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DIRECTLY DECOMPOSABLE CONGRUENCES IN VARIETIES WITH NULLARY OPERATIONS

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A class of algebras $\mathscr{C}$ has *directly decomposable congruences* if for any two algebras $\mathfrak{A}, \mathfrak{B} \in \mathscr{C}$ and each congruence $\Theta \in Con(\mathfrak{A} \times \mathfrak{B})$ there exist congruences $\Theta_1 \in Con(\mathfrak{A})$, $\Theta_2 \in Con(\mathfrak{B})$ such that $\Theta = \Theta_1 \times \Theta_2$. Varieties of algebras with directly decomposable congruences were characterized by a rather complicated Mal'cev condition in [4]. A simpler Mal'cev condition characterizing direct decomposability of congruences in the case of permutable or 3-permutable variety can be found in [2] or [3]. The aim of this paper is to show how the original Mal'cev condition (derived by G. A. Fraser and A. Horn) can be simplified in the case of a variety with a nullary operation and what other varieties can satisfy the modified definition of decomposability.

It was proved in [4] that a class $\mathscr{C}$ has directly decomposable congruences if and only if for any two $\mathfrak{A}, \mathfrak{B} \in \mathscr{C}$ and any $a_1, a_2 \in \mathfrak{A}$, $b_1, b_2 \in \mathfrak{B}$,

$$\Theta([a_1, b_1], [a_2, b_2]) = \Theta(a_1, a_2) \times \Theta(b_1, b_2).$$

In other words, $\mathscr{C}$ has directly decomposable congruences if and only if it has directly decomposable principal congruences. This property is used in the next definition:

Definition. *Let $\mathscr{C}$ be a class of algebras of the same type containing the nullary operation c. $\mathscr{C}$ has c-directly decomposable congruences if for each $\mathfrak{A}, \mathfrak{B} \in \mathscr{C}$ and each $x_1 \in \mathfrak{A}$, $x_2 \in \mathfrak{B}$,*

$$\Theta([c, c], [x_1, x_2]) = \Theta(c, x_1) \times \Theta(c, x_2).$$

Lemma. *Let $\mathscr{V}$ be a variety with a nullary operation c. The following conditions are equivalent:*

(a) *$\mathscr{V}$ has c-directly decomposable congruences;*
(b) *for every $\mathfrak{A}, \mathfrak{B} \in \mathscr{V}$ and each $a \in \mathfrak{A}$, $b, d \in \mathfrak{B}$,*

$$\langle [c, d], [a, d] \rangle \in \Theta([c, c], [a, b]).$$

Proof. (a) ⇒ (b): Since

$$\langle [c, d], [a, d] \rangle \in \Theta(c, a) \times \omega_B \subseteq \Theta(c, a) \times \Theta(c, b) = \Theta([c, c], [a, b]),$$

the implication (a) ⇒ (b) is evident.

(b) ⇒ (a): Apply (b) onto $\mathfrak{B} \times \mathfrak{A}$; we obtain

$$\langle [c, e], [b, e] \rangle \in \Theta([c, c], [b, a]).$$

Using the canonical homomorphism $\mathfrak{B} \times \mathfrak{A} \to \mathfrak{A} \times \mathfrak{B}$, we have immediately

$$\langle [e, c], [e, b] \rangle \in \Theta([c, c], [a, b]).$$

From it and (b) we obtain

$$\begin{aligned} \Theta(c, a) \times \omega_B &\subseteq \Theta([c, c], [a, b]) \\ \omega_A \times \Theta(c, b) &\subseteq \Theta([c, c], [a, b]). \end{aligned}$$

The transitivity implies

$$\Theta(c, a) \times \Theta(c, b) \subseteq \Theta([c, c], [a, b]).$$

The converse inclusion is evident.

Theorem. *Let $\mathscr{V}$ be a variety with a nullary operation c. The following conditions are equivalent*:

(1) *$\mathscr{V}$ has c-directly decomposable congruences*;

(2) *there exist $(2 + n)$-ary polynomials $p_1, \ldots, p_m$, unary polynomials $q_1, \ldots, q_n$ and binary polynomials $r_1, \ldots, r_n$ such that*

$$\begin{aligned} &c = p_1(c, x, q_1(x) \ldots, q_n(x)) \\ &x = p_m(x, c, q_1(x), \ldots, q_n(x)) \\ &p_i(x, c, q_1(x), \ldots, q_n(x)) = p_{i+1}(c, x, q_1(x), \ldots, q_n(x)) \\ &\qquad\qquad \text{for } i = 1, \ldots, m - 1 \\ &z = p_1(c, y, r_1(y, z), \ldots, r_n(y, z)) \\ &z = p_m(y, c, r_1(y, z), \ldots, r_n(y, z)) = \\ &\qquad p_{i+1}(c, y, r_1(c, y, r_1(y, z), \ldots, r_n(y, z)) \\ &\qquad\qquad \text{for } i = 1, \ldots, m - 1. \end{aligned}$$

Proof. (1) ⇒ (2): Let $\mathscr{V}$ be a variety with a nullary operation c which has c-directly decomposable congruences. Let $\mathfrak{A} = \mathfrak{F}_1(x)$ or $\mathfrak{B} = \mathfrak{F}_2(y, z)$ be free algebras in $\mathscr{V}$ with free generators x or y, z, respectively. By the Lemma, we have clearly

$$\langle [c, z], [x, z] \rangle \in \Theta([c, c], [x, y]).$$

Then, by the Mal'cev Lemma (see, e. g., [5]), there exist $(2 + n)$-ary polynomials $p_1, \ldots, p_m$ such that

$$[c, z] = p_1([c, c], [x, y], v_1, \ldots, v_n)$$
$$[x, z] = p_m([x, y], [c, c], v_1, \ldots, v_n)$$
$$p_i([x, y], [c, c], v_1, \ldots v_n) = p_{i+1}([c, c], [x, y], v_1, \ldots, v_n) \quad \text{for } i = 1, \ldots, m - 1,$$

where $v_i \in \mathfrak{A} \times \mathfrak{B} = \mathfrak{F}_1(x) \times \mathfrak{F}_2(y, z)$. Hence, there exist unary polynomials q_i and binary polynomials r_i such that

$$v_i = [q_i(x), r_i(y, z)].$$

Putting these terms instead of v_i into the foregoing identities, we obtain (2).

