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KOROVKIN THEORY IN BANACH * ALGEBRAS

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ABSTRACT. In this paper the author investigates the Korovkin closures in a class of noncommutative Banach-*-algebras. The universal Korovkin closure of a *-subalgebra with respect to Schwarz maps is nothing but its closure with respect to the norm of the enveloping C^*-algebra in the case of liminal algebras. This yields equivalent conditions for such an algebra to possess a finite universal Korovkin system.

1. Introduction

Let $\mathcal{A}$ be a Banach-*-algebra (i.e. a complex Banach algebra with an isometric involution), and $T \subset \mathcal{A}$ a non-empty subset. Let $\mathcal{P}$ be a class of positive operators $\mathcal{A} \to \mathcal{C}$, where $\mathcal{C}$ is a C*-algebra. Then let us define

$\mathrm{Kor}^u_{\mathcal{A},\mathcal{P}}(T) = \{x \in \mathcal{A} \mid$ if $(P_i)_i$ is a net of operators $P_i\colon \mathcal{A} \to \mathcal{C}$ in $\mathcal{P}$, where $\mathcal{C}$ is a C*-algebra, and if $S\colon \mathcal{A} \to \mathcal{C}$ is a *-homomorphism such that $\|P_i y - Sy\| \to 0$ for all $y \in T$, then $\|P_i x - Sx\| \to 0\}$.

An interesting case is $\mathrm{Kor}^u_{\mathcal{A},\mathcal{P}}(T) = \mathcal{A}$ for then to prove convergence $P_i x \to Sx$ for all $x \in \mathcal{A}$ it suffices to show $P_i y \to Sy$ for all y in the test set T. $\mathcal{A}$ is said to have a finite universal Korovkin system if there is a finite subset $T \subset \mathcal{A}$ such that its universal Korovkin-closure $\mathrm{Kor}^u_{\mathcal{A},\mathcal{P}}(T)$ coincides with $\mathcal{A}$.

In [Bec1] it has been shown that $\mathrm{Kor}^u_{\mathcal{A},\mathcal{P}}(B) = B$ for all J*-subalgebras B of a dual C*-algebra $\mathcal{A}$, where $\mathcal{P}$ stands for the class of all positive operators which are norm bounded by 1 (in this case the symbol $\mathcal{P}$ will be left out). The same result also holds when $\mathcal{P}$ is the class $\mathcal{S}$ of Schwarz-maps, i.e. continuous and linear maps $P\colon \mathcal{A} \to \mathcal{C}$ which satisfy $P(x)^*P(x) \le P(x^*x)$ and $P(x)^* = P(x^*)$ for all $x \in \mathcal{A}$ and B is a C*-subalgebra. This result will be extended to type I C*-algebras in the second and third paragraph, and to a more general class of Banach-*-algebras in the fourth paragraph. It will be convenient to treat the

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case of liminal and type I C*-algebras differently since in the liminal case we will get some further information about the surjective Korovkin closure which will be explained now.

Define the surjective Korovkin closure $\mathrm{Kor}^s_{\mathcal{A},\mathcal{P}}(T)$ by considering only surjective *-homomorphisms in the above definition. In the same way we may define the dense Korovkin closure $\mathrm{Kor}^d_{\mathcal{A},\mathcal{P}}(T)$ by restricting attention to *-homomorphisms having dense image (this is of course the same as the surjective Korovkin closure if $\mathcal{A}$ is a C*-algebra). If we only approximate the *-homomorphism $S = \mathrm{id}_{\mathcal{A}}$ in which case $\mathcal{A}$ necessarily must be a C*-algebra, we write $\mathrm{Kor}_{\mathcal{A},\mathcal{P}}(T)$. It is not clear whether this coincides with the universal Korovkin closures defined above, but of course it contains them.

It may be extracted from [Rob] or taken from [Bec2] that

$$\mathrm{C}^*(T) \subset \mathrm{Kor}^u_{\mathcal{A},\mathcal{S}}\left(T \cup \{t^*t, tt^* \mid t \in T\}\right),$$

where $\mathrm{C}^*(T)$ is the C*-subalgebra generated by T. We will achieve equality for type I C*-algebras in Corollary 3.5, this partially answers a question posed in [Alt]. The analog

$$\mathrm{J}^*(T) \subset \mathrm{Kor}^u_{\mathcal{A}}\left(T \cup \{t^* \circ t \mid t \in T\}\right)$$

also holds (here $\mathrm{J}^*(T)$ is the closed Jordan-*-algebra generated by T and $\circ$ is the Jordan product). This can be proved along the lines of [Pr] and [LN] or taken from [Bec2].

Those algebras, which possess a finite universal Korovkin system will be characterized as expected, i.e. they should be finitely generated in some sense. When these results are applied to the case of a commutative Banach-*-algebra, some known results will follow quite easily. This is demonstrated among other things in the last paragraph.

