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ON GRANDE'S PROBLEM CONCERNING $\mathcal{B}_1^*$ FUNCTIONS

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ABSTRACT. A function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ belongs to the $\mathcal{B}_1^*$ class if and only if for each non-empty perfect set $P \subset \mathbb{R}$, $f|_P$ is quasi continuous on some portion of P.

Let us establish some terminology to be used. We consider only real functions defined on subspaces of the real line $\mathbb{R}$. For a function $f\colon X \to \mathbb{R}$ we denote by $C(f)$ the set of all points at which f is continuous. Moreover, for a given $x \in X$, let $\operatorname{osc}_x(f)$ denote the oscillation of f at x. (Recall that $x \in C(f)$ if and only if $\operatorname{osc}_x(f) = 0$.)

For $A \subset X$ we denote by $\mathrm{cl}_X(A)$ and $\mathrm{int}_X(A)$ (or $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ if X is fixed) the closure and the interior of A in X, respectively.

A subset A of X is said to be a *portion* of X if and only if $A = J \cap X \neq \emptyset$ for some open interval $J \subset \mathbb{R}$. Let $A \subset X$. A maximal (with respect to inclusions) portion of A contained in X is called a "component" of A in X.

For $x \in \mathbb{R}$ and $\varepsilon > 0$ we denote by $\overline{B}(x, \varepsilon)$ the closed ball in $\mathbb{R}$ centred at x and with the radius ε.

DEFINITION 1. *A function f defined on X is said to be quasi-continuous at a point $x_0 \in X$ if and only if for every $\varepsilon > 0$ and for every neighbourhood $U \subset X$ of x_0 there exists an open set $V \subset X$ such that $\emptyset \neq V \subset U$ and $|f(x) - f(x_0)| < \varepsilon$ for every $x \in V$ ([SK]).*

f is said to be quasi-continuous if and only if it is quasi-continuous at each point $x \in X$.

The set of all points at which f is quasi-continuous will be denoted by $Q(f)$.

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DEFINITION 2. *A function $f\colon \mathbb{R} \to \mathbb{R}$ belongs to the $\mathcal{B}_1^*$ class if and only if for each non-empty perfect set $P \subset \mathbb{R}$ there exists a portion of P on which $f|_P$ is continuous* ([ROM], *see also* [Z] *and* [BK], *for the history of the class* $\mathcal{B}_1^*$).

Recall that $f\colon \mathbb{R} \to \mathbb{R}$ is a Baire one function if and only if $C(f|_P) \neq \emptyset$ for each non-empty perfect set $P \subset \mathbb{R}$ (see e.g. [KK, Theorem 1, p. 301]).

Moreover, the following fact is known (and easy to obtain).

FACT 1. *A function $f\colon \mathbb{R} \to \mathbb{R}$ is of the first Baire class if and only if $Q(f|_P)$ is non-void for each non-empty perfect set $P \subset \mathbb{R}$.*

Z. Grande [ZG] posed the problem whether the analogous characterization holds for the class $\mathcal{B}_1^*$. In this note we give an affirmative answer to this question.

LEMMA 1. *Assume that P_0 is a non-void perfect set, a function $f\colon P_0 \to \mathbb{R}$ is quasi-continuous, and $C(f)$ contains no portion of P_0. Then there exists a non-empty perfect set $P \subset P_0$ such that $Q(f|_P)$ contains no portion of P.*

Proof. For each $x \in P_0 \setminus C(f)$ let $\varepsilon_x = \operatorname{osc}_x(f) > 0$. In the whole proof of this lemma, $\mathrm{cl}(A)$, $\mathrm{int}(A)$ and "component" of A mean the closure, the interior and a component of A in P_0, for any $A \subset P_0$.

First choose $x \in P_0 \setminus C(f)$ and define

$$A_1 = \{x\}, \qquad P_1 = P_0 \setminus \mathrm{int}\left(f^{-1}\left(\overline{B}(f(x), \varepsilon_x/2)\right)\right).$$

Then

(i$_1$) $x \in P_1 \setminus \left(C(f) \cup Q(f|_{P_1})\right)$,

(ii$_1$) $\mathrm{int}(P_1)$ is dense in P_1 (this follows easily from the quasi-continuity of f).

Now let $(I_n)_n$ be the sequence of all "components" of the set $\mathrm{int}(P_1)$. For each n choose a point $x_n \in I_n \setminus C(f)$ and define

$$A_2 = \{x_n :\ n \in N\}, \qquad P_2 = P_1 \setminus \bigcup_n \Big(I_n \cap \mathrm{int}\left(f^{-1}\left(\overline{B}(f(x_n), \varepsilon_{x_n}/2)\right)\right)\Big).$$

Note that

(i$_2$) $A_1 \cup A_2 \subset P_2 \setminus \left(C(f) \cup Q(f|_{P_2})\right)$,

(ii$_2$) $\mathrm{int}(P_2)$ is dense in P_2.

In this way we choose by induction two sequences $(P_n)_n$ and $(A_n)_n$ of sets such that

(i$_n$) $\bigcup_{i \le n} A_i \subset P_n \setminus \left(C(f) \cup Q(f|_{P_n})\right)$, (ii$_n$) $\operatorname{int}(P_n)$ is dense in P_n,

(iii$_n$) if J is a "component" of P_n then $\operatorname{card}(J \cap A_n) = 1$.

Set

$$P = \operatorname{cl}\left(\bigcup_{n=1}^{\infty} A_n\right).$$

Then P is a non-empty perfect subset of P_0 and $f|_P$ is quasi-continuous at no point of $\bigcup_n A_n \subset P$. But $\bigcup_n A_n$ is dense in P, so $f|_P$ is quasi continuous on no portion of P. □

THEOREM 1. *A function $f\colon \mathbb{R} \to \mathbb{R}$ belongs to the $\mathcal{B}_1^*$ class if and only if for each non-empty perfect set $P \subset \mathbb{R}$, $f|_P$ is quasi continuous on some portion of P.*

Proof. Obviously only the implication " $\Longleftarrow$ " needs to be proved. Suppose that $f \notin \mathcal{B}_1^*$. Then there exists a non-empty perfect set $P_0 \subset \mathbb{R}$ such that $f|_{P_0}$ is continuous on no portion of P_0. Hence $P_0 \setminus C(f)$ is dense in P_0 and, by Lemma 1, there exists a non-empty perfect set P such that $f|_P$ is quasi-continuous on no portion of P; this is a contradiction. □

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