(2) ⇒ (1): Let $\mathscr{V}$ be a variety with a nullary operation c and satisfying identities (2). Let $a \in \mathfrak{A}$ and $b, d \in \mathfrak{B}$. Putting a, b, d into (2), we obtain

$$\langle [c, d], [a, d] \rangle \in \Theta([c, c], [a, b]).$$

By the Lemma, this implies (1).

Clearly, every variety $\mathscr{V}$ with a nullary operation c which has directly decomposable congruences has also c-directly decomposable congruences. The following example shows that there are also varieties having c-directly decomposable congruences but have not directly decomposable congruences.

Example 1. Let $\mathscr{V}$ be a *variety of join semilattices* with a nullary operation 0 (the lest element). Then $\mathscr{V}$ has 0-directly decomposable congruences.

We can put $n = m = 2$ and
$$p_1(x_1, x_2, x_3, x_4) = x_1 \vee x_3$$
$$p_2(x_1, x_2, x_3, x_4) = x_2 \vee x_4,$$

$$q_1(x) = 0,\ q_2(x) = x,\ r_1(y, z) = r_2(y, z) = z.$$

Then

$$p_1(0, x, q_1(x), q_2(x)) = 0 \vee 0 = 0$$
$$p_2(x, 0, q_1(x), q_2(x)) = 0 \vee x = x$$
$$p_1(x, 0, q_1(x), q_2(x)) = x \vee 0 = x = p_2(0, x, q_1(x), q_2(x))$$
$$p_1(0, y, r_1(y, z), r_2(y, z)) = 0 \vee z = z$$
$$p_2(y, 0, r_1(y, z), r_2(y, z)) = 0 \vee z = z$$
$$p_1(y, 0, r_1(y, z), r_2(y, z)) = y \vee z = p_2(0, y, r_1(y, z), r_2(y, z)).$$

On the contrary, the variety $\mathscr{V}$ has not directly decomposable congruences. It can be easily shown if we take, e.g., the two-element join semilattice $\mathfrak{J} = \{0, x\}$ and put $\mathfrak{S} = \mathfrak{J} \times \mathfrak{J}$. Then clearly the principal congruence $\Theta = \Theta([0, x], [x, 0])$ on $\mathfrak{S}$ is not directly decomposable.

Example 2. Let $\mathscr{V}$ be a variety of *implication algebras*, i.e. the variety with one binary and one nullary operation denoted by . and 1, satisfying the identities

$$(a.b).a = a$$
$$(a.b).b = (b.a).a$$
$$a.(b.c) = b.(a.c)$$
$$a.a = 1,$$

see, e.g., [1]. Put $n = m = 2$ and

$$p_1(x_1, x_2, x_3, x_4) = x_1.x_3, \quad p_2(x_1, x_2, x_3, x_4) = x_2.x_4$$
$$q_1(x) = 1, \quad q_2(x) = x, \quad r_1(y, z) = r_2(y, z) = z.$$

Clearly

$$p_1(1, x, q_1(x), q_2(x)) = 1.1 = 1$$
$$p_1(x, 1, q_1(x), q_2(x)) = x.1 = 1 = x.x = p_2(1, x, q_1(x), q_2(x))$$
$$p_2(x, 1, q_1(x), q_2(x)) = 1.x = x$$

$$p_1(1, y, r_1(y, z), r_2(y, z)) = 1.z = z$$
$$p_1(y, 1, r_1(y, z), r_2(y, z)) = y.z = p_2(1, y, r_1(y, z), r_2(y, z))$$
$$p_2(y, 1, r_1(y, z), r_2(y, z)) = 1.z = z.$$

Hence, $\mathscr{V}$ has 1-directly decomposable congruences. We can show that $\mathscr{V}$ has in general no directly decomposable congruences. Take the three-element implication algebra $\mathfrak{J} = \{a, b, 1\}$ with $a.b = b$, $b.a = a$, $1.x = x$ and $x.y = 1$ for any other combination of $x, y \in \{a, b, 1\}$. Put $\mathfrak{A} = \mathfrak{J} \times \mathfrak{J}$ and let $\Theta = \Theta([a, 1], [1, b])$. Clearly Θ is not directly decomposable.

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ПРЯМОЕ РОЗЛОЖЕНИЕ КОНГРУЭНЦИЙ В МНОГООБРАЗИЯХ С НУЛЯРНЫМИ ОПЕРАЦИЯМИ

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Резюме

Дается условие Мальцева для многообразия $\mathscr{V}$ с нулярной операцией c, удовлетворяющие следующему условию для главных конгруэнций:

$$\Theta(c, x) = \Theta_1 \times \Theta_2 \qquad (\Theta_1 \in Con(\mathfrak{A}),\ \Theta_2 \in Con(\mathfrak{B}))$$

для каждово элемента $x \in \mathfrak{A} \times \mathfrak{B}$ и любых $\mathfrak{A}, \mathfrak{B} \in \mathscr{V}$.