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ERGODIC THEOREMS FOR LINEAR OPERATORS ON $C(X)$ WITH STRICT TOPOLOGY

JAROSLAV MOHAPL

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ABSTRACT. If X is a completely regular Hausdorff space and if $C(X)$ is provided by a suitable locally convex topology $\mathcal{T}$, then there is a $1-1$ correspondence between the continuous linear operators on $(C(X),\mathcal{T})$ and the integral operators defined by kernels on $X \times M_{\Theta}(X)$, where $\Theta \in \{t,\tau,\sigma\}$ according to the selection of $\mathcal{T}$. This fact is used for study of certain asymptotic properties of solutions of evolution equations and for comparison of the Statistical Ergodic Theorem with more recent results.

Introduction

Let X be a topological space determining the linear space $C(X)$ of continuous functions, $C(X)$ be provided by any locally convex Hausdorff topology and let $T\colon C(X) \to C(X)$ be a continuous linear operator. Then the Yoshida version of the Statistical Ergodic Theorem (SET), [11], states, that if the sequence $\Big(\frac{1}{n}\sum\limits_{k=1}^{n} T^k f\Big)_{n\in\mathbb{N}}$ has a weakly convergent subsequence for a given $f \in C(X)$, then the sequence itself converges in the strong topology. In order to explore the information from this theorem as much as possible we may naturally ask. As long as we have $C(X)$ and its dual determining the weak topology. What is the strongest topology in which $C(X)$ has the a priori given dual?

A theoretical answer to this question yields the known Mackey theorem [6]. Nevertheless, the practically most interesting duals are $M_t(X)$, $M_\tau(X)$ and $M_\sigma(X)$, which evokes a new problem: How to describe the strong Mackey topology by means of the better manageable uniformly tight, τ-smooth and σ-smooth subsets of $M_t(X)$, $M_\tau(X)$ and $M_\sigma(X)$, respectively? And is such a description possible at all?

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These and many related problems were broadly solved since the early 70's [1], [2], [3], [13], [14] and not always the above formulated questions have an affirmative solution. But if a measure theoretical description of the strong Mackey topology exists, then, as we are going to show, there is a $1-1$ correspondence between the continuous linear operators on $C(X)$ and the integral operators defined by kernels on $X \times M_\Theta(X)$, where $\Theta \in \{t, \tau, \sigma\}$, provided $C(X)$ is equipped by the strict topology yielding as dual $M_\Theta(X)$.

Combination of the representation theorems for continuous linear operators with a modified version of the SET yields an interesting tool for study of asymptotic properties of solutions of evolution equations (compare with [5]) and allows to compare the SET with more recent results in [10], [8].

1. Preliminaries

As to the terminology used below we refer the reader to [11], [12], [13]. In the whole paper X denotes a completely regular Hausdorff space, $\mathcal{B}(X)$ denotes the algebra of all Baire sets, $C(X)$ denotes the set of all real, bounded, continuous functions on X and $M_t(X)$, $M_\tau(X)$, $M_\sigma(X)$ denote the classes of all regular measures on $\mathcal{B}(X)$ with finite variation which are tight, τ-smooth and σ-smooth, respectively. The non-negative parts of $C(X)$ and $M_\Theta(X)$ are denoted by $C^+(X)$ and $M_\Theta^+(X)$, $\Theta \in \{t, \tau, \sigma\}$, respectively.

The bounded set $M \subset M_\sigma(X)$ ($M \subset M_\tau(X)$) is called uniformly σ-smooth (τ-smooth) if for each decreasing sequence (net) $(f_\alpha) \subset C^+(X)$ with $|f_\alpha(x)| \le c$ for all α and $\lim_\alpha f_\alpha(x) = 0$ for all $x \in X$ $\lim_\alpha \sup_{m \in M} |mf_\alpha| = 0$. The bounded set $M \subset M_t(X)$ is called uniformly tight if for each $\varepsilon > 0$ there exists a compact K, $K \subset X$, such that $|m|K^c < \varepsilon$ for all $m \in M$. Here and below $mf = \int f(x)\, m(\mathrm{d}x)$, $|m| = m^+ - m^-$, where m^+ and m^- are the positive and negative parts of m, respectively.

The pairs $\big(C(X), M_\Theta(X)\big)$, $\Theta \in \{t, \tau, \sigma\}$, will be studied in duality by means of the bilinear form $(\cdot, \cdot)$ defined by $(f, m) = mf$. If $M \subset M_\sigma(X)$, then we can define a seminorm $\|\cdot\|_M$ on $C(X)$ by $\|f\|_M = \sup_{m \in M} |(f, m)|$ for each $f \in C(X)$. The open balls $\{f : \|f\|_M < \varepsilon,\ f \in C(X)\}$, $\varepsilon > 0$, define various topologies on $C(X)$. Denoting by δ_x the Dirac (point) measure with mass concentrated in $x \in X$, one can describe these topologies:

p – the topology of pointwise convergence which is generated by the seminorms $\big\{\|\cdot\|_M : M$ consists of a finite class of Dirac measures$\big\}$,

$\mathcal{K}$ – the topology of uniform convergence on compact sets which is generated by the seminorms $\big\{\|\cdot\|_M : M = \{\delta_x : x \in K\},\ K \subset X,\ K$ is compact$\big\}$,

β_0 – the topology which is generated by the seminorms $\big\{\|\cdot\|_M : M \subset M_t(X)$,

M is convex and uniformly tight $\}$,

β – generated by $\{\|\cdot\|_M : M \subset M_\tau(X)$, M is convex and uniformly τ-smooth $\}$,

β_1 – generated by $\{\|\cdot\|_M : M \subset M_\sigma(X)$, M is convex and uniformly σ-smooth $\}$, and

u – the topology of uniform convergence, which is defined by the norm $\|\cdot\|_M$, where $M = \{\delta_x : x \in X\}$. This norm is simply denoted $\|\cdot\|$.