2. Liminal C*-algebras

Let $\mathrm{P}(\mathcal{A})$ be the set of pure states of a C*-algebra $\mathcal{A}$, and let $\tau = \sigma(\mathcal{A}, \mathrm{span}\,\mathrm{P}(\mathcal{A}))$ be the weak topology which is induced by the pure states on $\mathcal{A}$. Then τ is a locally convex Hausdorff topology and the involution * is τ-continuous (since $f(x^*) = \overline{f(x)}$ for all states f). Let us prove that the multiplication is separately continuous: For this consider $x_i \to x$ and $y \in \mathcal{A}_1$, where $\mathcal{A}_1$ is the C^*-algebra obtained from $\mathcal{A}$ by adjoining a unit. If $f \in \mathrm{P}(\mathcal{A})$, then the GNS-construction shows that $f(y^* \cdot y) \in \mathbb{R}^+ \cdot \mathrm{P}(\mathcal{A}) \subset \mathrm{span}\,\mathrm{P}(\mathcal{A})$, and so $y^*x_iy \to y^*xy(\tau)$. Now the formula $yx_i = \frac{1}{4}\sum_{k=0}^{3} i^k(y+i^k)^*x_i(y+i^k)$ proves the claim. Conclusion: If $B \subset \mathcal{A}$ is a J*-subalgebra, then so is $\overline{B}^\tau$.

Let $\{\rho_i \mid i \in I\}$ be a complete system of representatives of unitary equivalence classes of irreducible representations of $\mathcal{A}$. If J is a subset of I, then define $\pi_J := \bigoplus_{j\in J} \rho_j$. Then $\pi_a := \pi_I$ is called the atomic representation of $\mathcal{A}$ ([Pd, 4.3.7]). All the π_J may be considered as subrepresentations of π_a in the obvious way, in particular they all act on the atomic Hilbert space $H_a := \bigoplus_{i\in I} H_i$. So we identify $L(H_i)$ with a subspace of $L(H_a)$. Note that the weak operator topology (WOT) of $L(H_i)$ coincides with the WOT of $L(H_a)$ restricted to $L(H_i)$. Let $\mathcal{J}$ be the set of finite subsets of I. Then it is a simple matter to show that $\lim_{J\in\mathcal{J}} \pi_J(x) = \pi_a(x)$ for all $x \in \mathcal{A}$ in the strong operator topology.

LEMMA 2.1. *Let $(x_i)_i$ be a net in $\mathcal{A}$ and $x \in \mathcal{A}$. Then the following are equivalent:*

(i) $x_i \to x$ *with respect to* τ.
(ii) $\pi_J(x_i) \to \pi_J(x)$ *in the weak operator topology for all* $J \in \mathcal{J}$.
(iii) $\rho_j(x_i) \to \rho_j(x)$ *in the weak operator topology for all* $j \in J$.

Proof. The equivalence of (ii) and (iii) is trivial. To prove that (iii) follows from (i) observe that we have $\rho_j = \pi_f$ for some pure state f, where π_f denotes the GNS-representation associated to f. Let ξ_f be the corresponding cyclic vector. If then $\xi \in H_f$ there is a $y \in \mathcal{A}$ such that $\pi_f(y)\xi_f = \xi$. Then $\langle \rho_j(x_i)\xi, \xi\rangle = f(y^* x_i y) \to f(y^* x y) = \langle \rho_j(x)\xi, \xi\rangle$, i.e. we have convergence in the weak operator topology. The reverse implication is proved in a similar way using the fact that if f is a pure state, then π_f must be unitarily equivalent to one of the ρ_j's.

COROLLARY 2.2. *If $B \subset \mathcal{A}$ is a* J*-*subalgebra, and if $J \in \mathcal{J}$, then we have* $\pi_J(\overline{B}^\tau) \subset \overline{\pi_J(B)}^{\mathrm{WOT}}$.

COROLLARY 2.3. *If $\pi_a(x_i) \to \pi_a(x)$ with respect to WOT, then $x_i \to x$ with respect to τ.*

LEMMA 2.4. *Let $B \subset \mathcal{A}$ be a* J*-*subalgebra and $x \in \mathcal{A} \setminus \overline{B}^\tau$. Then there is a $J \in \mathcal{J}$ such that $\pi_J(x) \notin \pi_J(\overline{B}^\tau)$.*

Proof. First let us consider the case, where x is selfadjoint. Let us assume that $\pi_J(x) \in \pi_J(\overline{B}^\tau) \subset \overline{\pi_J(B)}^{\mathrm{WOT}}$ for all finite subsets J of I. Let r be a positive number greater than $\|x\|$. Then $\|\pi_J(x)\| < r$ for all $J \in \mathcal{J}$. Since J*-subalgebras are closed with respect to functional calculus of selfadjoint elements, the proof of Kaplansky's density theorem which is given in ([Pd, Th. 2.3.3]) tells us that $\{y \in \pi_J(B) \mid y = y^*, \ \|y\| \le r\}$ is dense in

