Mathematica Slovaca

Péter Kiss

Pure powers and power classes in recurrence sequences

Mathematica Slovaca, Vol. 44 (1994), No. 5, 525--529

Persistent URL: http://dml.cz/dmlcz/136626

Math. Slovaca, **44** (1994), No. 5, 525–529

Mathematica Slovaca
©1994
Mathematical Institute
Slovak Academy of Sciences

In memory of Professor Štefan Znám

PURE POWERS AND POWER CLASSES IN RECURRENCE SEQUENCES

PÉTER KISS[1]

(*Communicated by Stanislav Jakubec*)

ABSTRACT. Let G be a linear recursive sequence of order k satisfying the recursion $G_n = A_1 G_{n-1} + \cdots + A_k G_{n-k}$. In case $k = 2$ it is known that there are only finitely many perfect powers in such a sequence. Ribenboim and McDaniel proved for sequences with $k = 2$, $G_0 = 0$ and $G_1 = 1$ that in general for a term G_n there are only finitely many terms G_m such that $G_m G_n = x^2$ for some integer x. In the general case, with some restrictions, we show that for any n there exists a number q_0, depending on G and n, such that the equation $G_n G_x = w^q$ in integers x, w, q has no solution with $x > n$ and $q > q_0$.

Let $R = R(A, B, R_0, R_1)$ be a second order linear recursive sequence defined by

$$R_n = AR_{n-1} + BR_{n-2} \qquad (n > 1),$$

where A, B, R_0 and R_1 are fixed rational integers. In the sequel we assume that the sequence is not a degenerate one, i.e. α/β is not a root of unity, where α and β denote the roots of the polynomial $x^2 - Ax - B$.

The special cases $R(1,1,0,1)$ and $R(2,1,0,1)$ of the sequence R are called the *Fibonacci* and the *Pell sequence*, respectively.

The squares and other pure powers in sequences R were investigated by many authors. For the Fibonacci sequence Cohn [2] and Wylie [22] showed that a *Fibonacci number* F_n is a square only when $n = 0, 1, 2$, or 12. Pethő [11], London and Finkelstein [8], [9] proved that F_n is a full cube

AMS Subject Classification (1991): Primary 11B37.

Key words: Recursive sequences, Perfect powers in sequences.

[1] Research (partially) supported by Hungarian National Foundation for Scientific Research, grant No. 1641.

only if $n = 0, 1, 2$, or 6. From a result of Ljunggren [7] it follows that a *Pell number* is a square only if $n = 0, 1$, or 7, and Pethő [12] showed that these are the only perfect powers in the Pell sequence. Similar, but more general results were shown by McDaniel and Ribenboim [10], Robbins [18], [19] Cohn [3], [4], [5], and Pethő [14]. A general result was obtained by Shorey and Stewart [20]:

Any non degenerate binary recurrence sequence contains only finitely many pure powers which can be effectively determined.

This result also follows from a result of Pethő [13].

Another type of problems was studied by Ribenboim and McDaniel. For a sequence R we say that the terms R_m, R_n are in the same *square-class* if there exists a non zero integer x such that

$$R_m R_n = x^2 .$$

A square-class is called *trivial* if it contains only one element.

Ribenboim [15] proved that in the Fibonacci sequence the square-class of a Fibonacci number F_m is trivial, i.e. the equation

$$F_m F_y = x^2$$

has no solution in non-zero integers x and $y \neq m$, if $m \neq 1, 2, 3, 6$, or 12 and for the *Lucas sequence* $L(1, 1, 2, 1)$ the square-class of a Lucas number L_m is trivial if $m \neq 0, 1, 3$ or 6. For more general sequences $R(A, B, 0, 1)$, with $(A, B) = 1$, Ribenboim and McDaniel [16] obtained that each square-class is finite and its elements can be effectively computable (see also Ribenboim [17]).

For general recursive sequences of order larger than two we have fewer results.