In addition to the above mentioned we will use the usual weak and weak-star topologies arising from the duality of $\big(C(X), M_\Theta(X)\big)$, $\Theta \in \{t, \tau, \sigma\}$ and denoted by w and w^*, respectively. The topologies β_0, β, β_1 are also called strict topologies. Because of the $1-1$ correspondence between the Dirac measures δ_x and $x \in X$, the definitions of p, $\mathcal{K}$ and u agree with the commonly used definitions. By the words M *is bounded* we mean that $\sup\limits_{m\in M} |m|X$ is a finite number.

Since X is a Hausdorff space, its one point sets are closed and compact, hence, $\delta_x \in M_t^+(X)$ for each $x \in X$ and therefore $p \subseteq \mathcal{K}$. If $M = \{\delta_x : x \in K\}$, $K \subset X$, K is compact and $\overline{M} = \Big\{m : m = \sum\limits_{i=1}^n \alpha_i \delta_{x_i}$, $0 \le \alpha_i \le 1$, $\sum\limits_{i=1}^n \alpha_i = 1$, $x_i \in K$ for all i, $0 \le i \le n$, $n \in \mathbb{N}\Big\}$, then $\overline{M}$ is uniformly tight, convex and $M \subset \overline{M}$, hence, $\mathcal{K} \subseteq \beta_0$.

We are going to show that $\beta_0 \subseteq \mathcal{K}$ on u-bounded sets. Let $(f_\alpha) \subset C(X)$ be a net such that $\mathcal{K} - \lim\limits_\alpha f_\alpha = f \in C(X)$, $\|f_\alpha\| \le 1$ for all α and let M be uniformly tight, with $\sup\limits_{m\in M} |m|X = c$. Then to each $\varepsilon > 0$ there exists $K \subset X$, K compact, and α_0 so that $|m|K^c < \varepsilon/4$ for all $m \in M$ and $\sup\limits_{x\in K} |f(x) - f_\alpha(x)| < \varepsilon/(2c)$ for all $\alpha \ge \alpha_0$. Since

$$\sup_{m\in M} |(f - f_\alpha, m)| \le 2|m|K^c + \sup_{x\in K} |f(x) - f_\alpha(x)|c < \varepsilon \qquad \text{for all} \quad \alpha \ge \alpha_0,$$

it is clear that β_0- $\lim\limits_\alpha f_\alpha = f$.

If M is uniformly tight, with $\sup\limits_{m\in M} |m|X = c$, then for each decreasing net $(F_\alpha) \subset \mathcal{B}(X)$ of closed sets with empty intersection $\lim\limits_\alpha \sup\limits_{m\in M} |m|F_\alpha = 0$ (note that (F_α^c) is an open covering of each compact $K \subset X$ and must contain a finite subcovering of K). Now if $(f_\alpha) \subset C(X)$, $\|f_\alpha\| \le 1$ and $f_\alpha \downarrow 0$, then if we put $F_\alpha = \{x : f_\alpha(x) \ge \varepsilon\}$,

$$\lim_\alpha \sup_{m\in M} |mf_\alpha| \le \lim_\alpha \sup_{m\in M} |m|F_\alpha + c\varepsilon \le c\varepsilon.$$

This shows that $\beta_0 \subseteq \beta$ and obviously $\beta \subseteq \beta_1 \subseteq u$.

Due to the known Mackey theorem [6] the strongest locally convex linear topology on $C(X)$ yielding the dual $M_\Theta(X)$, $\Theta \in \{t, \tau, \sigma\}$, agrees with the topology of uniform convergence on the convex w^*-compact subsets of $M_\Theta(X)$. This topology is called the strong Mackey topology. To prove that β_0, β, and β_1 agree with the strong Mackey topology, it is necessary and sufficient that each relatively w^*-compact convex subset of $M_\Theta(X)$ is bounded, uniformly tight, τ-smooth and σ-smooth, respectively.

From Alaoglu's theorem and the representation theorems in [12] it is easy to derive, that the uniformly tight, τ-smooth and σ-smooth subsets are relatively w^*-compact in $M_t(X)$, $M_\tau(X)$ and $M_\sigma(X)$, respectively. But, the converse assertion is not always true.

Immediately from Varadarjan's result [12, ch. 2, Thm. 28] it follows that β_1 agrees always with the strong Mackey topology. In [13] it is shown that if X is a metric space, then β agrees with the strong Mackey topology. If X is a complete metric space or a locally compact space, then $M_\tau(X) = M_t(X)$ and $\beta_0 = \beta$ (see [3]). If X is a separable space, then $M_\tau(X) = M_\sigma(X)$ and $\beta = \beta_1$.

