Mathematica Slovaca

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Mathematica Slovaca, Vol. 38 (1988), No. 4, 341--344

Persistent URL: http://dml.cz/dmlcz/136474

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REMARK ON AN INTEGRAL OF M. MATLOKA

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Recently M. Matloka has constructed a Riemann type integral for functions $f\colon \langle a, b\rangle \to L(R)$, where $L(R)$ is a special set of so-called fuzzy numbers. Of course, the set $L(R)$ has a natural algebraic and topological structure: it becomes an ordered space and simultaneously a metric space. This remark contains an abstract point of view of the Matloka theory. We give assumptions under which the corresponding generalizations of the Riemann-Matloka integral have the expected properties. Recall that the space $L(R)$ with his usual operations does not form a linear space. Therefore our point of view may be useful.

First we shall consider an ordered structure.

1. Assumptions. There is given a partially ordered set A satisfying the following properties:

1.1. A is a boundedly complete lattice.

1.2. There is given a commutative and associative operation $+$ on A with a neutral element O, preserving the ordering (i.e. $x \leqq y \Rightarrow x + z \leqq y + z$).

1.3. There is given a multiplication of elements of A by real numbers, associative, preserving the ordering (i.e. $x \leqq y, c > 0, d < 0 \Rightarrow cx \leqq cy, dx \geqq dy$) and such that $1x = x$.

2. Definition. *If $f\colon \langle a, b\rangle \to A$ is a bounded function and $D = \{x_0, \ldots, x_n\}$ is a decomposition of $\langle a, b\rangle$, then we first define the lower and upper sums*

$$\bar{S}(f, D) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}),\ \underline{S}(f, D) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$

and then the lower and upper integrals

$$(U)\int_a^b f(x)\,\mathrm{d}x = \inf\{\bar{S}(f, D);\ D \text{ is a partition of } \langle a, b\rangle\},$$

$$(L)\int_a^b f(x)\,\mathrm{d}x = \sup\{\underline{S}(f, D);\ D \text{ is a partition of } \langle a, b\rangle\}.$$

The function f is integrable if $(U)\int_a^b f(x)\,\mathrm{d}x = (L)\int_a^b f(x)\,\mathrm{d}x$.

The common value will be denoted by $(O)\int_a^b f(x)\,\mathrm{d}x$.

3. Proposition. *If f, g are integrable functions and α, β are real numbers, then $\alpha f + \beta g$ is integrable, too, and*

$$(O)\int_a^b (\alpha f(x) + \beta g(x))\,\mathrm{d}x = \alpha(O)\int_a^b f(x)\,\mathrm{d}x + \beta(O)\int_a^b g(x)\,\mathrm{d}x.$$

If $f \leqq g$, then $(O)\int_a^b f(x)\,\mathrm{d}x \leqq (O)\int_a^b g(x)\,\mathrm{d}x$.

Proof. It is straightforward.

4. Proposition. *If f is integrable on $\langle a, b\rangle$ and $c \in (a, b)$, then f is integrable on $\langle a, c\rangle$ and $\langle c, b\rangle$ and*

$$(O)\int_a^b f(x)\,\mathrm{d}x = (O)\int_a^c f(x)\,\mathrm{d}x + (O)\int_c^b f(x)\,\mathrm{d}x.$$

Proof. It follows from the inequalities

$$(U)\int_a^b f(x)\,\mathrm{d}x \geqq (U)\int_a^c f(x)\,\mathrm{d}x + (U)\int_c^b f(x)\,\mathrm{d}x \geqq$$

$$\geqq (L)\int_a^c f(x)\,\mathrm{d}x + (L)\int_c^b f(x)\,\mathrm{d}x \geqq (L)\int_a^b f(x)\,\mathrm{d}x.$$

Now the second point of view.

5. Assumptions. Let (A, d) be a complete metric space satisfying the following conditions:

5.1. There is given a commutative and associative operation + on A with a neutral element and satisfying the identities

$$d(a + b, c + d) \leqq d(a, c) + d(b, d) \quad \text{and} \quad d(a, b) \leqq d(a + c, b + c).$$

5.2. There is given a multiplication of elements of A by real numbers such that $0\,a = 0$ and the identities $\lambda(a + b) = \lambda a + \lambda b$, $d(\lambda a, \lambda b) = |\lambda| d(a, b)$ are satisfied.

6. Definition. *Let (A, d) be a metric space satisfying the assumptions 5. A function $\langle a, b\rangle \to A$ is called integrable if there is $I \in A$ such that to every $\varepsilon > 0$ there is $\delta > 0$ such that for every decomposition D with the norm $\|D\| < \delta$ we have $d(S(f, D), I) < \varepsilon$ ($S(f, D)$ is an arbitrary integral sum). the element I will be denoted by* $\int_a^b f(x)\,\mathrm{d}x$.