Let $G = G(A_1, \dots, A_k, G_0, \dots, G_{k-1})$ be a kth order linear recursive sequence of rational integers defined by

$$G_n = A_1 G_{n-1} + A_2 G_{n-2} + \dots + A_k G_{n-k} \qquad (n > k - 1),$$

where $A_1, \dots, A_k$ and $G_0, \dots, G_{k-1}$ are not all zero integers. Denote by $\alpha = \alpha_1, \alpha_2, \dots, \alpha_s$ the distinct zeros of the polynomial $x^k - A_1 x^{k-1} - A_2 x^{k-2} - \dots - A_k$. Assume that $\alpha, \alpha_2, \dots, \alpha_s$ has multiplicity $1, m_2, \dots, m_s$ respectively, and $|\alpha| > |\alpha_i|$ for $i = 2, \dots, s$. In this case, as it is known, the terms of the sequence can be written in the form

$$G_n = a\alpha^n + r_2(n)\alpha_2^n + \dots + r_s(n)\alpha_s^n \qquad (n \geq 0), \tag{1}$$

where r_i $(i = 2, \ldots, s)$ are polynomials of degree $m_i - 1$ and the coefficients of the polynomials and a are elements of the algebraic number field $\mathbf{Q}(\alpha, \alpha_2, \ldots, \alpha_s)$. Under some natural conditions Shorey and Stewart [20] proved that the sequence G does not contain qth powers if q is large enough. This result follows also from [6] and [21], where more general theorems are presented.

The purpose of this note is to show a result, similar to those mentioned above, for general sequences.

Theorem. *Let G be a kth order linear recursive sequence satisfying the above conditions. Assume that $a \neq 0$ and $G_i \neq a\alpha^i$ for $i > n_0$. Then for any integer n, with $G_n \neq 0$, there exists a number q_0, depending only on n and the sequence, such that the equation*

$$G_n G_x = w^q \tag{2}$$

in positive integers x, w, q has no solution with $x > n$ and $q > q_0$.

For the proof of our theorem we need a result due to Baker [1].

Lemma. *Let $\gamma_1, \ldots, \gamma_v$ be non-zero algebraic numbers. Let $M_1, \ldots, M_v$ be upper bounds for the heights of $\gamma_1, \ldots, \gamma_v$, respectively. We assume that M_v is at least 4. Further let $b_1, \ldots, b_{v-1}$ be rational integers with absolute values at most B and let b_v be a non-zero rational integer with absolute value at most B'. We assume that B' is at least three. Let L be defined by*

$$L = b_1 \log \gamma_1 + \cdots + b_v \log \gamma_v ,$$

where the logarithms are assumed to have their principal values. If $L \neq 0$, then

$$|L| > \exp\bigl(-C(\log B' \log M_v + B/B')\bigr) ,$$

where C is an effectively computable positive number depending only on the numbers $M_1, \ldots, M_{v-1}$, $\gamma_1, \ldots, \gamma_v$, and v (see [1; Theorem 1] with $\delta = 1/B'$).

Proof of the theorem. We can suppose that $n > n_0$ and n is sufficiently large since by [20] or [6] it follows that for any given d the equation

$$dG_x = w^q$$

implies that $q < q_0$. We can also assume, without loss of generality, that the terms of the sequence G are positive.

Let x, w and q be integers satisfying (2). Then by (1)

$$w^q = a\alpha^x\left(1 + r_2(x)\frac{1}{a}\left(\frac{\alpha_2}{\alpha}\right)^x + \dots\right)G_n\,, \tag{3}$$

and so

$$c_1\frac{x}{q} < \log w < c_2\frac{x}{q} \tag{4}$$

follows with some $c_1, c_2 > 0$, which depend on the sequence G, since $r_2(x)(\alpha_2/\alpha)^x \to 0$ as $x \to \infty$ and $\log G_n \approx n\log|\alpha| + \log|a| < c_3 x$. Using that $x > n_0$ and the properties of the logarithm function by (3), with some $c_4 > 0$, we have

$$L = \left|\log \frac{w^q}{G_n a\alpha^x}\right| < \mathrm{e}^{-c_4 x}\ . \tag{5}$$

On the other hand, by Lemma with $v = 4$, $M_4 = w$ and $B' = q$, we obtain the estimate

$$L = |q\log w - \log G_n - \log a - x\log\alpha| > \mathrm{e}^{-C(\log q \log w + x/q)}\,, \tag{6}$$

where $C > 0$ depends on n. By (5) and (6), using (4) we obtain

$$c_4 x < C(\log q \log w + c_5 \log w) < c_6 \log q \log w\,,$$

from which

$$x < c_7 \log q \log w \tag{7}$$

follows with some $c_5, c_6, c_7 > 0$. By (4) and (7), it follows that

$$q\log w < c_2 x < c_8 \log q \log w\,,$$

and so

$$q < c_8 \log q\,,$$

which is impossible if $q > q_0 = q_0(n)$.