Many ergodic theorems (cf. [10]) state that if Q is an ergodic probability kernel on $X \times \mathcal{B}(X)$, then $p\text{-}\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=1}^{n} Q^k f$ exists in $C(X)$ for each $f \in C(X)$ while the SET deals with the w-compact classes of functions. This leads to the question about the agreement of p and w topologies, especially of their compact subsets. Using the relation $\delta_x \in M_\Theta(X)$, $\Theta \in \{t, \tau, \sigma\}$, we can prove that $p \subseteq w$. In [14] the question what is the greatest subset $Z \subset M_\sigma(X)$ with the property $C \subset C(X)$, *C is uniformly bounded and relatively compact in p, then C is relatively compact in $\sigma\big(C(X), Z\big)$* is in detail solved. If we restrict our attention only to the sequentially compact subsets, then we obtain from the bounded convergence theorem for integrals that $p = w$ on the relatively sequentially compact sets.

2. Operators and kernels

A map Q defined on $X \times \mathcal{B}(X)$ is called a kernel if $Q(x, \cdot) \in M_\sigma(X)$ for each $x \in X$, $Q(\cdot, E)$ is measurable for each fixed $E \in \mathcal{B}(X)$ and $\sup\limits_{x \in X} |Q|(x, X) < \infty$. We are going to write $Qf(x)$ instead of $\int f(y)\, Q(x, \mathrm{d}y)$ provided f is integrable and mQ is the measure relating to each $E \in \mathcal{B}(X)$ $\int Q(x, E)\, m(\mathrm{d}x)$ if $m \in M_\sigma(X)$. We say that Q has the Feller property (FP) if $Qf \in C(X)$ for each $f \in C(X)$. The kernel Q is said to have the strong FP (shortly SFP) if $Q(\cdot, E) \in C(X)$ for each $E \in \mathcal{B}(X)$ and Q is said to have the SFP in restricted sense (shortly SFPR) if $\lim\limits_\alpha \operatorname{var}\big(Q(x_\alpha, \cdot) - Q(x, \cdot)\big) = 0$ whenever $(x_\alpha) \subset X$,

$x \in X$, and $\lim_{\alpha} x_\alpha = x$.

In the whole paper T is assumed to be a u-continuous linear map, $T\colon C(X) \to C(X)$. We will call T positive if $f \in C^+(X)$ implies $Tf \in C^+(X)$. If given T there exists a kernel Q such that $Tf = Qf$ on $C(X)$, then we will say that Q represents T.

PROPOSITION 2.1. *If T is β_1-continuous, then there exists a kernel Q with FP representing T. If in addition T is β (β_0)-continuous, then $Q(x,\cdot) \in M_\tau(X)$ ($Q(x,\cdot) \in M_t(X)$) for each $x \in X$.*

Proof. If T is β_1 (β)-continuous, then for each decreasing sequence (net) $(f_\alpha) \subset C(X)$ with $p\text{-}\lim_{\alpha} f_\alpha = 0$ and $\|f_\alpha\| \le 1$ for all α $p\text{-}\lim_{\alpha} Tf_\alpha = 0$. This can be proved using the definition of β_1 (β), the continuity of T, and the fact that for each $x \in X$ $Tf(x) = (Tf, \delta_x)$ and $(\cdot\,, \delta_x)$ is a continuous linear functional.

Similarly if T is β_0-continuous and $(f_\alpha) \subset C(X)$ is a net with $\mathcal{K}\text{-}\lim_{\alpha} f_\alpha = 0$ and $\|f_\alpha\| \le 1$ for all α, then, since β_0 and $\mathcal{K}$ agree on the u-bounded sets, $\beta_0\text{-}\lim_{\alpha} f_\alpha = 0$ and therefore $p\text{-}\lim_{\alpha} Tf_\alpha = 0$.

Applying the representation theorems for bounded linear functionals from [12] we obtain that to each $x \in X$ there exists a measure $Q(x,\cdot) \in M_\Theta(X)$, where $\Theta \in \{t, \tau, \sigma\}$ according to the topology defining the continuity of T, such that $Tf(x) = \int f(y)\, Q(x, \mathrm{d}y)$ for all $f \in C(X)$.

It remains to prove the measurability of Q with respect to x. If $G \in \mathcal{B}(X)$ is open, then there exists a sequence $(f_n) \subset C^+(X)$ such that $\chi_G = \lim_{n\to\infty} f_n$ pointwise on X. By the bounded convergence theorem $Q(x,G) = \lim_{n\to\infty} Qf_n(x)$ for each $x \in X$. As $(Qf_n) \subset C(X)$, $Q(x,G)$ is measurable. But the open Baire sets generate $\mathcal{B}(X)$, which allows a simple completion of the proof.

