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PERIODIC SOLUTIONS OF THE THIRD ORDER PARAMETRIC DIFFERENTIAL EQUATIONS INVOLVING LARGE NONLINEARITIES

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ABSTRACT. Sufficient conditions are carried out for the existence of the periodic solutions to the triad of parametric third order equations. The main endeavour is that involved nonlinearities $a(x'')$ or $b(x')$ or $c(x)$ be not restricted too much. The uniqueness and the nonstability criteria are established in the special cases as well.

Introduction

This paper deals with the existence of periodic solutions to the following triad of differential equations

$$x''' + L(t,x) = a(x''), \tag{1}$$
$$x''' + L(t,x) = b(x'), \tag{2}$$
$$x''' + L(t,x) = c(x), \tag{3}$$

where $L(t,x) := f(t)x'' + g(t)x' + h(x) + p(t) + q(t,x,x',x'')$.
In the entire text we assume that the functions $f(t) \in C^1(\mathbf{R}^1)$, $g(t) \in C^2(\mathbf{R}^1)$, $p(t) \in C(\mathbf{R}^1)$ and $q(t,x,y,z) \in C(\mathbf{R}^4)$ are *ω-periodic in t*. Furthermore, let $h(x) \in C^1(\mathbf{R}^1)$ and $a(z), b(y), c(x) \in C(\mathbf{R}^1)$.

The main emphasis is focussed on the *nonlinearities* $a(z), b(y), c(x)$ in order to relax the conditions imposed on them as much as possible. Although several previous contributions, related usually to the equation $x''' + L(t,x) = 0$ with the constants $f(t) \equiv \hat{a}$, $g(t) \equiv \hat{b}$, are improved, extended or completed here, especially when $\hat{b} < 4\pi^2/\omega^2$ (cf. e.g. [1]–[10]), this could not be performed so simply if the equations would comprise a t variable in the functions a, b, c; even when comparing the results established by the standard methods only (cf. e.g. [11], [12]).

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Recently, mainly the equation $x''' + \hat{a}x'' + \hat{b}x' + h(t,x) = p(t) + q(t,x,x',x'')$ has been treated (see e.g. [13]–[18]) with special respect to put the restrictions concerning the quotient $h(t,x)/x$ more liberal. There are also known some earlier results (see e.g. [19], [20]) for the third order equations with the general right-hand side, but the obtained criteria are not very explicit, and thus practically not comparable with ours.

The existence of an *ω-periodic solution* of (1), (2), (3) is proved via the *(Poincaré) periodic boundary problem*, i.e. in order that the solution $x(t)$ of the given equations satisfies the conditions

$$x^{(j)}(0) = x^{(j)}(\omega), \qquad j = 0,1,2 \quad (\omega > 0). \tag{P}$$

It is clear that such a solution can be extended on the whole real axis in the ω-periodic way.

In order to perform this, we apply the following standard *Leray-Schauder alternative* (cf. [3, p. 103]).

Proposition. *If all ω-periodic solutions of the one-parametric family of equations*

$$x''' + \mu L(t,x) + (1-\mu)ex = \mu w(x,x',x''), \qquad \mu \in \langle 0,1\rangle, \tag{S}$$

where $w(x,y,z)$ denotes $a(z)$ or $b(y)$ or $c(x)$ and e is a suitable nonzero real, are uniformly a priori bounded together with their first and second derivatives, independently of $\mu \in \langle 0,1\rangle$, and the linear equation, resulting from (S) *for $\mu = 0$, has no nontrivial ω-periodic solution, then the equation obtained from* (S) *for $\mu = 1$ admits a harmonic.*

Remark. One can readily check that the second requirement follows immediately for every nonzero e, and consequently, we restrict ourselves to verify the first condition only.

The main tool for proving the a priori estimates required in Proposition will be, besides the well-known *Schwarz inequality*, the following *Wirtinger-type inequality* (cf. e.g. [5], [10], [16] and the references included)

$$\int_0^\omega [x^{(k)}]^2(t)\,\mathrm{d}t \le \omega_0^2 \int_0^\omega [x^{(k+1)}]^2(t)\,\mathrm{d}t, \qquad k = 1,2, \tag{W}$$

where $x(t) \in C^3\langle 0,\omega\rangle$ satisfies (P), $\omega_0 = \omega/2\pi$.

