Ján Šipoš

Primitive idempotents in convergence semigroups

*Mathematica Slovaca*, Vol. 37 (1987), No. 3, 263--265

Persistent URL: http://dml.cz/dmlcz/136452

Math. Slovaca 37, 1987, No. 3, 263—265

# PRIMITIVE IDEMPOTENTS IN CONVERGENCE SEMIGROUPS

JÁN ŠIPOŠ

In [2], Koch raised the question whether the primitive idempotents of a compact semigroup with zero form a closed set. This question has been answered affirmatively for a wide class of semigroups (even in a semitopological case) by Chow [1]. His theorem states:

**Theorem 1.** *If $S$ is a semitopological semigroup with zero in which the set $E$ of idempotents of $S$ is commutative and closed, then the set $P$ of primitive idempotents is closed.*

We show, by giving an example, that for convergence semigroups Koch's question cannot be answered affirmatively in general. (Unfortunately the convergence structure of this semigroup cannot be topologized.) For this, we shall need the notion of a convergence space and the notion of a convergence semigroup.

A *convergence space* $F$ is a set $F$ with a distinguished class of sequences $\{a_n\}$ ($a_n \in F$) which are called *convergent*. We assume that to each convergent sequence there corresponds a unique element $a$ of $F$, called *the limit* of the sequence and denoted by $a = \lim_n a_n$ (or simply $a_n \to a$) such that $\lim_n a_n = a$ if $a_n = a$ for $n = 1, 2, \dots$. We assume also that if $a_n \to a$ then $a_{n_k} \to a$, where $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$.

We do not assume that this convergence is determined by a topology. However, it is clear that a topological space is also a convergence space.

The *sequential closure* $vA$ of a set $A \subset F$ is the set of all limits of all convergment sequences $\{a_n\}$ taking their values in $A$ (i.e. $a_n \in A$). If $vA = A$ we say that $A$ is *sequentially closed.*

A *convergence semigroup* $S$ is a semigroup provided with a convergence structure in which multiplication is continuous, i.e., if $a_n \to a$ and $b_n \to b$, then $a_n b_n \to ab$ (the elements $a_n$, $b_n$, $a$ and $b$ being in $S$).

A convergence semigroup $S$ is called *sequentially compact* iff every sequence $\{a_n\}$ of elements from $S$ contains a convergent subsequence.

We can express our result as follows:

**Theorem 2.** *There exists a sequentially compact semigroup $S$ with zero, in which the set $P$ of primitive idempotents of $S$ is not closed.*

We present now our exxample:

3. Example Let $S$ be a semigroup generated by the set $\{0, e, f, e_1, e_2, \ldots\}$ defined by the following equations: $ee = e$, $ff = ef = fe = f$, $e_i e_i = e_i$ for $i = 1, 2, \ldots$, $x0 = 0x = 0$ for $x \in \{e, f, 0, e_1, e_2, \ldots\}$. Then the set of idempotents of $S$ is

$$E = \{0, e, f, e_1, e_2, \ldots\}.$$

The set $P$ of the primitive idempotents of $S$ is

$$P = \{0, f, e_1, e_2, \ldots\}.$$

(Recall that 0 is considered as a primitive idempotent.)

Define now a convergence structure on $\{e, f, e_1, e_2, \ldots\}$. We put $e_{n_k} \to e$ if $\{n_k\}$ is a subsequence of $\{1, 2, 3, \ldots\}$ and put $\lim_n c_n = c$ if $c_n = c$ for "almost all" $n$, $c$ being in $\{e, f, e_1, e_2, \ldots\}$. Note that this convergence is a topological one, since in fact it is the one point compactification of the discrete topological space $\{f, e_1, e_2, \ldots\}$ with the point $e$.

Now if $a_n = a_{1,n} \,.\, a_{2,n} \,.\, \ldots \,.\, a_{k,n}$, $a = a_1 \,.\, a_2 \,.\, \ldots \,.\, a_k (a_i, a_{i,n} \in E - \{0\})$ and $a_{i,n_j} \to$ $\to a_i$ for $i = 1, 2, \ldots, k$ then we put $\lim_n a_n = a$.

For $a \in S - \{0\}$ let $h(a)$ be the length of the shortest word in the free semigroup generated by $\{e, f, e_1, e_2, \ldots\}$ factorizing to $a$, for instance

$$h(efe_1ff) = h(fe_1f) = 3.$$

In addition we put $h(0) = \infty$. We say that $a_n \to 0$ if $\lim_n h(a_n) = \infty$.

It is easy to see that the set $S$ with the convergence defined above becomes a convergence space.

**3.1.** Let us show at first that $S$ is sequentially compact with respect to this convergence structure.

Let $\{a_n\}$ be a sequence of elements of $S$.

a) If $h(a_n) \to \infty$ then clearly $a_n \to 0$.

b) Let $h(a_n) \nrightarrow \infty$ then there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ with $h(a_{n_k}) =$ $= m$ (where $m$ is a fixed integer). Put $b_k = a_{n_k}$ then $b_k = b_{1,k} \,.\, b_{2,k} \,.\, \ldots \,.\, b_{m,k}$ where $b_{i,j} \in \{e, f, e_1, e_2, \ldots\}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots$. Assume first that $m = 2$. Let $b_{1,k_1} \to b_1$ and $b_{2,k_1} \to b_2$, then $\{b_{k_{l_i}}\}$ is a convergent subsequence of $\{a_n\}$. In general one can proceed by induction, hence $S$ is a sequentially compact convergence space.

**3.2.** Let us turn our attention to the continuity of the multiplication in $S$. Let $a_n \to a$ and $b_n \to b$, if $a = 0$ then, since $h(a_n) \to \infty$ involves $h(a_n b_n) \to \infty$, we get $a_n b_n \to 0 = a \,.\, b$.

If $a_n \to a$ and $b_n \to b$ with $a \neq 0 \neq b$ then $a_n b_n \to ab$ follows immediatelly from the definition of the convergence in $S$.

**3.3.** It is clear that $e \in vP = v\{0, f, e_1, e_2, \dots\}$. Hence $P$ is not closed.

Note that our convergence is not a topological one, i.e., there exists a set $A \subset S$ with $v(vA) \supseteqq vA$. In fact it is sufficient to take

$$A = \{e_i(ee_j)^i;\ i, j = 1, 2, \dots\},$$

then $e \in v(vA) - vA$.

## REFERENCES


[1] CHOW, H. L.: Primitive idempotents in semigroups. Semigroup Forum, Vol. 10 (1975), 373—374.

[2] KOCH, R. J.: Remarks on primitive idempotents in compact semigroups with zero. Proc. Amer. Math. Soc. 5 (1954), 828—833.



*Katedra matematiky a deskriptívnej geometrie*
*Stavebnej fakulty SVŠT*
*Radlinského 11*
*813 68 Bratislava*


## ПРИМИТИВНЫЕ ИДЕМПОТЕНТЫ В ПОЛУГРУППАХ СХОДИМОСТИ


Ján Šipoš



Резюме

В статье показано, что примитивные идемпотенты компактной полугруппы сходимости в общем случае не образуют замкнутое множество.