PROPOSITION 2.2. *Let Q be a kernel and $\mathcal{T}$ be the strong Mackey topology, which is defined by $M_\Theta(X)$, $\Theta \in \{t, \tau, \sigma\}$. Then Q represents a $\mathcal{T}$-continuous operator T if and only if Q has the FP and $mQ \in M_\Theta(X)$ for each $m \in M_\Theta(X)$.*

Proof. If $M \subset M_\Theta(X)$ is w^*-compact, then each net $(m_\alpha) \subset M$ has a subnet (m_{α_β}) such that $w^*\text{-}\lim_{\beta} m_{\alpha_\beta} = m \in M$. If $Qf \in C(X)$ and $mQ \in M_\Theta(X)$ for each $m \in M_\Theta(X)$ and $f \in C(X)$, then $w^*\text{-}\lim_{\beta} m_{\alpha_\beta} Q = mQ$, which means that $QM = \{\overline{m} : \overline{m} = mQ\,,\ m \in M\}$ is w^*-compact. Hence, if T is represented by Q, then $\|Tf\|_M = \|f\|_{QM}$ for each $f \in C(X)$. This proves continuity of T.

Conversely let T be continuous in $\mathcal{T}$. Since $(Tf,m) = (f,mQ)$ for each $f \in C(X)$ and $m \in M_\Theta(X)$, the linear functional $(\cdot,mQ)$ is continuous in the strong Mackey topology. But the dual of $\big(C(X),\mathcal{T}\big)$ agrees with $M_\Theta(X)$, hence, $mQ \in M_\Theta(X)$.

PROPOSITION 2.3. *T can be represented by a kernel Q with FP if and only if it is β_1-continuous. If β (β_0) agree with the strong Mackey topology, then T can be represented by a kernel with FP and $Q(x,\cdot) \in M_\tau(X)$ ($Q(x,\cdot) \in M_t(X)$) for each $x \in X$ if and only if T is β (β_0)-continuous.*

P r o o f. See Proposition 2.1, 2.2, and the Preliminaries.

PROPOSITION 2.4. *If (X,d) is a metric space, then $T\colon \big(C(X),\beta_0\big) \to \big(C(X),\beta_0\big)$ is a compact operator if and only if the kernel Q representing T has the SFPR.*

P r o o f. Let T be a compact operator represented by the kernel Q (by Proposition 2.3 Q always exists since T is β_1 continuous). We base our decisions on the fact that $\mathcal{K} = \beta_0$ on the u-bounded sets. The unit ball $B \subset \big(C(X),u\big)$ is by T mapped on a u-bounded set B'. For any compact set $\mathcal{K} \subset X$ and $f \in C(X)$ we denote by f_K the restriction of f to K. If B' is a relatively compact subset determined by $\mathcal{K}$, then $\{f_K : f \in B'\}$ is relatively compact in $\big(C(K),\|\cdot\|_K\big)$. From Ascoli's theorem we can derive that to each $\varepsilon > 0$ there exists $\delta > 0$ with the property $x,y \in K$, $d(x,y) < \delta$, then $\operatorname{var}\big(Q(x,\cdot) - Q(y,\cdot)\big) \le \varepsilon$. Each convergent sequence in X can be considered as a compact set, hence the SFPR is proved.

To verify the converse assertion we use the indirect proof, assuming that Q has the SFPR but B' is not relatively $\mathcal{K}$-compact. Then there exists a compact K, $K \subset X$, such that $\{f_K : f \in B'\}$ is not compact and therefore does not satisfy Ascoli's theorem. Then there exists $\varepsilon > 0$ such that to each natural n there exists a couple $x_n, y_n \in K$ for which $d(x_n,y_n) < \dfrac{1}{n}$ and $\operatorname{var}\big(Q(x_n,\cdot) - Q(y_n,\cdot)\big) \le \varepsilon$. Since K is compact, we can find a subsequence (n_k) and $x \in K$ such that $x_{n_k} \to x$ and $y_{n_k} \to x$ as $k \to \infty$.

$$\begin{aligned}\varepsilon &\le \operatorname{var}\big(Q(x_{n_k},\cdot) - Q(y_{n_k},\cdot)\big)\\ &\le \operatorname{var}\big(Q(x_{n_k},\cdot) - Q(x,\cdot)\big) + \operatorname{var}\big(Q(y_{n_k},\cdot) - Q(x,\cdot)\big),\end{aligned}$$

and the SFPR allows to make the sum in the right arbitrarily small. This contradiction completes the proof.

PROPOSITION 2.5. *Let X be a separable space and let T be a positive operator. If Q is the kernel representing T, then Q has the SFP if and only if T maps the u-bounded subsets of $C(X)$ onto the relatively p-compact subsets.*

Proof. Since p-compact and p-sequentially compact subsets of $C(X)$ agree if X is separable, we can work only with bounded sequences.