Hence, we proceed by the standard method developed by Villari in [2]. More concretely, the same technique is applied here to equation (3), similarly as it was performed, e.g., in [3], [5], [10] and many other papers (see also the references included and survey article [6]) for much simpler equations than (3). For equations (1) and (2) the appropriate modification of such a manner is employed in the analogous way.

Concerning the notation, we use for the sake of brevity the symbol $q[t]$ for the composed function $q(t,x(t),x'(t),x''(t))$. At last, the "plus" constant, say M^+, means the following

$$M^+ := \begin{cases} M & \text{for} \quad M > 0 \\ 0 & \text{for} \quad M \le 0. \end{cases}$$

1. Equation (1)

Theorem 1. *Let the following assumptions be satisfied:*

(i) $\int\limits_0^\omega p(t)\,\mathrm{d}t = 0$,

(ii) $\exists\, H'$ *(a nonnegative constant)*: $|h'(x)| \le H'$ *for all* x,

(iii) $\exists\, \alpha,\beta,\gamma$ *(nonnegative constants)*: $|q(t,x,y,z)| \le \alpha|z| + \beta|y| + \gamma$, *uniformly for all* t,x, *where* $(\omega_0 := \omega/2\pi)$
$\Omega := 1 - \left[\alpha\omega_0 + (\beta + M^+)\omega_0^2 + H'\omega_0^3 + \frac{1}{2}G_2\omega_0^4\right] > 0$,
$M := \max\limits_{t\in\langle 0,\omega\rangle}\left[\frac{1}{2}f'(t) + g(t)\right]$, $G_2 := \max\limits_{t\in\langle 0,\omega\rangle}[-g''(t)]$,

(iv) $\exists\, R$ *(a positive constant)*: $h(x)\operatorname{sgn} x > h$ *or* $h(x)\operatorname{sgn} x < -h$ *for* $|x| > R$, *where* $h := A + \gamma + (N+\beta)D' + \alpha D''$,
$A := \max\limits_{|z|\le D''}|a(z)|$, $D'' := \omega(P+\gamma)/\Omega$, $D' := \omega_0 D''$,
$P := \max\limits_{t\in\langle 0,\omega\rangle}|p(t)|$, $N := \max\limits_{t\in\langle 0,\omega\rangle}|f'(t) - g(t)|$.

Then equation (1) *admits an* ω*-periodic solution.*

Proof. Consider

$$x''' + \mu L(t,x) + (1-\mu)ex = \mu a(x''), \tag{S$_1$}$$

where $\mu \in \langle 0,1\rangle$ is a parameter and $e \ne 0$ is a suitable constant. Let $x(t)$ be a solution of (S_1)-(P).

Substituting $x(t)$ into (S_1), multiplying it by $x'''(t)$ and integrating the obtained identity from 0 to ω, we obtain

$$\int\limits_0^\omega [x''']^2(t)\,\mathrm{d}t + \mu\int\limits_0^\omega L(t,x(t))x'''(t)\,\mathrm{d}t = 0.$$

Integrating the first, the second and the third term in $L(t,x(t))x'''(t)$ by parts, we get by means of (W) and the Schwarz inequality that [cf. (ii)–(iv)]

$$\int_0^\omega [x''']^2(t)\,\mathrm{d}t \le \left[\frac{1}{2}G_2\omega_0^4 + M^+\omega_0^2 + H'\omega_0^3 + (\alpha\omega_0 + \beta\omega_0^2)\right] \cdot$$

$$\cdot \int_0^\omega [x''']^2(t)\,\mathrm{d}t + (P+\gamma)\sqrt{\omega}\sqrt{\int_0^\omega [x''']^2(t)\,\mathrm{d}t},$$

i.e. [see (iii)]

$$\int_0^\omega [x''']^2(t)\,\mathrm{d}t \le \omega(P+\gamma)^2/\Omega^2 := D_3^2.$$