$\{y \in \overline{\pi_J(B)}^{\text{WOT}} \mid y = y^*,\ \|y\| \le r\}$. Hence, if $\mathcal{U}$ is the set of all convex WOT-neighbourhoods of $0 \in L(H_a)$, then there is a $T_{J,U} = \pi_J(b_{J,U}) \in \pi_J(B)$ such that the norm of $T_{J,U}$ is less than r and $T_{J,U} - \pi_J(x) \in \frac{1}{3}U$, where $U \in \mathcal{U}$. Since $\pi_J(b_{J,U}) \in \pi_J(\mathrm{C}^*(b_{J,U})) \subset \pi_J(B)$, we may assume that the norm of $b_{J,U}$ is less than r.

Now if $\xi = \bigoplus_{i\in I} \xi_i \in H_a$ is given one easily computes $\|(\pi_a(b_{J,U}) - \pi_J(b_{J,U}))\xi\|^2 \le r^2 \sum_{i\in I\setminus J} \|\xi_i\|^2$, and so $\pi_a(b_{J,U}) - \pi_J(b_{J,U}) \to 0$ with respect to WOT for each fixed $U \in \mathcal{U}$.

Now consider the net $(b_{J,U})_{J,U}$, where $\mathcal{J} \times \mathcal{U}$ carries the product order. Then

$$\begin{aligned}\pi_a(b_{J,U}) - \pi_a(x) &= \pi_a(b_{J,U}) - \pi_J(b_{J,U}) + T_{J,U} - \pi_J(x) + \pi_J(x) - \pi_a(x)\\ &\in \tfrac{1}{3}U + \tfrac{1}{3}U + \tfrac{1}{3}U = U\end{aligned}$$

if J is big enough. This proves $\pi_a(b_{J,U}) \to \pi_a(x)$ and by the above corollary we may conclude $b_{J,U} \to x$ with respect to τ.

Now let $x \in \mathcal{A}\setminus\overline{B}^{\tau}$ be arbitrary. If $\pi_J(\overline{B}^{\tau})$ contained $\pi_J(x)$ for all $J \in \mathcal{J}$, then it also would contain the real and imaginary part of these $\pi_J(x)$, and by what we have proved above, the real and imaginary part of x would belong to $\overline{B}^{\tau}$ and so would x itself, a contradiction.

Theorem 2.5. *Let $\mathcal{A}$ be a liminal* C**-algebra, $B \subset \mathcal{A}$ a* J**-subalgebra. Then* $\mathrm{Kor}^u_{\mathcal{A}}(B) \subset \overline{B}^{\tau}$. *If B is a *-subalgebra, then* $\mathrm{Kor}^u_{\mathcal{A},\mathcal{S}}(B) \subset \overline{B}^{\tau}$.

Proof. If $x \notin \overline{B}^{\tau}$, the above lemma gives us a finite subset J of I such that $\pi_J(x) \notin \pi_J(B)$. But $\pi_J(B) \subset \pi_J(\mathcal{A}) \subset \mathcal{C}\big(\bigoplus_{j\in J} H_j\big)$ since $\mathcal{A}$ is liminal (the $\mathcal{C}$ stands for compact operators), and so $\pi_J(\mathcal{A})$ is a dual C*-algebra. Therefore we know (see introduction) $\pi_J(x) \notin \mathrm{Kor}^u_{\pi_J(\mathcal{A})}(\pi_J(B)) \supset \pi_J(\mathrm{Kor}^u_{\mathcal{A}}(B))$, this inclusion is trivial. So this gives us the desired result $x \notin \mathrm{Kor}^u_{\mathcal{A}}(B)$. If B is a *-subalgebra, then the same arguments apply to the universal Korovkin closure with respect to Schwarz-maps.

Corollary 2.6. *Let $\mathcal{A}$ be a liminal* C**-algebra, B a *-subalgebra. Then we have* $\mathrm{Kor}^u_{\mathcal{A}}(B) = \mathrm{Kor}^u_{\mathcal{A},\mathcal{S}}(B) = \overline{B}$ *(norm closure).*

We have $B \subset \mathrm{Kor}^u_{\mathcal{A}}(B) \subset \mathrm{Kor}^u_{\mathcal{A},\mathcal{S}}(B) \subset \overline{B}^{\tau}$ by the theorem above. Since Korovkin-closures clearly are norm closed, we have to show, that $\overline{B} = \overline{B}^{\tau}$.

But this is a simple application of the Stone-Weierstraß theorem for type I C^*-algebras ([Dxm, 11.1.8]). This result will be generalized in the next section.