Let Q have the SFP. As X is separable, we can find a dense countable subset $(x_n) \subset X$ and define the measure m by $m = \sum_{n \geq 1} 2^{-n} Q(x_n)$. For each $x \in X$ $Q(x,\cdot)$ is absolutely continuous with respect to m, hence, due to the Radon-Nikodým theorem $Q(x,E) = \int_E h(x,y)\, m(\mathrm{d}y)$ for all $E \in \mathcal{B}(X)$. Let $(f_n) \subset C(X)$, $\|f_n\| \leq 1$ for all n. $C(X) \subset L^2(X,m)$, hence, we can find a subsequence (f_{n_k}) converging weakly (in $L^2(X,m)$) to some $f \in L^2(X,m)$. If we define h_p by $h_p = \min(h,p)$, p natural, then $h_p(x,\cdot) \in L^2(X,m)$ for each p and x and therefore $\lim_{k\to\infty} \int f_{n_k} h_p(x,\cdot)\, m = \int f h_p(x,\cdot)\, m$ while $\lim_{p\to\infty} \int h_p(x,\cdot)\, m = \int h(x,\cdot)\, m$ by the monotone convergence theorem for each $x \in X$. As

$$
\begin{aligned}
&|Qf(x) - Qf_{n_k}(x)| \\
&\leq 2 \int \big(h(x,y) - h_p(x,y)\big)\, m(\mathrm{d}y) + \int \big(f(y) - f_{n_k}(y)\big) h_p(x,y)\, m(\mathrm{d}y),
\end{aligned}
$$

we can conclude that $\lim_{k\to\infty} Qf_{n_k}(x) = Qf(x)$. We knew that $\|f_n\| \leq 1$. Consequently $|f(x)| \leq 1$ a.s. m and we can consider $|f(x)| \leq 1$ on the whole X. Now using the SFP it is easy to show that $Qf \in C(X)$.

Conversely, let T map the u-bounded subsets of $C(X)$ on the relatively sequentially p-compact ones. If $G \in \mathcal{B}(X)$ is open, then there exists an increasing sequence $(f_n) \subset C(X)$ such that $\lim_{n\to\infty} f_n = \chi_G$ and therefore p-$\lim_{n\to\infty} Qf_n = Q(\cdot,G) \in C(X)$. Consequently $Q(\cdot,E) \in C(X)$ if E is open or closed. As $Q(x,E) = \sup_{F \subseteq E} Q(x,F) = \inf_{G \supseteq E} Q(x,G)$, where G and F are open and closed Baire sets, respectively, for each $E \in \mathcal{B}(X)$, $Q(\cdot,E)$ is at the same time upper and lower semicontinuous, hence, continuous.

3. Stability of solutions of evolution equations

In this section we are going to work with a set $(T_t)_{t\geq 0}$ of mappings with properties:

i) $T_t \colon C(X) \to C(X)$ is a linear operator for each $t \geq 0$,
ii) $T_t \circ T_s = T_{t+s}$ for all $s,t \geq 0$ and $T_t = I$ (identity) for $t = 0$.

If $\mathcal{T}$ is a locally convex Hausdorff topology on $C(X)$, then

iii) $\mathcal{T}\text{-}\lim_{t\to t'} T_t f = T_{t'} f$ for all $t' \geq 0$ and $f \in C(X)$,

iv) to each $\mathcal{T}$-continuous seminorm q there exists a $\mathcal{T}$-continuous seminorm q' such that

$$q(T_t f) \leq q'(f) \qquad \text{for all} \quad t > 0\,, \;\; f \in C(X)\,. \tag{$*$}$$

In [11, ch. 9] $(T_t)_{t\geq 0}$ is called a uniformly continuous (shortly uc) C_0-semigroup. The linear operator A, $Af = \lim_{h\downarrow 0} h^{-1}(T_h - I)f$, is defined on $\mathcal{D}(A) \subset C(X)$, where $\mathcal{D}(A)$ consists of all $f \in C(X)$ for which the limit exists. The operator A is called the infinitesimal generator of $(T_t)_{t\geq 0}$ and it has the properties:

v) $\mathcal{D}(A) \subset C(X)$ is dense in $\big(C(X), \mathcal{T}\big)$,

vi) the operator $(I - n^{-1}A)^{-m}$ exists for each $n, m \in \mathbb{N}$ and to each $\mathcal{T}$-continuous seminorm q there exists a $\mathcal{T}$-continuous seminorm q' such that

$$q\big((I - n^{-1}A)^{-m} f\big) \leq q'(f) \qquad \text{for all} \quad n, m \in \mathbb{N}\,, \;\; f \in C(X)\,, \tag{$*'$}$$

vii) the function $u(t,x) = T_t f(x)$, where $f \in \mathcal{D}(A)$, satisfies the evolution equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Au \quad \text{and} \quad u(0,x) = f(x) \quad \text{for all} \quad x \in X\,. \tag{$**$}$$

The famous result by Yosida [11, ch. 9] states that if A is a linear operator and if $\mathcal{D}(A) \subset C(X)$, then v) and vi) are sufficient for A to be an infinitesimal generator of a C_0-semigroup provided the topology $\mathcal{T}$ is sequentially complete.

Throughout this section we are going to assume that $\mathcal{T}$ denotes one of the topologies β_0, β and β_1, respectively. Under this assumption we can use Proposition 2.1 and identify the uc C_0-semigroup $(T_t)_{t\geq 0}$ with a class $(Q^t)_{t\geq 0} \subset M_\Theta(X)$ of kernels with $\Theta \in \{t, \tau, \sigma\}$ according to the choice of $\mathcal{T}$. Obviously $Q^{t+s} = \int Q^t(y,\cdot)\, Q^s(x, \mathrm{d}y)$ for all $t, s \geq 0$, $Q^0(x,\cdot) = \delta_x$ for all $x \in X$, $Q^t f(x)$ is continuous in $t \geq 0$ and in $x \in X$ for each $f \in C(X)$ etc.