Applying (W) again, we come to

$$\int_0^\omega [x'']^2(t)\,\mathrm{d}t \le \omega_0^2 D_3^2 := D_2^2,$$
$$\int_0^\omega [x']^2(t)\,\mathrm{d}t \le \omega_0^2 D_2^2 := D_1^2.$$

Since (according to Rolle's theorem) there exist points $t_1, t_2 \in (0,\omega)$, such that $x'(t_1) = 0 = x''(t_2)$, we arrive at the inequalities

$$|x'(t)| \le \int_0^\omega |x''(t)|\,\mathrm{d}t \le \sqrt{\omega\int_0^\omega [x'']^2(t)\,\mathrm{d}t} = \sqrt{\omega}D_2 := D', \tag{4}$$

$$|x''(t)| \le \int_0^\omega |x'''(t)|\,\mathrm{d}t \le \sqrt{\omega\int_0^\omega [x''']^2(t)\,\mathrm{d}t} = \sqrt{\omega}D_3 := D''. \tag{5}$$

Now, substituting $x(t)$ into (S_1) and integrating from 0 to ω, we obtain [cf. (i)]

$$\int_0^\omega [\mu h(x(t)) + (1-\mu)ex(t)]\,\mathrm{d}t = \mu\int_0^\omega \left\{a(x''(t)) - [g(t) - f'(t)]x'(t) - q[t]\right\}\mathrm{d}t.$$

If $\min\limits_{t\in\langle 0,\omega\rangle} |x(t)| > R$, then choosing $e \neq 0$ in order to be [see (iv)] $eh(x)x > 0$, we get by means of (iii), (iv), (4), (5) that

$$\int_0^{\omega} \left|h\big(x(t)\big)\right| \mathrm{d}t \leq h\omega,$$

when multiplying the foregoing identity by $\operatorname{sgn} ex$. This, however, leads to the contradiction with (iv). Therefore

$$\min_{t\in\langle 0,\omega\rangle} |x(t)| \leq R,$$

and consequently

$$|x(t)| \leq R + \int_0^{\omega} |x'(t)| \,\mathrm{d}t \leq R + \sqrt{\omega} D_1 := D \qquad \text{for} \quad t \in \langle 0,\omega\rangle. \tag{6}$$

It follows from (4), (5), (6) that $\sum\limits_{j=0}^{2} |x^{(j)}(t)| \leq D + D' + D''$ holds for every ω-periodic solution of (S_1), independently of $\mu \in (0,1\rangle$.

Hence, applying Proposition, the proof is completed.

2. Equation (2)

Theorem 2. *Let the following assumptions be satisfied:*

(i) $\int\limits_0^{\omega} p(t)\,\mathrm{d}t = 0$,

(ii) $\exists\, \varepsilon$ *(a positive constant):* $|f(t)| \geq \varepsilon$ *for* $t \in \langle 0,\omega\rangle$,

(iii) $\exists\, H_1$ *(a constant):* $h'(x) \operatorname{sgn} f(t) \leq H_1$ *for all* x,

(iv) $\exists\, \alpha,\beta,\gamma$ *(nonnegative constants):* $|q(t,x,y,z)| \leq \alpha|z| + \beta|y| + \gamma$, *uniformly for all* t, x, *where* ($\omega_0 := \omega/2\pi$)
$\Omega := \varepsilon - \omega_0^2\Big(\frac{1}{2}G_1^+ + H_1^+\Big) - (\alpha + \beta\omega_0) > 0$,
$G_1 := \max\limits_{t\in\langle 0,\omega\rangle} [g'(t) \operatorname{sgn} f(t)]$,

(v) $\exists\, R$ *(a positive constant):* $h(x) \operatorname{sgn} x \operatorname{sgn} f(t) < -h$ *for* $|x| > R$, *where*
$h := B + \gamma + (\beta + N)D' + \alpha D_2/\sqrt{\omega}$,
$N := \max\limits_{t\in\langle 0,\omega\rangle} |f'(t) - g(t)|$, $B := \max\limits_{|y|\leq D'} |b(y)|$,
$D' := \sqrt{\omega} D_2$, $D_2 := \sqrt{\omega}(P + \gamma)/\Omega$, $P := \max\limits_{t\in\langle 0,\omega\rangle} |p(t)|$.