This contradiction proves our theorem.

REFERENCES

[1] BAKER, A.: *A sharpening of the bounds for linear forms in logarithms II*, Acta Arith. **24** (1973), 33–36.

[2] COHN, J. H. E.: *On square Fibonacci numbers*, J. London Math. Soc. **39** (1964), 537–540.

[3] COHN, J. H. E.: *Squares in some recurrent sequences*, Pacific J. Math. **41** (1972), 631–646.

[4] COHN, J. H. E.: *Eight Diophantine equations*, Proc. London Math. Soc. **16** (1966), 153–166.

[5] COHN, J. H. E.: *Five Diophatine equations*, Math. Scand. **21** (1967), 61–70.

[6] KISS, P.: *Differences of the terms of linear recurrences*, Studia Sci. Math. Hungar. **20** (1985), 285–293.

[7] LJUNGGREN, W.: *Zur Theorie der Gleichung* $x^2+1=Dy^4$, Avh. Norske Vid Akad. Oslo. **5** (1942).

[8] LONDON, J.—FINKELSTEIN, R.: *On Fibonacçi and Lucas numbers which are perfect powers*, Fibonacci Quart. **7** (1969), 476–481, 487 (Errata ibid **8** (1970), 248).

[9] LONDON, J.—FINKELSTEIN, R.: *On Mordell's Equation* $y^2-k=x^3$, Bowling Green University Press, 1973.

[10] McDANIEL, W. L.—RIBENBOIM, P.: *Squares and double-squares in Lucas sequences*, C.R. Math. Rep. Acad. Sci. Canada. **14** (1992), 104–108.

[11] PETHŐ, A.: *Full cubes in the Fibonacci sequence*, Publ. Math. Debrecen. **30** (1983), 117–127.

[12] PETHŐ, A.: *The Pell sequence contains only trivial perfect powers.* In: Sets, Graphs and Numbers. Colloq. Math. Soc. János Bolyai 60, North-Holland, Amsterdam-New York, 1991, pp. 561–568.

[13] PETHŐ, A.: *Perfect powers in second order linear recurrences*, J. Number Theory. **15** (1982), 5–13.

[14] PETHŐ, A.: *Perfect powers in second order recurrences.* In: Topics in Classical Number Theory, Akadémiai Kiadó, Budapest, 1981, pp. 1217–1227.

[15] RIBENBOIM, P.: *Square classes of Fibonacci and Lucas numbers*, Portugal. Math. **46** (1989), 159–175.

[16] RIBENBOIM, P.—McDANIEL, W. L.: *Square classes of Fibonacci and Lucas sequences*, Portugal. Math. **48** (1991), 469–473.

[17] RIBENBOIM, P.: *Square classes of* $(a^n-1)/(a-1)$ *and* a^n+1, Sichuan Daxue Xunebar. **26** (1989), 196–199.

[18] ROBBINS, N.: *On Fibonacci numbers of the form* px^2*, where* p *is prime*, Fibonacci Quart. **21** (1983), 266–271.

[19] ROBBINS, N.: *On Pell numbers of the form* PX^2*, where* P *is prime*, Fibonacci Quart. **22** (1984), 340–348.

[20] SHOREY, T. N.—STEWART, C. L.: *On the Diophantine equation* $ax^{2t}+bx^ty+cy^2=d$ *and pure powers in recurrence sequences*, Math. Scand. **52** (1983), 24–36.

[21] SHOREY, T. N.—STEWART, C. L.: *Pure powers in recurrence sequences and some related Diophatine equations*, J. Number Theory **27** (1987), 324–352.

[22] WYLIE, O.: *In the Fibonacci series* $F_1=1$*,* $F_2=1$*,* $F_{n+1}=F_n+F_{n-1}$ *the first, second and twelfth terms are squares*, Amer. Math. Monthly **71** (1964), 220–222.

Received January 17, 1994

Eszterházy K. Teacher's Training College
Department of Mathematics
Leányka u. 4
H-3301 Eger
Hungary