The topologies β_0, β and β_1 are generally not sequentially complete. The spaces X for which $\big(C(X), \mathcal{K}\big)$ is sequentially complete are called k_R-spaces. These spaces are more general than the so-called k-spaces. X is a k-space if the condition that all intersections of $F \subset X$ with compact subsets of X are closed sets implies that F itself is closed. To k-spaces belong the locally compact spaces, spaces with the first axiom of countability, spaces complete in sense of Čech, etc. It can be proved that if X is a k_R space, then β_0, β and β_1 are sequentially complete, β_0 and β have the same convergent sequences, and if X is a D-space, then β_0, β and β_1 have the same convergent sequences (see [13] and references there). The introduced observations may be summarized:

THEOREM 3.1. *Let $\mathcal{T} \in \{\beta_0, \beta, \beta_1\}$. If X is a k_R-space and if A is a linear operator with $\mathcal{D}(A) \subset C(X)$ and properties* v), vi), *then the solution u of the equation $(**)$ exists for each $f \in \mathcal{D}(A)$ and it can be expressed in the form $u(t,x) = \int f(y)\, Q^t(x, \mathrm{d}y)$, where $(Q^t)_{t\geq 0}$ is a class of kernels representing a uc C_0-semigroup of linear operators on $\big(C(X), \mathcal{T}\big)$.*

Now we are going to extend the SET from [11, ch. 8]. In connection with Theorem 3.1 it can be considered as a criterion for study of stability of the solutions of $(**)$ provided A is an infinitesimal operator of a uc C_0-semigroup.

THEOREM 3.2. *Let $\mathcal{T} \in \{\beta_0, \beta, \beta_1\}$ and let $(T_t)_{t\geq 0}$ be a uc C_0-semigroup of linear operators on $\big(C(X), \mathcal{T}\big)$. If for each $f \in \mathcal{D}(A)$ $\Big(\frac{1}{n}\sum\limits_{k=1}^{n} T_k f\Big)_{n\in\mathbb{N}}$ has a w-convergent subsequence, then there exists a linear operator U, $U\colon C(X) \to C(X)$, such that u, defined by $u(t,x) = T_t f(x)$, satisfies*

viii) $\mathcal{T}\text{-}\lim \frac{1}{t}\int\limits_0^t u(s,\cdot)\,\mathrm{d}s = Uf$ *for each* $f \in \mathcal{D}(A)$,

ix) $U = U^2 = T_t U = U T_t$ *for all* $t \geq 0$,

x) $\mathcal{R}(U) = \mathcal{N}(I - T_t)$, $\mathcal{N}(U) = \mathcal{R}(I-T_t)^{\mathrm{cl}} = \mathcal{R}(I-U)$ *for each* $t \geq 0$.

P r o o f. $(T_t)_{t\geq 0}$ is a C_0-semigroup, hence, we can to each $\mathcal{T}$-continuous seminorm q find a $\mathcal{T}$-continuous seminorm q' such that

$$q\left(\frac{1}{n}\sum_{k=1}^{n} T_k f\right) \leq q'(f) \qquad \text{for all} \quad n \in \mathbb{N},\ f \in \mathcal{D}(A). \qquad (*'')$$

Therefore we can apply the SET from [11, ch. 8] and show that there exists a continuous linear operator U_0 on $\big(C(X), \mathcal{T}\big)$ with the property

$$\mathcal{T}\text{-}\lim \frac{1}{n}\sum_{k=1}^{n} T_k f = U_0 f \qquad \text{for all} \quad f \in C(X).$$

By $\int\limits_0^t u(s,\cdot)\,\mathrm{d}s$, $t \geq 0$, we understand the function $\int\limits_0^t u(s,x)\,\mathrm{d}s$ with variable $x \in X$ and integral considered, say, in the Riemann sense. If $(f_n) \subset \big(C(X), \mathcal{T}\big)$ converges to zero then to each continuous seminorm q we can find a seminorm q' such that

$$q\left(\int_0^1 T_t f_n\,\mathrm{d}t\right) \leq \int_0^1 q(T_t f_n)\,\mathrm{d}t \leq q'(f_n),$$

and the last term converges to zero if n tends to infinity. This and linearity of T_t explain the relation

$$\mathcal{T}\text{-}\lim \frac{1}{n}\int_0^n T_t f \, \mathrm{d}t = \mathcal{T}\text{-}\lim \int_0^1 T_t\Big(\frac{1}{n}\sum_{k=1}^{n} T_k f\Big) \, \mathrm{d}t = \int_0^1 T_t U_0 f \, \mathrm{d}t \, .$$

If we put $Uf = \int_0^1 T_t U_0 f \, \mathrm{d}t$, then for each continuous seminorm q

$$q\bigg(\frac{1}{t}\int_0^t T_s f \, \mathrm{d}s - Uf\bigg) \le q\bigg(\frac{1}{n}\int_0^n T_s f \, \mathrm{d}s - Uf\bigg) + \frac{1}{n} q\int_n^t T_s f \, \mathrm{d}s$$

for each $n \le t$, and the last expression tends to zero if n tends to infinity. So viii) is proved, and the proof of ix) and x) can be done by the same arguments as in [11, ch. 8].

