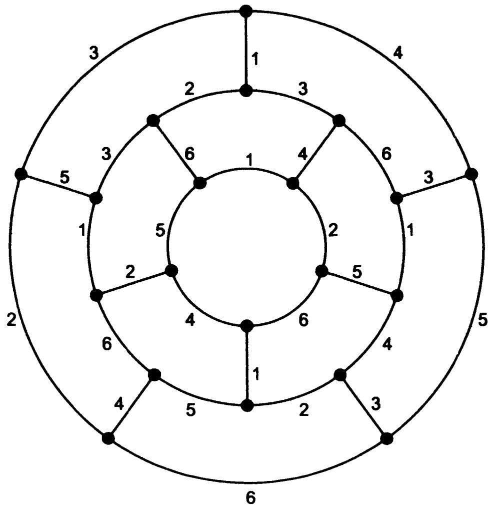

Figure 1.

An immediate question arises: For which pairs (k, d) does there exist a d-regular graph G with $\binom{k}{d}$ vertices and $\operatorname{obs}(G) = k$? Besides the trivial case $k = d = 0$ covered by K_1, such a pair requires $k \in [3, \infty)$ and $d \in [2, k-1]$. The complete answer depends only on the parity of $\binom{k-1}{d-1}$.

3.1. THEOREM. *If $k \in [3, \infty)$ and $d \in [2, k-1]$, then there exists a fully observable d-regular graph with observability k if and only if $\binom{k-1}{d-1} \equiv 0 \pmod 2$.*

Proof.

1. Let G be a d-regular graph with $|V(G)| = \binom{k}{d}$ and $\operatorname{obs}(G) = k$. For $\varphi \in \operatorname{Obs}_k(G)$ and $i \in [1, k]$ the number of colour classes containing i is $\binom{k-1}{d-1}$; on the other hand, it is equal to $2|\varphi^{-1}(i)|$, hence $\binom{k-1}{d-1}$ must be even.

2. Now suppose $\binom{k-1}{d-1} \equiv 0 \pmod 2$.

(a) If $d = 2$ and k is odd, then, by Theorem 2.3, the cycle $C_{\binom{k}{2}}$ is a fully observable graph with the pair of parameters $(k, 2)$.

(b) For $d = k - 1 \equiv 0 \pmod 2$ use the complete graph K_k and the statement of Theorem 2.1 showing admissibility of the pair $(k, k-1)$.

(c) Finally we shall treat the situation $k \in [5, \infty)$, $d \in [3, k-2]$. Denote by G the complete graph on the vertex set U of all d-element subsets of $[1, k]$. Our task will be done by finding k edge-disjoint matchings $M_1, \dots, M_k$ in G such that for each $i \in [1, k]$ the vertices of M_i are exactly those containing i then $H = \big(U, M^{(k)}\big)$, where

$$M^{(j)} = \bigcup_{i=1}^{j} M_i \qquad \text{for} \quad j \in [0, k]$$

is a d-regular subgraph of G, the proper k-edge-colouring φ of H defined by $\varphi^{-1}(i) = M_i$ for every $i \in [1, k]$ fulfils $\mathrm{Im}_u(\varphi) = u$ for all $u \in U$, hence it belongs to $\mathrm{Obs}_k(H)$ and H is fully observable.

Take $i \in [1, k]$ and suppose M_j is determined for each $j \in [1, i-1]$. Let U_i be the set of all vertices of U containing i, and consider the graph $G_i = G\langle U_i\rangle - M^{(i-1)}$ created from the graph induced in G on vertices of U_i by omitting edges of all up to now constructed matchings. This graph has $\binom{k-1}{d-1}$ vertices, which is, due to $2 \le d-1 \le k-3$, not less than $\binom{k-1}{k-3} = \binom{k-1}{2} = \dfrac{(k-1)(k-2)}{2} \ge \dfrac{4d}{2} = 2d$. The number of edges of $M^{(i-1)}$ incident with a vertex $u \in U_i$ is

$$|u \cap [1, i-1]| \le d - 1\,,$$

hence the neighbourhood N_u of u in the graph G_i has at least $2d-1-(d-1) = d$ vertices.

If M_i' is a maximum matching of G_i, it can be shown that M_i' is perfect. Assume it is not; then it omits at least two vertices x, y of U_i (remember that $|U_i|$ is even). Evidently, each neighbour of x in G_i belongs to an edge of M_i'. Thus, if $\mu(u)$ is the counterpart of $u \in N_x$ with respect to M_i', then $|\mu(N_x)| = |N_x| \ge d$. As N_y omits at most $d-1$ vertices of $U_i - \{y\}$, it necessarily meets $\mu(N_x)$, and there exists $u \in N_x$ such that $\mu(u) \in N_y$. But in such a case the matching $M_i' - \{u, \mu(u)\} \cup \big\{\{x, u\}, \{y, \mu(u)\}\big\}$ contradicts the maximality of M_i'. The perfect matching M_i' can be used in the role of M_i, and the proof follows. □

Till now our effort to construct a non-fully observable graph fulfilling nevertheless all known necessary conditions to be fully observable has not been successful (it could correspond only to the case (c) in the proof of Theorem 3.1). This leads us to the following hypothesis.

3.2. CONJECTURE. *If $k \in [5, \infty)$, $d \in [3, k-2]$ and $\binom{k-1}{d-1} \equiv 0 \pmod 2$, then every d-regular graph on $\binom{k}{d}$ vertices is fully observable.*

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* *Faculty of Management*
University of South Bohemia
Jarošovská 1117/II
CZ–377 01 Jindřichův Hradec
CZECH REPUBLIC
E-mail: cerny_j@jh.jcu.cz

** *Department of Geometry and Algebra*
Faculty of Science
Šafárik University
Jesenná 5
SK–041 54 Košice
SLOVAKIA
E-mail: hornak@turing.upjs.sk
sotak@kosice.upjs.sk