Then equation (2) *admits an* ω*-periodic solution.*

P r o o f. Consider

$$x''' + \mu L(t,x) + (1-\mu)ex = \mu b(x'), \tag{S_2}$$

where $\mu \in \langle 0,1\rangle$ is a parameter and $e \neq 0$ is a suitable constant again. Let $x(t)$ be a solution of (S_2)–(P).

Substituting $x(t)$ into (S_2), multiplying it by $x''(t)$ and integrating the obtained identity from 0 to ω, we have

$$\mu \int_0^\omega L\big(t,x(t)\big)x''(t)\,\mathrm{d}t = (1-\mu)e\int_0^\omega [x']^2(t)\,\mathrm{d}t.$$

Integrating the second and the third terms in $L\big(t,x(t)\big)x''(t)$ by parts, we get

$$\mu\left\{\int_0^\omega f(t)[x'']^2(t)\,\mathrm{d}t - \int_0^\omega \Big[\frac{1}{2}g'(t) + h'\big(x(t)\big)\Big][x']^2(t)\,\mathrm{d}t + \int_0^\omega [p(t)+q[t]]\,x''(t)\,\mathrm{d}t\right\}$$
$$= (1-\mu)e\int_0^\omega [x']^2(t)\,\mathrm{d}t.$$

Choosing e in order to be $ef(t) < 0$, we get furthermore by means of (W) and the Schwarz inequality, when multiplying the last relation by $\operatorname{sgn} f(t)$ that [cf. (ii)–(iv)]

$$\varepsilon\int_0^\omega [x'']^2(t)\,\mathrm{d}t \le \int_0^\omega |f(t)|[x'']^2(t)\,\mathrm{d}t$$
$$\le \operatorname{sgn} f(t)\int_0^\omega \Big[\frac{1}{2}g'(t) + h'\big(x(t)\big)\Big][x']^2(t)\,\mathrm{d}t + \int_0^\omega |p(t)+q[t]||x''(t)|\,\mathrm{d}t$$
$$\le \Big[\omega_0^2\Big(\frac{1}{2}G_1^+ + H_1^+\Big) + (\alpha+\beta\omega_0)\Big]\int_0^\omega [x'']^2(t)\,\mathrm{d}t + (P+\gamma)\sqrt{\omega}\sqrt{\int_0^\omega [x'']^2(t)\,\mathrm{d}t},$$

i.e. [see (iv)]

$$\int_0^\omega [x'']^2(t)\,\mathrm{d}t \le \omega\ P+\gamma)^2/\Omega\ \ \cdot\!- D_2^2.$$

Applying (W) again, we come to

$$\int_0^\omega [x']^2(t)\,\mathrm{d}t \le \omega_0^2 D_2^2 := D_1^2,$$

and consequently

$$|x'(t)| \le \int_0^\omega |x''(t)|\,\mathrm{d}t \le \sqrt{\omega} D_2 := D', \tag{7}$$

with respect to the existence of a point $t_1 \in (0,\omega)$ such that $x'(t_1) = 0$, implied by Rolle's theorem.

Now, substituting $x(t)$ into (S_2) and integrating from 0 to ω, we obtain [cf. (i)]

$$\int_0^\omega \big[\mu h\big(x(t)\big) + (1-\mu) e x(t)\big]\,\mathrm{d}t = \mu \int_0^\omega \big\{b\big(x'(t)\big) - \big[g(t) - f'(t)\big]x'(t) - q[t]\big\}\,\mathrm{d}t.$$

Because of $ef(t) < 0$, we would get for $\min_{t\in\langle 0,\omega\rangle} |x(t)| > R$ that [see (iv), (v), (7)]

$$\int_0^\omega \big|h\big(x(t)\big)\big|\,\mathrm{d}t \le h\omega,$$

when multiplying the foregoing identity by $\operatorname{sgn} x$ and $\operatorname{sgn} f(t)$, a contradiction to (v). Therefore

$$\min_{t\in\langle 0,\omega\rangle} |x(t)| \le R,$$

and consequently

$$|x(t)| \le R + \int_0^\omega |x'(t)|\,\mathrm{d}t \le R + \sqrt{\omega} D_1 := D \qquad \text{for} \quad t \in \langle 0,\omega\rangle. \tag{8}$$