COROLLARY 3.3. *Let X be a separable metric space, let $\mathcal{T} = \beta_0$ and let $(Q^t)_{t\ge 0}$ be a class of kernels representing a uc C_0-semigroup $(T_t)_{t\ge 0}$ of linear operators on $\big(C(X), \mathcal{T}\big)$. If for some $s > 0$ Q^s has the SFPR, then the assertion of Theorem 3.2 holds.*

P r o o f. By Proposition 2.4 T_s represented by a kernel with SFPR must be a compact operator, hence, $T_n = T_{n-s} \circ T_s$ is compact for each $n \ge s$. Therefore for $n_0 \ge s$

$$\frac{1}{n}\sum_{k=1}^{n} T_k f = T_{n_0}\Bigg(\frac{1}{n}\sum_{k=n_0}^{n} T_{k-n_0} f\Bigg) + \frac{1}{n}\sum_{k=1}^{n_0} T_k f$$

if $f \in C(X)$ and holding f fixed we obtain that $\lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{k=1}^{n_0} T_k f = 0$. Since X is separable, the compact and sequentially compact subsets of $\big(C(X), \mathcal{T}\big)$ agree, and $\Big(T_{n_0}\Big(\frac{1}{n}\sum\limits_{k=n_0}^{n} T_{k-n_0}\Big)\Big)_{n\in\mathbb{N}}$ has a w-convergent subsequence.

R e m a r k 3.4. If the operators in Corollary 3.3 are positive, then SFPR can be replaced by SFP and β_0 by β_1.

THEOREM 3.5. *Let X be a separable coherent space, and let $(P^t)_{t\ge 0}$ be a class of probability kernels representing a uc C_0-semigroup $(T_t)_{t\ge 0}$ of linear operators on $\big(C(X), \mathcal{T}\big)$. If for some $s > 0$ P^s has the SFP, and if for some $r > 0$ P^r is aperiodic, then there exists a unique probability measure $\pi \in M_\sigma(X)$ such that $\pi P^t = \pi$ and*

xi) $\lim \operatorname{var}\big(P^t(x,\cdot) - \pi\big) = 0$.

Proof. P^t is a probability kernel if it is non-negative and $P^t(x,X) = 1$ for all $x \in X$. Due to the Remark 3.5 and Corollary 3.4 viii)−x) hold and for each $x \in X$ there exists a measure $\pi_x \in M_\sigma(X)$ such that $Uf(x) = \pi_x f$. To show that π is unique and independent of x, it suffices to verify that each P^t-invariant function is constant almost sure with respect to π. Let $B \in \mathcal{B}(X)$ be a P^t-invariant set, i.e. let $P^t(x,B) = \chi_B(x)$ for all $t \geq 0$ and $x \in X$. For some $s > 0$ P^s has the SFP, hence $\chi_B \in C(X)$. But X is coherent, and so or $B = X$ or $B - \emptyset$, i.e. χ_B is constant. Because each P^t-invariant function can be approximated by linear combinations of characteristic functions of P^t-invariant sets, each such function is constant. Now π is unique by [11, ch. 13, §3].

Relation $(*'')$ and SET guarantee that Q, defined by $Q = P^r$, where P^r is the aperiodic kernel, belongs to a positively recurrent Markov chain with invariant measure equal to π. By [10, Corollary 6.3], $\lim \operatorname{var}\bigl(Q^n(x,\cdot) - \pi\bigr) = 0$ for each $x \in X$. Equations $P^t = Q^n P^{t-nr}$, $\pi = \pi P^{t-nr}$ allow to complete the proof.

We recall that a positive kernel Q is said to be irreducible if there exists a measure $m \in M_\sigma^+(X)$ such that for each $x \in X$ and $E \in \mathcal{B}(X)$ with $mE > 0$ $\sum\limits_{n\geq 1} Q^n(x,E) > 0$. If Q is irreducible, then there exists an integer p, $0 < p < \infty$, called period of Q, and a collection of disjoint sets $\{E_1,\dots,E_p\} \subset \mathcal{B}(X)$ with the properties $Q\chi_{E_{i+1}} = \chi_{E_i}$ and $Q^p\chi_{E_i} = \chi_{E_i}$ for all $1 \leq i \leq p$ (see [10, ch. 2]).

If X is separable, then $M_\sigma(X) = M_\tau(X)$, consequently $\beta_1 = \beta_0$ and the β_0- and β_1-convergent sequences agree. As $\mathcal{K} = \beta_0$ on the bounded subsets of $C(X)$ (see Section 1 and the remarks preceeding Theorem 3.1), the convergence in viii), which was proved under the assumptions of Theorem 3.2 and Corollary 3.3, is uniform on the compact sets.

4. Discussion

The two additional Feller properties SFP and SFPR were introduced in [4] and in connection with evolution equations they were studied in [5]. The study was restricted to locally compact separable metric spaces and among many remarkable properties studied there one can find Proposition 2.4 and the question under which condition xi) in Theorem 3.5 holds.