7. Proposition. *If f, g are integrable, then $\alpha f + \beta g$ is integrable too and*

$$\int_a^b (\alpha f(x) + \beta g(x))\,\mathrm{d}x = \alpha \int_a^b f(x)\,\mathrm{d}x + \beta \int_a^b g(x)\,\mathrm{d}x.$$

8. Proposition. *If $(f_n)_n$ is a sequence of integrable functions converging uniformly on f, then f is integrable and*

$$\lim_{n \to \infty} d\left(\int_a^b f(x)\,\mathrm{d}x,\ \int_a^b f_n(x)\,\mathrm{d}x\right) = 0.$$

9. Proposition. *If f is integrable on $\langle a, b\rangle$, then it is integrable on $\langle a, c\rangle$ and $\langle c, b\rangle$ and*

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x.$$

Proof. The only interesting point is to prove that f is integrable on $\langle a, c\rangle$. It follows by the following Bolzano-Cauchy criterion: $\forall \varepsilon > 0\ \exists \delta > 0\ \forall D_1, D_2$: $\|D_1\| < \delta \wedge \|D_2\| < \delta \Rightarrow d(S(f, D_1), S(f, D_2)) < \varepsilon$. Indeed, if this condition is satisfied, then we can choose to $\varepsilon = \dfrac{1}{n+1}$ corresponding δ_n and then put $A_n = \{S(f, D);\ \|D\| < \max_{i \le n} \delta_i\}$. Then diam $\bar{A}_n < \dfrac{1}{n}$ and the element I can be obtained by $\{I\} = \bigcap_{n=1}^{\infty} \bar{A}_n$.

10. Examples. The most interesting example is the set $L(R)$ of all fuzzy numbers, i.e. functions μ: $R \to \langle 0, 1\rangle$ satisfying the following properties:

1. There is $x_0 \in R$ such that $\mu(x_0) = 1$.
2. There is a compact set $K \subset R$ such that $\{x;\ \mu(x) > 0\} \subset K$.
3. For every $\alpha \in (0, 1\rangle$ the set $\mu_\alpha = \{x;\ \mu(x) \geqq \alpha\}$ is convex.
4. μ is upper semicontinuous, i.e. $\{x;\ \mu(x) < \alpha\}$ is open for every $\alpha \in \langle 0, 1\rangle$.

It follows that $\mu_\alpha = \langle a_\alpha, b_\alpha\rangle$ for every $\alpha \in (0, 1\rangle$. If $\nu_\alpha = \langle c_\alpha, d_\alpha\rangle$; then we define $\mu \leqq \nu$ if $a_\alpha \leqq c_\alpha$, $b_\alpha \leqq d_\alpha$ for every α and we define $\mu + \nu$ by $(\mu + \nu)_\alpha = \langle a_\alpha + c_\alpha,\ b_\alpha + d_\alpha\rangle$ and $\lambda\mu$ by $(\lambda\mu)_\alpha = \langle \lambda a_\alpha, \lambda b_\alpha\rangle$ for $\lambda \geqq 0$, $(\lambda\mu)_\alpha = \langle \lambda b_\alpha, \lambda a_\alpha\rangle$ for $\lambda < 0$. It is not difficult to see that $L(R)$ satisfies the assumptions 1. Another example of a set A satisfying these assumptions is any boundedly complete linear lattice.

If we define $d(\mu, \nu) = \sup\{d(\mu_\alpha, \nu_\alpha);\ \alpha \in \langle 0, 1\rangle\}$, where $d(\mu_\alpha, \nu_\alpha) = \max\{|c_\alpha - a_\alpha|, |d_\alpha - b_\alpha|\}$, then also the assumptions 5 are satisfied. Another example satisfying 5 is any Banach space with $d(a, b) = \|a - b\|$.

REFERENCES

[1] MATLOKA, M.: On an integral of fuzzy mappings. Busefal. 31, 1987, 45 55.
[2] MATLOKA, M.: On a fuzzy integral. In: Proc. Polish Symp. Interval and Fuzzy Mathematics. Poznań 1987, 163—170.

Received March 24, 1987

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ЗАМЕТКА ОБ ИНТЕГРАЛЕ М. МАТЛОКИ

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Резюме

В теории Матлоки изучаются отображения с значениями в множестве $L(R)$ так называемых нечётких чисел. В настоящей работе показано, что множество $L(R)$ можно заменить упорядоченным пространством или метрическим пространством.