At last, substituting $x(t)$ into (S_2), multiplying it by $x'''(t)$ and integrating from 0 to ω, we obtain in a similar way as above that [cf. (iv), (v)]

$$\int_0^\omega [x''']^2(t)\,\mathrm{d}t \le \int_0^\omega |x'''(t)|\{[\alpha + |f(t)|]\,|x''(t)| + E\}\,\mathrm{d}t$$

$$\le \left\{\int_0^\omega [x''']^2(t)\,\mathrm{d}t \int_0^\omega [\alpha + |f(t)|]^2 [x'']^2(t)\,\mathrm{d}t\right\}^{\frac{1}{2}} + \sqrt{\omega}E\sqrt{\int_0^\omega [x''']^2(t)\,\mathrm{d}t}$$

$$\le [(\alpha + F)D_2 + \sqrt{\omega}E]\sqrt{\int_0^\omega [x''']^2(t)\,\mathrm{d}t},$$

where $E := B+\gamma+H+P+(G+\beta)D'$, $F := \max_{t\in\langle 0,\omega\rangle} |f(t)|$, $G := \max_{t\in\langle 0,\omega\rangle} |g(t)|$, $H := \max_{|x|\le D} |h(x)|$.

Hence, $\int_0^\omega [x''']^2(t)\,\mathrm{d}t \le [(\alpha + F)D_2 + \sqrt{\omega}E]^2 := D_3^2$,
and consequently

$$|x''(t)| \le \int_0^\omega |x'''(t)|\,\mathrm{d}t \le \sqrt{\omega}D_3 := D'', \tag{9}$$

with respect to the existence of a point $t_2 \in (0,\omega)$ such that $x''(t_2) = 0$, implied by Rolle's theorem.

Summarizing (7), (8), (9), we arrive at the same conclusion as in the proof of Theorem 1, when applying Proposition.

3. Equation (3)

Theorem 3. *Let the following assumptions be satisfied:*

(i) $\int_0^\omega p(t)\,\mathrm{d}t = 0$,

(ii) $\exists\, \alpha, \beta, \gamma$ *(nonnegative constants)*: $|q(t,x,y,z)| \le \alpha|z| + \beta|y| + \gamma$, *uniformly for all* t, x, *where* ($\omega_0 := \omega/2\pi$)
$\Omega := 1 - \alpha\omega_0 - (K^+ + \beta)\omega_0^2 > 0$,
$K := \max_{t\in\langle 0,\omega\rangle} \left[g(t) - \frac{1}{2}f'(t)\right]$,

(iii) $\exists R$ *(a positive constant)*: $h(x)\operatorname{sgn} x > h$ *or* $h(x)\operatorname{sgn} x < -h$ *for* $R \le |x| \le D$,
$h := C + \gamma + (\beta + N)D' + \alpha D_2/\sqrt{\omega}$,
$N := \max_{t\in\langle 0,\omega\rangle} |f'(t) - g(t)|$, $C := \max_{|x|\le D} |c(x)|$,
$D := R + \sqrt{\omega}\omega_0 D_2$, $D_2 := \omega\omega_0^2(P+\gamma)^2/\Omega^2$, $D' := \sqrt{\omega} D_2$,
$P := \max_{t\in\langle 0,\omega\rangle} |p(t)|$.

Then equation (3) *admits an* ω*-periodic solution.*

P r o o f. Instead of $x''' + \mu L(t,x) + (1-\mu)ex = \mu c(x)$, consider

$$x''' + \mu\big[f(t)x'' + g(t)x' + h^*(x) + p(t) + q(t,x,x',x'')\big] + (1-\mu)ex = \mu c^*(x), \quad (\mathrm{S}_3^*)$$

where

$$h^*(x) := \begin{cases} h(x) & \text{for } |x| \le D \\ h(D\operatorname{sgn} x) & \text{for } |x| \ge D, \end{cases} \qquad c^*(x) := \begin{cases} c(x) & \text{for } |x| \le D \\ c(D\operatorname{sgn} x) & \text{for } |x| \ge D, \end{cases}$$

$\mu \in \langle 0,1\rangle$ is a parameter and $e \ne 0$ is a suitable constant. Let $x(t)$ be a solution of (S_3^*)-(P).