Remark. Clearly $\mathrm{Kor}^u_{\mathcal{A},\mathcal{P}}(T) \subset \mathrm{Kor}^s_{\mathcal{A},\mathcal{P}}(T)$ and the above proof shows equality if T is a *-subalgebra, since the *-homomorphism which has been used in [Bec1] to show $\pi_J(x) \notin \mathrm{Kor}^u_{\pi_J(\mathcal{A})}\big(\pi_J(B)\big)$ is the identity map on $\pi_J(\mathcal{A})$, which clearly is surjective. Therefore we arrived at the slightly stronger result $\mathrm{Kor}^u_{\mathcal{A}}(B) = \mathrm{Kor}^s_{\mathcal{A}}(B) = \overline{B}$ if B is a *-subalgebra of $\mathcal{A}$.

3. Type I C^*-algebras

Things are pretty much easier if we restrict to unital C^*-algebras and C^*-subalgebras containing this unit.

PROPOSITION 3.1. *Let $\mathcal{A}$ be a unital C^*-algebra and $B \subset \mathcal{A}$ a nuclear C^*-subalgebra containing the unit element of $\mathcal{A}$. Then* $\mathrm{Kor}_{\mathcal{A}}(B) = B$.

Proof. Since B is nuclear, there are $k_n \in \mathbb{N}$ and unital completely positive maps $R_n \colon B \to \mathrm{M}_{k_n}$ and $S_n \colon \mathrm{M}_{k_n} \to B$ such that $S_n \circ R_n$ converges to id_B pointwise in the norm topology, see [L] for a survey on nuclearity. By Arveson's extension theorem for completely positive maps (see [Arv], or [Pl, Th. 6.5]), there are completely positive maps $\overline{R_n} \colon \mathcal{A} \to \mathrm{M}_{k_n}$ which extend R_n. Then $P_n := S_n \circ \overline{R_n}$ is a completely positive map $\mathcal{A} \to B \subset \mathcal{A}$ which is norm bounded by one and obviously $\|P_n x - x\| \to 0$ if and only if $x \in B$, hence the proposition.

In order to apply this proposition we must look for those unital C^*-algebras which only have nuclear C^*-subalgebras. By [Bl] these are exactly the type I C^*-algebras.

COROLLARY 3.2. *Let $\mathcal{A}$ be a unital type I C^*-algebra and B a C^*-subalgebra containing the unit element. Then* $\mathrm{Kor}_{\mathcal{A}}(B) = B$.

In order to get rid of the unit element we prove

LEMMA 3.3. *Let $\mathcal{A}$ be a C^*-algebra, $T \subset \mathcal{A}$. Let $\mathcal{A}_1$ be the C^*-algebra where a unit element has been adjoined. Then* $\mathrm{Kor}^u_{\mathcal{A}}(T) = \mathrm{Kor}^u_{\mathcal{A}_1}(\{1\} \cup T) \cap \mathcal{A}$.

Proof. First let $x \in \mathrm{Kor}^u_{\mathcal{A}}(T)$. Let $(P_i)_i$ be a net of positive linear contractions $\mathcal{A}_1 \to \mathcal{C}$ and $S \colon \mathcal{A}_1 \to \mathcal{C}$ a *-homomorphism such that $P_i y$ converges to Sy for all $y \in \{1\} \cup T$. Then restrict this situation to $\mathcal{A}$ and conclude $P_i x \to Sx$. Thus we have proved $\mathrm{Kor}^u_{\mathcal{A}}(T) \subset \mathrm{Kor}^u_{\mathcal{A}_1}\big(\{1\} \cup T\big)$.

Conversely let us consider $x \in \mathrm{Kor}^u_{\mathcal{A}_1}\big(\{1\} \cup T\big) \cap \mathcal{A}$. Let $(P_i)_i$ be a net of positive linear contractions $\mathcal{A} \to \mathcal{C}$ and S a corresponding *-homomorphism such that $P_i y \to Sy$ for all $y \in T$. P_i and S may be extended to unital maps of the same kind $\overline{P_i}, \overline{S} \colon \mathcal{A}_1 \to \mathcal{C}_1$. Then one easily concludes $x \in \mathrm{Kor}^u_{\mathcal{A}}(T)$.

THEOREM 3.4. *Let $\mathcal{A}$ be a type* I *C*-algebra and $B \subset \mathcal{A}$ a C*-subalgebra. Then we have $\mathrm{Kor}_{\mathcal{A}}(B) = B$.*

This obviously is a consequence of the last lemma and the last corollary.

Remark. In the first part of the proof of the above lemma we had to restrict the *-homomorphism S. This restriction in general is not surjective. So we cannot say anything about the surjective Korovkin closure as we could in the liminal case.

COROLLARY 3.5. *Let $\mathcal{A}$ be a type* I *C*-algebra, $T \subset \mathcal{A}$. Then*

$$\mathrm{Kor}^u_{\mathcal{A},\mathcal{S}}\big(T \cup \{t^*t, tt^* \mid t \in T\}\big) = \mathrm{C}^*(T).$$

One inclusion has been mentioned in the introduction, the other one is a consequence of the above theorem.

