Mathematica Slovaca

Štefan Varga

Minimum variance quadratic unbiased estimation of variance components

Mathematica Slovaca, Vol. 36 (1986), No. 2, 163--170

Persistent URL: http://dml.cz/dmlcz/136420

Math. Slovaca 36, 1986, No. 2, 163—170

MINIMUM VARIANCE QUADRATIC UNBIASED ESTIMATION OF VARIANCE COMPONENTS

ŠTEFAN VARGA

Introduction

Let us consider the linear model

$$\mathbf{Y} = \mathbf{X}\beta + \mathbf{e}, \tag{1}$$

where $\mathbf{X}$ is a given $n \times p$-matrix, β an unknown p-vector of parameters and $\mathbf{e}$ a random n-vector with expectation zero and covariance matrix

$$D(\mathbf{e}) = D(\mathbf{Y}) = \theta_1\mathbf{V}_1 + \ldots + \theta_m\mathbf{V}_m = \mathbf{V}_\theta. \tag{2}$$

The matrices $\mathbf{V}_i$ $(i = 1, 2, \ldots, m)$ are known symmetric $n \times n$-matrices. We are interested in the estimation of the unknown parameter vector $\theta = (\theta_1, \ldots, \theta_m)$ belonging to the set $\mathcal{O}$ of all $\theta \in \mathcal{R}^m$ such that $\mathbf{V}_\theta$ becomes positive definite (*p. d.*).

Assume that the matrix $\mathbf{S}$ containing prior values of elements of the covariance matrix $\mathbf{V}_\theta$ is known (the (i, j)-*th* element of the matrix $\mathbf{S}$ is a prior value of the (i, j)-*th* element of the matrix $\mathbf{V}_\theta$ for $i, j = 1, 2, \ldots, n$).

A quadratic estimation of the linear function

$$q = \sum_{i=1}^{m} f_i\theta_i = \mathbf{f}'\theta \tag{3}$$

of θ will be considered in the form

$$\hat{q}(\mathbf{Y}, \mathbf{V}, \mathbf{S}) = \mathbf{Y}'\mathbf{A}(\mathbf{V}, \mathbf{S})\mathbf{Y}, \tag{4}$$

where the matrix $\mathbf{V}$ is defined by

$$\mathbf{V} = \left\langle \begin{array}{l} \mathbf{V}_1 + \ldots + \mathbf{V}_m \\ \alpha_1\mathbf{V}_1 + \ldots + \alpha_m\mathbf{V}_m \end{array} \right. \tag{5}$$

in dependence on whether a prior value $\alpha = (\alpha_1, \ldots, \alpha_m)'$ of $\theta = (\theta, \ldots, \theta)'$ is unknown (first row) or known (second row).

A natural question is how the knowledge of the matrix **S** contributes to estimating the variance components $\theta_1, \ldots, \theta_m$ and the function (3).

1. Symbols and auxiliary statements

Let $(\mathcal{A}, \langle .,. \rangle)$ be a Hilbert space of symetric $n \times n$ matrices, $\langle .,. \rangle$ denotes the inner product given by $\langle \mathbf{A}, \mathbf{B} \rangle = tr\, \mathbf{AB}$, $\mathbf{A}, \mathbf{B} \in \mathcal{A}$; here $tr\, \mathbf{C}$ denotes the trace of the matrix **C**. The matrix **A** in (4) is from $\mathcal{A}$.

The natural estimator of the function (3) in the linear model (1) is defined by the formula

$$\mathbf{e}_*' \sum_{i=1}^{m} \lambda_i \mathbf{V}^{-1/2} \mathbf{V}_i \mathbf{V}^{-1/2} \mathbf{e}_* \tag{6}$$

(see *(5.4.3)* in [4]), where $\mathbf{e}_* = \mathbf{V}^{-1/2}\mathbf{e}$ and the vector $\lambda = (\lambda_1, \ldots, \lambda_m)'$ is any solution of the linear system

$$\mathbf{M}\lambda = \mathbf{f}. \tag{7}$$

The (i, j)-*th* element of the matrix **M** is $\{\mathbf{M}\}_{i,j} = tr\, \mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{V}_j$ and $\mathbf{f} = (f_1, \ldots, f_m)'$.