Substituting $x(t)$ into (S_3^*), multiplying it by $x'(t)$ and integrating the obtained identity from 0 to ω, we get

$$\int_0^\omega [x'']^2(t)\,\mathrm{d}t = \mu \int_0^\omega \big[f(t)x''(t) + g(t)x'(t) + p(t) + q[t]\big]x'(t)\,\mathrm{d}t.$$

Integrating the first term on the right-hand side of the last relation by parts, we get by means of (W) and the Schwarz inequality that [cf. (ii), (iii)]

$$\int_0^\omega [x'']^2(t)\,\mathrm{d}t \le \int_0^\omega \Big[g(t) - \frac{1}{2}f'(t)\Big][x']^2(t)\,\mathrm{d}t + \int_0^\omega |p(t) + q[t]||x'(t)|\,\mathrm{d}t$$

$$\le \big[\alpha\omega_0 + (K^+ + \beta)\omega_0^2\big]\int_0^\omega [x'']^2(t)\,\mathrm{d}t + \sqrt{\omega}\omega_0(P+\gamma)\sqrt{\int_0^\omega [x'']^2(t)\,\mathrm{d}t},$$

i.e. [see (ii)]

$$\int_0^\omega [x'']^2(t)\,\mathrm{d}t \le \omega\omega_0^2(P+\gamma)^2/\Omega^2 := D_2^2.$$

Applying (W) again, we obtain

$$\int_0^\omega [x']^2(t)\,\mathrm{d}t \le \omega_0^2 D_2^2 := D_1^2,$$

and consequently

$$|x'(t)| \le \sqrt{\omega} D_2 := D' \tag{10}$$

with respect to the existence of a point $t_1 \in (0,\omega)$ such that $x'(t_1) = 0$, implied by Rolle's theorem.

Now, substituting $x(t)$ into (S_3^*) and integrating from 0 to ω, we obtain [cf. (i)]

$$\int_0^\omega \left[\mu h^*\big(x(t)\big) + (1-\mu)ex(t)\right] \mathrm{d}t = \mu \int_0^\omega \left\{\left[f'(t) - g(t)\right]x'(t) + c^*\big(x(t)\big) - q[t]\right\} \mathrm{d}t$$

If $\min\limits_{t\in\langle 0,\omega\rangle} |x(t)| > R$, then choosing $e \ne 0$ in order to be [see (iii)] $eh^*(x)x > 0$, we get by means of (ii), (iii), (10) that

$$\int_0^\omega \left|h^*\big(x(t)\big)\right| \mathrm{d}t \le h\omega,$$

when multiplying th fore oing identity by $\operatorname{sgn} x$. This, however, leads to the contradiction with (iii). Tl ıefore,

$$\min_{t\in\langle 0,\omega\rangle} |x(t)| \le R,$$

and consequently

$$|x(t)| \le R + \sqrt{\omega} D_1 := D \qquad \text{for} \quad t \in \langle 0,\omega\rangle. \tag{11}$$

Finally, substituting $x(t)$ into (S_3^*), multiplying it by $x'''(t)$ and integrating from 0 to ω, we obtain in a similar vay as above that [cf. (ii), (iii)],

$$\int_0^\omega [x''']^2(t)\,\mathrm{d}t \le \left[(\alpha + F)D_2 + \sqrt{\omega}E\right]^2 := D_3^2,$$

wh re $E := C^* + \gamma + H^* + P + (G + \beta)D'$, $\quad F := \max\limits_{t\in\langle 0,\omega\rangle} |f(t)|$,

$G := \max\limits_{t\in\langle 0,\omega\rangle} |g(t)|$, $\quad C^* := C$, $\quad H^* := \max\limits_{|x|\le D} |h^*(x)|$.