Next let us attack the question which type I C*-algebras possess finite universal Korovkin systems.

LEMMA 3.6. *Let T be a Jordan subalgebra of the associative algebra $\mathcal{A}$. Assume $x_1 \ldots x_n \in T$ and $\{x_i x_j \mid i,j = 1 \ldots n\} \subset T$. Then T already contains the algebra which is generated by $\{x_1, \ldots, x_n\}$.*

Proof. Let us prove inductively that T contains all products of length less than or equal to m which may be formed out of $\{x_1, \ldots, x_n\}$. This holds by assumption for $m = 1$ and $m = 2$. Now consider $y_1, \ldots, y_m \in \{x_1, \ldots, x_n\}$, where $m \geq 3$. Then $z := y_2 \ldots y_{m-1} \in T$ by induction hypothesis, and for the same reason $y_1 z, z y_m \in T$. But then T also must contain $y_1 \ldots y_m = y_m \circ y_1 z - y_m y_1 \circ z + y_1 \circ z y_m$, and this finishes the proof.

THEOREM 3.7. *Let $\mathcal{A}$ be a type* I *C*-algebra. Then the following are equivalent:*

(i) *$\mathcal{A}$ possesses a finite universal Korovkin system with respect to all positive contractions.*
(ii) *$\mathcal{A}$ possesses a finite universal Korovkin system with respect to all Schwarz maps.*
(iii) *$\mathcal{A}$ is a finitely generated C*-algebra.*

Proof. The implication (i) $\Longrightarrow$ (ii) is trivial, since $\mathrm{Kor}^u_{\mathcal{A}}(T) \subset \mathrm{Kor}^u_{\mathcal{A},\mathcal{S}}(T)$ always holds.

To prove (ii) $\Longrightarrow$ (iii) let T be a finite universal Korovkin set with respect to Schwarz maps. Then $T_0 := T \cup \{t^*t, tt^* \mid t \in T\}$ is finite and $\mathcal{A} = \mathrm{Kor}^u_{\mathcal{A},\mathcal{S}}(T) \subset \mathrm{Kor}^u_{\mathcal{A},\mathcal{S}}(T_0) = \mathrm{C}^*(T)$ by the above corollary.

Finally (iii) $\Longrightarrow$ (i). If T is a finite set generating $\mathcal{A}$, then we may assume that T consists of selfadjoint elements only. Then $T_1 := T \cup \{t_1t_2 \mid t_1, t_2 \in T\}$ is also finite and $\mathcal{A} = \mathrm{C}^*(T_1) = \mathrm{J}^*(T_1) \subset \mathrm{Kor}^u_{\mathcal{A}}\left(T_1 \cup \{t^* \circ t \mid t \in T_1\}\right)$.

4. Liminal Banach-*-Algebras

In this section let $\mathcal{A}$ be a Banach-*-algebra. If $x \in \mathcal{A}$, then $\|x\|_* := \sup_{\pi \in R} \|\pi(x)\| \leq \|x\|$, where R is the class of all Hilbert space representations of $\mathcal{A}$, see ([Dxm, 1.3.7]). Then $N := \{x \in \mathcal{A} \mid \|x\|_* = 0\}$ is a closed two-sided *-ideal, and $\mathcal{A}/N$ may be completed to a C*-algebra $\overline{\mathcal{A}/N}$, which will be called the enveloping C*-algebra. $\mathcal{A}$ is said to be liminal if and only if its enveloping C*-algebra is.

Let $\mathcal{F}$ be the set of positive functionals on $\mathcal{A}$ which satisfy $f(x^*) = \overline{f(x)}$ and $|f(x)|^2 \leq K_f f(x^*x)$ for all $x \in \mathcal{A}$, where K_f is some constant depending on f.

For every $f \in \mathcal{F}$ we have $f = \langle \pi_f(\cdot)\xi_f, \xi_f \rangle$ by the well-known GNS-construction ([Rick, 4.5.12]), and so $f(N) = 0$, hence f defines a positive linear functional $\tilde{f}$ on $\mathcal{A}/N$ which can be extended to a positive linear functional $\overline{f}$ on $\overline{\mathcal{A}/N}$. In the same way we can define a representation $\tilde{\pi}_f$ of $\mathcal{A}/N$. This representation may be extended to a representation $\overline{\pi_f}\colon \overline{\mathcal{A}/N} \to L(H_f)$, then $\overline{f}(x) = \langle \overline{\pi_f}(x)\xi_f, \xi_f \rangle$ extends $\tilde{f}$ to a positive functional on $\overline{\mathcal{A}/N}$.

Conversely if $g \in \left(\overline{\mathcal{A}/N}\right)'$ is a positive functional, then $g \circ \rho \in \mathcal{F}$, where ρ is the canonical map onto the quotient algebra. And so we see that $f \leftrightarrow \overline{f}$ is a bijective affine correspondence between $\mathcal{F}$ and $\left(\overline{\mathcal{A}/N}\right)'_+$.

