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Mathematica Slovaca, Vol. 42 (1992), No. 5, 615--619

Persistent URL: http://dml.cz/dmlcz/136568

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Mathematica Slovaca

METRIC RESULTS ON A NEW NOTION OF DISCREPANCY

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ABSTRACT. We prove a law of the iterated logarithm for a new notion of discrepancy of point sequences in the s-dimensional unit cube and give the connection to the usual discrepancy.

1. Introduction

In the theory of uniform distribution discrepancy is used to quantify the distribution behaviour of a given point sequence. The usual notion of discrepancy of a sequence $x_1, x_2, \dots, x_N$ of points in the unit cube I^s is given by

$$D_N^{\mathcal{A}}(x_1, \dots, x_N) = \sup_A \left| \frac{1}{N} \sum_{n=1}^{N} \chi_A(x_n) - \lambda(A) \right|, \tag{1.1}$$

where χ_A denotes the characteristic function of the set A and λ is the usual s-dimensional Lebesgue measure. The supremum is taken over a system $\mathcal{A}$ of subsets of I^s, e.g. boxes, cubes, balls or convex sets (cf. [K-N], [Hl]).

In a forthcoming paper Sobol and Nushdin [S-N] study a new notion of discrepancy, which seems to be more suitable for computational applications. We slightly modify their definition: we consider a partition $\mathcal{P} = \{\mathcal{A}_j\}$ of $\mathcal{A} = \bigcup_{j \in J} \mathcal{A}_j$ into disjoint classes $\mathcal{A}_j$ of sets of equal measure (j running through an index set J). For instance we put all translations of one cube or box into each set $\mathcal{A}_j$ or we gather all boxes of measure r into sets $\mathcal{A}_r$. Then we define

$$D_N^{\mathcal{P}}(x_1, \dots, x_N) = \frac{1}{N} \max_{j \in J} \left(\max_{A \in \mathcal{A}_j} \sum_{n=1}^{N} \chi_A(x_n) - \min_{B \in \mathcal{A}_j} \sum_{n=1}^{N} \chi_B(x_n) \right) \tag{1.2}$$

AMS Subject Classification (1991): Primary 60F15. Secondary 11K05.
Key words: Uniform distribution, Discrepancy, Law of the iterated logarithm.
[1)] The author is supported by the Austrian Science Foundation (Project Nr. P8274 PHY).

(Sobol and Nushdin consider dyadic boxes only). It can be seen by simple arguments, that

$$D_N^{\mathcal{A}} \le D_N^{\mathcal{P}} \le 2D_N^{\mathcal{A}} . \tag{1.3}$$

We use a very general form of the law of the iterated logarithm due to Philipp [Ph] to obtain a precise estimate for this notion of discrepancy, which is valid for almost all point sequences in the unit cube. Our result shows where in the interval $\left[D_N^{\mathcal{A}}, 2D_N^{\mathcal{A}}\right]$ $D_N^{\mathcal{P}}$ is most likely to be.

2. The Theorem

THEOREM 1. *Let $\mathcal{P}$ be a partition of the system $\mathcal{A}$ of all boxes or of all cubes in the unit cube I^s. Then*

$$\limsup_{N\to\infty} \frac{\sqrt{N} D_N^{\mathcal{P}}}{\sqrt{2\log\log N}} = \sigma$$

for almost all sequences in the unit cube, where

$$\sigma = \sup_{j\in J} \sup_{A,B\in\mathcal{A}_j} \sqrt{\lambda(A \bigtriangleup B)} ;$$

$A \bigtriangleup B$ denotes the symmetric difference of the two sets A and B.

In the following corollaries we will compute the constants σ for some special sets $\mathcal{A}_j$.

COROLLARY 1. *Let*

$$D_N^{\mathcal{T}}(x_1,\dots,x_N) = \frac{1}{N} \max_{0<r<1} \left(\max_{\boldsymbol{t}} \sum_{n=1}^{N} \chi_{\boldsymbol{t}+A_r}(x_n) - \min_{\boldsymbol{t}} \sum_{n=1}^{N} \chi_{\boldsymbol{t}+A_r}(x_n) \right),$$

where $A_r = [0,r)^s$. Then for almost all sequences

$$\limsup_{N\to\infty} \frac{\sqrt{N} D_N^{\mathcal{T}}}{\sqrt{2\log\log N}} = \frac{1}{\left(2 - 2^{\frac{1}{s-1}}\right)^{\frac{s-1}{2}}}$$

holds; the expression on the right hand side tends to $\frac{1}{\sqrt{2}}$ as s tends to ∞.