Since a point $t_2 \in (0,\omega)$ exists such that $x''(t_2) = 0$, according to Rolle's theorem, we arrive at

$$|x''(t)| \leq \sqrt{\omega} D_3 := D''. \tag{12}$$

Hence, applying Proposition to the equation resulting from (S_3^*) for $\mu = 1$, an ω-periodic solution $x(t)$ exists with respect to (10), (11), (12), which obviously satisfies (3) as well, because of $|x(t)| \leq D$ for $t \in \langle 0,\omega \rangle$.

This completes the proof.

4. Uniqueness and stability result

Let us conclude by the consideration of the special cases of (1), (2), (3), namely

$$x''' + a(x'') + \hat{b}\, x' + \hat{c}\, x = p(t), \tag{1_0}$$

$$x''' + \hat{a}\, x'' + b(x') + \hat{c}\, x = p(t), \tag{2_0}$$

$$x''' + \hat{a}\, x'' + \hat{b}\, x' + c(x) = p(t), \tag{3_0}$$

where $p(t)$ is ω-periodic again.

These equations have been treated by G. Villari [1], who has found (besides the uniqueness and nonstability criteria), with the superfluous requirement concerning the existence of a bounded solution, sufficient conditions for an ω-periodic solution as follows

$$\hat{b} < 0 \ \wedge \ \hat{c} > 0 \ \wedge \ [a(z_1) - a(z_2)](z_1 - z_2) < 0 \quad \text{for } z_1 \neq z_2, \tag{I}$$

$$\hat{a} < 0 \ \wedge \ \hat{c} > 0 \ \wedge \ [b(y_1) - b(y_2)](y_1 - y_2) < 0 \quad \text{for } y_1 \neq y_2, \tag{II}$$

$$\hat{a} < 0 \ \wedge \ \hat{b} < 0 \ \wedge \ [c(x_1) - c(x_2)](x_1 - x_2) > 0 \quad \text{for } x_1 \neq x_2, \tag{III}$$

respectively.

Since a point $x \in \mathbb{R}^1$ exists necessarily such that $c(\bar{x}) = \bar{p} = \dfrac{1}{\omega}\displaystyle\int_0^\omega p(t)\,dt$ for (3_0) under (III) related to $c(x)$ [otherwise there would be a contradiction after the integration of (3_0)], it is clear that the assumption imposed on $c(x)$ in (III) can be regarded as a special case of the condition

$$\exists R \text{ (a positive constant)}: [c(x) - \bar{p}]x > 0 \qquad \text{for} \quad |x| > R.$$

Therefore, the following assertion represents the essential generalization of Villari's result in [1].

Theorem 4. *Let for* (1_0) *or* (2_0) *or* (3_0) *the conditions*

$$\hat{b} < \omega_0^{-2} \;\wedge\; \hat{c} \neq 0 \quad or \quad \hat{a}\hat{c} < 0 \quad or$$
$$\hat{b} < \omega_0^{-2} \;\wedge\; \exists R > 0\colon [c(x) - \bar{p}]x > 0 \quad or \quad [c(x) - \bar{p}]x < 0 \quad for \quad |x| > R,$$
$$where \quad \bar{p} = \frac{1}{\omega}\int_0^\omega p(t)\,\mathrm{d}t$$

be satisfied, respectively.
Then equation (1_0) *or* (2_0) *or* (3_0)*, respectively, admits an* ω*-periodic solution. If* (I) *or* (II) *or* (III) *is valid, moreover, then there exists exactly one nonstable (in the sense of Liapunov)* ω*-periodic solution of the above equations, while every bounded solution tends to it.*

Proof. The first part of our assertion can be proved just in the same way as in the theorems above, but with system (S) replaced by

$$\begin{aligned}
&x''' + \mu[a(x'') + \hat{b}\, x'\ ] + \hat{c}\, x \;= \mu p(t),\\
&x''' + \mu[\hat{a}\, x'' \ + b(x')] + \hat{c}\, x \;= \mu p(t),\\
&x''' + \mu[\hat{a}\, x'' \ + \hat{b}\, x' \ + c(x)] + (1-\mu)ex = \mu p(t), \quad \text{respectively,}
\end{aligned}$$

using the same approach (some criteria are simplified here). The second part follows immediately from Villari's result in [1].

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