Lemma 4.1. *Let $P\colon \mathcal{A} \to \mathcal{C}$ be a Schwarz map, where $\mathcal{C}$ is a C*-algebra. Then P may be extended uniquely to a Schwarz map $\overline{P}\colon \overline{\mathcal{A}/N} \to \mathcal{C}$.*

P r o o f . Let $f \in S(\mathcal{C})$, the state space of $\mathcal{C}$. Then $f \circ P \in \mathcal{F}$ and therefore $f(Px) = 0$ for all $x \in N$. Since $f \in S(\mathcal{C})$ is arbitrary, we see $P(N) = 0$. So P induces a map $\tilde{P}$ on $\overline{\mathcal{A}/N}$ which is easily seen to be a Schwarz map. Therefore we may assume w.l.o.g. that $N = 0$. We also may assume that $\mathcal{C} \subset L(H)$ for some Hilbert space H, just use an isometric representation for this. The claim now is that $\tilde{P}$ may be extended to a Schwarz map $\overline{P}\colon \overline{\mathcal{A}/N} \to \mathcal{C}$.

If $g \in \mathcal{C}'$, then g is a linear combination of positive functionals and so $g \circ P$ is a linear combination of elements in $\mathcal{F}$. This implies that there is a unique $\|\cdot\|_*$-continuous extension $\overline{g \circ P}$ of $g \circ P$.

Now let $x \in \mathcal{A}$. Define $\phi_x\colon H^2 \to \mathbb{C}$ by $\phi_x(\xi,\eta) := \overline{\xi \otimes \eta \circ P}(x)$, where $\xi \otimes \eta \in \mathcal{C}'$ is defined by $\xi \otimes \eta(y) = \langle y\xi, \eta\rangle$. It is easy to see that ϕ_x is a sesquilinear form on H. We claim, that it is continuous. For this let $(x_n)_n$ be a sequence in $\mathcal{A}$ such that $\|x_n - x\|_* \to 0$. If $g \in \mathcal{C}'$, then $\overline{g \circ P}$ is $\|\cdot\|_*$-continuous and so $\langle P(x_n)\xi, \eta\rangle = \overline{\xi \otimes \eta \circ P}(x_n)$ is a Cauchy sequence. Since WOT-Cauchy sequences are bounded by the uniform boundedness principle we see that $\|P(x_n)\|$ is bounded by a constant K, say. Now $|\phi_x(\xi,\eta)| = |\overline{\xi \otimes \eta \circ P}(x)| = \lim |\overline{\xi \otimes \eta \circ P}(x_n)| = \lim |\langle P(x_n)\xi, \eta\rangle| \leq K \cdot \|\xi\| \cdot \|\eta\|$, and so the continuity of ϕ_x is established.

But then there must be a $y \in L(H)$ such that $\phi_x(\xi,\eta) = \langle y\xi, \eta\rangle$. Define $\overline{P}(x) = y$. It is easy to see that $\overline{P}$ extends P and is linear. Since $\langle \overline{P}(x^*x)\xi, \xi\rangle = \overline{\xi \otimes \xi \circ P}(x^*x) \geq 0$, $\overline{P}$ is positive, hence continuous, hence a Schwarz map and uniquely determined. Moreover $\overline{P}(\overline{\mathcal{A}}) \subset \overline{P(\mathcal{A})} \subset \mathcal{C}$, and this finishes the proof.

THEOREM 4.2. *Let $\mathcal{A}$ be a liminal Banach-*-algebra, $B \subset \mathcal{A}$ a *-subalgebra. Then* $\mathrm{Kor}^u_{\mathcal{A},\mathcal{S}}(B) = \overline{B}^{\|\cdot\|_*}$.

P r o o f. Since B clearly is contained in the Korovkin closure, we can prove one inclusion by showing that the Korovkin closure in question is $\|\cdot\|_*$-closed. But this is very simple since all Schwarz maps and *-homomorphisms involved are $\|\cdot\|_*$-continuous by the above lemma and hence uniformly bounded by 1 with respect to the C*-norm. So we may assume $B = \overline{B}^{\|\cdot\|_*}$ and are left to show $\mathrm{Kor}^u_{\mathcal{A},\mathcal{S}}(B) \subset B$.