As an example let us consider $X = \mathbb{E} = (-\infty,\infty)$ with the Euclidean topology defining $\mathcal{B}(X)$ and the Lebesgue measure ℓ on $\mathcal{B}(X)$. If $\varrho^t(x,y) = \bigl(1/(2\pi t)\bigr)^{1/2}\exp\bigl\{(x-y)^2/(2t)\bigr\}$, then P^t, defined for $t > 0$ by $P^t(x,E) = \int\limits_E \varrho^t(x,y)\,\ell(\mathrm{d}y)$, have the SFPR.

It is well known, that the unique invariant measure for P^t is ℓ. But ℓ is only σ-finite, hence $(P^t)_{t\geq 0}$ does not define a uc C_0-semigroup with respect to β_0 (now $\beta_0 = \beta = \beta_1$). This means that assumption $(*'')$ cannot be excluded if we want viii) to hold.

Let us replace $\mathbb{E}$ by $\mathbb{R}$, the one-point compactification of $\mathbb{E}$. If the new point added to $\mathbb{E}$ is Δ, then $\mathbb{R} = \{\Delta\} \cup \mathbb{E}$. To each P^t we can define an extension $\overline{P^t}$ from $\mathcal{B}(\mathbb{E})$ to $\mathcal{B}(\mathbb{R})$ by

$$\overline{P^t}(x, E) = P^t(x, E \cap \mathbb{E}) \qquad \text{for} \quad x \in \mathbb{E}, \ \overline{P^t}(\Delta, E) = \delta_\Delta E,$$

where E are taken from $\mathcal{B}(\mathbb{R})$. Since the closed subsets of $\mathbb{R}$ agree with the compact subsets of $\mathbb{E}$ (except $\{\Delta\}$) and $\lim\limits_{t\uparrow\infty} P^t(x, K) = 0$ for each $x \in X$ and compact $K \subset E$, we can use the Alexandrov theorem [12, ch. 2, Thm. 2] and prove that $w^* - \lim\limits_{t\uparrow\infty} \overline{P^t}(x, \cdot) = \delta_\Delta$ for each $x \in X$. But the last equation implies viii) with $Uf = \int f(x)\, \delta_\Delta(\mathrm{d}x) = f(\Delta)$ for each $f \in C(\mathbb{R})$!

This convergence is remarkable for several reasons. First, $\overline{P^t}$ now has not the SFP, i.e. x'' can hold even if P^t has not the SFP. Second, the w^*-convergence cannot be replaced by the stronger setwise convergence (if $(m_n) \subset M_\sigma(X)$, then (m_n) converges to m_0 setwise if and only if $\lim\limits_{n\to\infty} m_n E = m_0 E$ for each $E \in \mathcal{B}(x)$). Third, if we denote by $M(\mathbb{E})$ the class of all regular measures on $\mathcal{B}(\mathbb{E})$ with finite variation (not necessarily σ-additive) and by $C_0(\mathbb{E})$ the class of all $f \in C(\mathbb{E})$ with compact support, then the w^*-convergence arising from the duality $\big(C_0(\mathbb{E}), M(\mathbb{E})\big)$ cannot be defined by a Hausdorff topology on $M(\mathbb{E})$. By the way, the last fact explains why P^t has two invariant measures, the σ-additive and σ-finite measure ℓ, and the purely additive probability measure δ_Δ.

Comparing the SET [10, ch. 8] (its extended version is given by Theorem 3.2) with the approach in [9], [10] one can say that, speaking about seminorms generating β_0, β, and β_1, the most limiting is the assumption $(*'')$ of uniform continuity of the averages. If $(*'')$ holds, then SET works for signed kernels with more than one invariant probability, while the recurrence approach in [10] works mainly for probability kernels admitting at most one invariant probability. Nevertheless, this problem can be eluded by restricting of the original kernel to suitable subsets of X. SET also does not require $\mathcal{B}(X)$ to be countably generated, while for [10] it is a substantial assumption. The power of the recurrence approach can be illustrated by

PROPOSITION 4.1. *Let X be a coherent separable space and let P be a probability kernel with SFP. Then or P has a unique σ-finite invariant measure, or for each positive bounded measurable function f $\sum\limits_{n\geq 1} P^n f(x) < \infty$.*

Proof. In the proof of Theorem 3.5 we have shown that under the given assumptions each P invariant function must be constant. Therefore by [10, Thm 3.8] the kernel is or dissipative or recurrent. If it is recurrent, then by [10, Corol. 5.2] the σ-finite measure exists.

By uniqueness we mean uniqueness to a multiple by a constant. SET yields a valuable information about the quality of convergence of the averages $\frac{1}{n}\sum_{k=1}^{n} P^k f(x)$ in the x variable, while in [10] we can find criteria for uniform convergence in f, $\|f\| \leq 1$. But if (and only if) there exists a constant $c > 0$ and a probability m so that $P(x, E) \geq cmE$ for all $E \in \mathcal{B}(X)$, then $\operatorname{var}\big(P^n(x,\cdot) - \pi\big) \leq q^n M$ for all $n \in \mathbb{N}$, $x \in X$ and some positive constants $M < \infty$, $q < 1$, i.e. the convergence is uniform both in x and f. Here $f \in C(X)$ and π is the invariant probability for P.

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