COROLLARY 2. *Let*

$$D_N^{\mathcal{R}}(x_1,\dots,x_N) = \frac{1}{N}\max_A\left(\max_t \sum_{n=1}^N \chi_{t+A}(x_n) - \min_t \sum_{n=1}^N \chi_{t+A}(x_n)\right),$$

where the maximum is taken over all boxes $A = \prod_{k=1}^{s}[0, r_k)$. *Then for almost all sequences*

$$\limsup_{N\to\infty} \frac{\sqrt{N} D_N^{\mathcal{R}}}{\sqrt{2\log\log N}} = 1$$

holds.

R e m a r k 1. Note that for the usual discrepancy almost surely

$$\limsup_{N\to\infty} \frac{\sqrt{N} D_N}{\sqrt{2\log\log N}} = \frac{1}{2}$$

holds (cf.[Ph]).

3. Proof of the Theorem

As indicated above the proof will use P h i l i p p 's uniform law of the iterated logarithm [Ph, Theorems 1.3.1., 1.3.2.] and his result on the usual discrepancy [Ph, Theorem 4.1.1.].

P r o o f. Let us note that the random variables $x_1, x_2, \dots$ are independent. Therefore we can use a simple version of the law of the iterated logarithm (cf.[Fe]) to obtain

$$\mathbb{P}\left(\limsup_{N\to\infty} \frac{\left|\sum_{n=1}^N \chi_A(x_n) - \sum_{n=1}^N \chi_B(x_n)\right|}{\sqrt{2N\log\log N}} = \sqrt{\lambda(A\triangle B)}\right) = 1 \tag{3.1}$$

for every pair of sets A, B with $\lambda(A) = \lambda(B)$ (not uniformly up to now!). We now use that the elements A of $\mathcal{A}_j$ indicated in the introduction are approximable by boxes $A' \subset A$, whose vertices are dyadic rationals with denominator 2^t, with an error $\lambda(A \setminus A') < 2s2^{-t}$. For the moment let us restrict (3.1) to these "dyadic boxes", then

$$\mathbb{P}\left(\limsup_{N\to\infty} \frac{\left|\sum_{n=1}^N \chi_{A'}(x_n) - \sum_{n=1}^N \chi_{B'}(x_n)\right|}{\sqrt{2N\log\log N}} = \sqrt{\lambda(A'\triangle B')}\right) = 1$$

uniformly in A' and B', as the countable intersection of sets of measure 1 has measure 1.

We now use the simple inequality

$$\sum_{n=1}^{N}\left(\chi_{A'}(x_n)-\chi_{B'}(x_n)-\chi_{B\setminus B'}(x_n)\right)\le\sum_{n=1}^{N}(\chi_A(x_n)-\chi_B(x_n))$$
$$\le\sum_{n=1}^{N}\left(\chi_{A'}(x_n)-\chi_{B'}(x_n)+\chi_{A\setminus A'}(x_n)\right)$$

and Philipp's uniform law of the iterated logarithm, which states that

$$\mathbb{P}\left(\limsup_{N\to\infty}\frac{\left|\sum_{n=1}^{N}\chi_A(x_n)-N\lambda(A)\right|}{\sqrt{2N\log\log N}}=\sqrt{\lambda(A)(1-\lambda(A))}\right)=1 \tag{3.2}$$

uniformly for all boxes $A\subset I^s$. Therefore we obtain that

$$\limsup_{N\to\infty}\left(\frac{\sum_{n=1}^{N}\chi_{A'}(x_n)-\sum_{n=1}^{N}\chi_{B'}(x_n)}{\sqrt{2N\log\log N}}-\frac{\sqrt{N}\lambda(B\setminus B')}{\sqrt{2\log\log N}}-\sqrt{\lambda(B\setminus B')}\right)$$
$$\le\limsup_{N\to\infty}\frac{\sum_{n=1}^{N}\chi_A(x_n)-\sum_{n=1}^{N}\chi_B(x_n)}{\sqrt{2N\log\log N}}$$
$$\le\limsup_{N\to\infty}\left(\frac{\sum_{n=1}^{N}\chi_{A'}(x_n)-\sum_{n=1}^{N}\chi_{B'}(x_n)}{\sqrt{2N\log\log N}}+\frac{\sqrt{N}\lambda(A\setminus A')}{\sqrt{2\log\log N}}+\sqrt{\lambda(A\setminus A')}\right)$$

almost sure. We now take $\lambda(B\setminus B')\le 2s2^{-t}<N^{-\frac{1}{2}}$ to finish the proof. □

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Received August 12, 1991

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