So consider $x \in \mathcal{A} \setminus B$. Let $\rho\colon \mathcal{A} \to \mathcal{A}/N$ be the canonical map. If we had $\rho(x) \in \overline{\rho(B)}$, then $\rho(x) = \lim \rho(x_n)$, $x_n \in B$ with respect to the C*-norm on $\mathcal{A}/N$. This implies $\|x - x_n\|_* = \|\rho(x) - \rho(x_n)\|_* \to 0$, and so $x \in \overline{B}^{\|\cdot\|_*} = B$. Therefore we must have $\rho(x) \notin \overline{\rho(B)}$, and this set coincides with $\mathrm{Kor}^u_{\overline{\mathcal{A}/N},\mathcal{S}}\big(\overline{\rho(B)}\big)$ by section 2. Hence there is a net $(P_i)_i$ of Schwarz maps $P_i\colon \overline{\mathcal{A}/N} \to \mathcal{C}$ and a *-homomorphism $S\colon \overline{\mathcal{A}/N} \to \mathcal{C}$ such that $P_i(z) \to S(z)$ for all $z \in \overline{\rho(B)}$, but $P_i(\rho(x)) \to S(\rho(x))$ does not hold. Now use the net $(P_i \circ \rho)_i$ and the *-homomorphism $S \circ \rho$ to conclude that $x \notin \mathrm{Kor}^u_{\mathcal{A},\mathcal{S}}(B)$.

Let $\sigma := \sigma(\mathcal{A}, \mathcal{F})$ be the initial topology induced by $\mathcal{F}$. Since B is convex, the usual arguments show $\overline{B}^{\|\cdot\|_*} = \overline{B}^{\sigma}$.

COROLLARY 4.3. *Let $\mathcal{A}$ be a liminal Banach-*-algebra. Then $\mathcal{A}$ has a finite universal Korovkin system if and only if it is finitely generated as a σ-closed *-algebra.*

R e m a r k s. An examination of the above proof immediately shows that we have the slightly stronger result $\mathrm{Kor}^d_{\mathcal{A},\mathcal{S}}(B) = \overline{B}^{\|\cdot\|_*}$.

Let us say that $\mathcal{A}$ is of type I if the enveloping C*-algebra is of type I. Then the same arguments which have been used to prove the above theorem yield $\mathrm{Kor}^u_{\mathcal{A},\mathcal{S}}(B) = \overline{B}^{\|\cdot\|_*}$, but whether this coincides with the dense Korovkin closure is unknown to me.

5. Examples

5.1. Commutative Banach-*-algebras.

Let $\mathcal{A}$ be a commutative Banach-*-algebra. Clearly $\mathcal{A}$ is liminal and so the results of the last section are applicable. In order to arrive at a nicer description let $\Delta^*_{\mathcal{A}}$ be the subset of the Gelfand spectrum $\Delta_{\mathcal{A}}$ which consists of all hermitian homomorphisms (i.e. $f(x^*) = \overline{f(x)}$). Obviously $\Delta^*_{\mathcal{A}} = \Delta_{\mathcal{A}} \cap \mathcal{F}$ is closed in $\Delta_{\mathcal{A}}$, furthermore $\Delta^*_{\mathcal{A}} \cup \{0\}$ is compact.

THEOREM 5.1. *Let $\mathcal{A}$ be a commutative Banach-*-algebra. Then the following are equivalent:*

(i) *$\mathcal{A}$ has a finite universal Korovkin system with respect to Schwarz maps.*
(ii) *Finitely many elements of $\mathcal{A}$ separate $\Delta^*_{\mathcal{A}} \cup \{0\}$.*

Proof. Let T be a finite universal Korovkin system, let $A^*(T)$ be the *-algebra generated by T. Then $\mathcal{A} = \mathrm{Kor}^u_{\mathcal{A},\mathcal{S}}(T) \subset \mathrm{Kor}^u_{\mathcal{A},\mathcal{S}}\left(A^*(T)\right) = \overline{A^*(T)}^{\sigma}$. So $A^*(T)$ must separate the points of $\mathcal{F}$, in particular it must separate the points of $\Delta^*_{\mathcal{A}} \cup \{0\}$, and so does T.

To prove the converse consider $\mathcal{F}_1 := \left\{f \in \mathcal{F} \mid |f(x)|^2 \leq f(x^*x)\right\}$ which is a convex and w^*-compact set. Observe that $\mathrm{ex}(\mathcal{F}_1) \subset \Delta^*_{\mathcal{A}}$ and this obviously is a Baire set in the w^*-topology of $\mathcal{F}_1$. By the well-known Choquet-Bishop-Meyer-de-Leeuw theorem there is a measure μ_f concentrated on $\Delta^*_{\mathcal{A}} \cup \{0\}$ such that $f(a) = \int_{\Delta^*_{\mathcal{A}}} \hat{a}(\phi)\, \mathrm{d}\mu_f(\phi)$ for all $a \in \mathcal{A}$.

Now let T be a finite set in $\mathcal{A}$ which separates the points of $\Delta^*_{\mathcal{A}} \cup \{0\}$. Then $A^*(T)^\wedge$ is a subalgebra of $\mathcal{C}_0(\Delta^*_{\mathcal{A}})$ which separates the points and does not vanish in a point. Since the elements of $\Delta^*_{\mathcal{A}}$ are hermitian, $A^*(T)^\wedge$ contains the conjugates of all its elements and so is dense in $\mathcal{C}_0(\Delta^*_{\mathcal{A}})$ by the Stone-Weierstraß theorem.

Now let $f, g \in \mathcal{F}_1$ satisfy $f|_{A^*(T)} = g|_{A^*(T)}$. Then for all $a \in A^*(T)$ we have $\int_{\Delta^*_{\mathcal{A}}} \hat{a}\, \mathrm{d}\mu_f = f(a) = g(a) = \int_{\Delta^*_{\mathcal{A}}} \hat{a}\, \mathrm{d}\mu_g$ and hence $\mu_f|_{\Delta^*_{\mathcal{A}}} = \mu_g|_{\Delta^*_{\mathcal{A}}}$, and finally $f = g$. So $A^*(T)$ separates the points of $\mathcal{F}$, and so $\mathcal{A} = \overline{A^*(T)}^{\sigma} = \mathrm{Kor}^u_{\mathcal{A},\mathcal{S}}\left(T \cup \{t^* \circ t \mid t \in T\}\right)$ which finishes the proof.

R e m a r k s . The universal Korovkin closure in [Alt] is defined in a different way, i.e. all C*-algebras $\mathcal{C}$ appearing in the definition of the Korovkin closure are supposed to be commutative. But since we have $\mathrm{Kor}^u_{\mathcal{A},\mathcal{S}}(B) = \mathrm{Kor}^d_{\mathcal{A},\mathcal{S}}(B)$ for subalgebras B these two notions of Korovkin closure coincide. This does not seem to be obvious.

This also makes clear why the more difficult proof for liminal C*-algebras which has been given in paragraph 2 is useful. It gives additional information about the surjective Korovkin closure, and this admitted a proof of the fact that the universal Korovkin closure of a liminal Banach-*-algebra coincides with the dense Korovkin closure which I do not know to hold in the type I case.

If $\mathcal{A}$ has a bounded approximate identity with bound 1, then it is easy to see that all positive linear contractions from $\mathcal{A}$ into commutative C*-algebras are in fact Schwarz maps. So the above results easily lead to Korovkin type theorems in commutative Banach-*-algebras with a bounded approximate identity (bound 1). For example in the end of section 2 of [Alt2] it is stated that the commutative C*-algebra $\mathcal{C}(X)$, where X is a compact Hausdorff space, has a finite universal Korovkin system if and only if finitely many functions separate the points of X. This also may be deduced from the facts stated above. Related questions are discussed for example in [Alt2], [Alt], [Pa].

5.2. Group Algebras.

Let $\mathcal{A}$ be a Banach-*-algebra having a bounded approximate identity. Then the set $\mathcal{F}$ of section 3 coincides with all positive functionals, and so $\Delta^*_{\mathcal{A}} = \Delta_{\mathcal{A}}$. If G is a compact group (or more generally a Moore group), then $L^1(G)$ only admits finite dimensional irreducible representations, and so $L^1(G)$ is liminal and the results of section 3 are applicable. If G is commutative, then $L^1(G)$ possesses a finite universal Korovkin set if and only if finitely many elements of $L^1(G)$ separate $\hat{G} \cup \{0\}$. If in addition $\hat{G}$ is totally disconnected, we know that $L^1(G)$ is a Stone-Weierstraß algebra ([Rud, 9.3]). This yields

THEOREM 5.2. *Let G be a locally compact abelian group such that $\hat{G}$ is totally disconnected. Then $L^1(G)$ has a finite universal Korovkin system if and only if $L^1(G)$ is a finitely generated Banach-*-algebra.*

In fact, such a theorem may be established for any semisimple commutative Banach-*-algebra which is generated by its idempotents, since such an algebra is a Stone-Weierstraß algebra (in fact, this has been used to prove the above Stone-Weierstraß result for $L^1(G)$ in [Rud]).

5.3. The Schatten Classes.

Let $\mathcal{C}_p$ be the pth Schatten class, $1 \le p \le \infty$, on a Hilbert space H, recall that $\mathcal{C}_\infty$ coincides with the ideal of compact operators. Then $\mathcal{C}_p$ does not possess a bounded approximate identity (if $p < \infty$). And indeed there are positive functionals not in $\mathcal{F}$. If $\mathcal{C}_p'$ is identified with $\mathcal{C}_q$ (where $\frac{1}{p} + \frac{1}{q} = 1$) in the usual way, then it is not hard to see that $\mathcal{F} = \mathcal{C}_p' \cap \mathcal{C}_1 = \mathcal{C}_1$. In the case $p = 1$ we have $\mathcal{C}_1' = L(H)$. It is easy to derive from this result that the enveloping C*-algebra is $\mathcal{C}_\infty$, hence liminal. The question, whether $\mathcal{C}_p$ has a finite universal Korovkin system with respect to Schwarz maps, does not depend on p. And the answer is, it has. Just take an irreducible operator $t \in \mathcal{C}_1$, then $\{t, t^*t, tt^*\}$ will be such a system for any p.

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