Definition 1.1. ([2]) A minimum norm invariant unbiased quadratic estimator (MINQUE(U, I)) of the function $\mathbf{f}'\theta$ is a statistic $\mathbf{Y}'\mathbf{AY}$, where the matrix **A** minimizes the expression $tr\, \mathbf{AVAV}$ in the class $\mathcal{A}_1$.

$$\mathcal{A}_1 = \{\mathbf{A} \in \mathcal{A}: \mathbf{AX} = \mathbf{0};\ tr\, \mathbf{AV}_i = \mathbf{f}_i,\ i = 1, 2, \ldots, m\} \tag{8}$$

Lemma 1.2. The MINQUE(U, I) of the function $\mathbf{f}'\theta$ is the statistic $\mathbf{Y}'\mathbf{AY}$, where $\mathbf{A} = \sum_{i=1}^{m} \delta_i \mathbf{Q}_V' \mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{Q}_V$, the expression $\mathbf{Q}_V = \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-}\mathbf{X}'\mathbf{V}^{-1}$ and the vector $\delta = (\delta_1, \ldots, \delta_m)'$ is any solution of the linear system $\mathbf{B}\delta = \mathbf{f}$. The (i, j)-*th* element of the matrix **B** is $tr\, \mathbf{V}_j\mathbf{Q}_V\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{Q}_V$. **I** is a unite matrix.

Proof. See [4], *Theorem 5.1.1* and *Note 2.*

Lemma 1.3. If $\mathbf{Y} \sim N_n(\mathbf{X}\beta, \sum_{i=1}^{m} \theta_i \mathbf{V}_i)$ and $\mathbf{A} \in \mathcal{A}_1$, then for the variance of the random variable $\mathbf{Y}'\mathbf{AY}$ the following holds

$$D_\theta(\mathbf{Y}'\mathbf{AY}) = 2\, tr\, \mathbf{AV}_\theta\mathbf{AV}_\theta. \tag{9}$$

Proof. See [1], *Theorem* 1.

2. Natural estimation and S-estimation

Using the transformation $\mathbf{e} = \mathbf{S}^{1/2}\varepsilon$ ($\varepsilon = \mathbf{S}^{-1/2}\mathbf{e}$) in the linear model (1), the natural estimator (6) of $\mathbf{f}'\theta$ is

$$\varepsilon'\mathbf{N}\varepsilon=\varepsilon'\sum_{i=1}^{m}\chi_i\mathbf{S}^{1/2}\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{S}^{1/2}\varepsilon, \tag{10}$$

where the vector $\chi=(\chi_1, \ldots, \chi_m)'$ is any solution of the linear system

$$\mathbf{M}\chi=\mathbf{f}. \tag{11}$$

The matrix $\mathbf{M}$ is defined in (7).

The considered quadratic estimator (4) with respect to the transformation $\mathbf{e}=\mathbf{S}^{1/2}\varepsilon$ is

$$\hat{q}(\mathbf{Y}, \mathbf{V}, \mathbf{S})=\mathbf{Y}'\mathbf{A}\mathbf{Y}=(\mathbf{X}\beta+\mathbf{S}^{1/2}\varepsilon)'\mathbf{A}(\mathbf{X}\beta+\mathbf{S}^{1/2}\varepsilon)=$$
$$=(\varepsilon', \beta')\begin{pmatrix}\mathbf{S}^{1/2}\mathbf{A}\mathbf{S}^{1/2} & \mathbf{S}^{1/2}\mathbf{A}\mathbf{X}\\ \mathbf{X}'\mathbf{A}\mathbf{S}^{1/2} & \mathbf{X}'\mathbf{A}\mathbf{X}\end{pmatrix}\begin{pmatrix}\varepsilon\\ \beta\end{pmatrix}. \tag{12}$$

The difference between the considered estimator (12) and the natural estimator (10) of the function $\mathbf{f}'\theta$ is

$$\mathbf{Y}'\mathbf{A}\mathbf{Y}-\varepsilon'\mathbf{N}\varepsilon=(\varepsilon', \beta')\begin{pmatrix}\mathbf{S}^{1/2}\mathbf{A}\mathbf{S}^{1/2}-\mathbf{N} & \mathbf{S}^{1/2}\mathbf{A}\mathbf{X}\\ \mathbf{X}'\mathbf{A}\mathbf{S}^{1/2} & \mathbf{X}'\mathbf{A}\mathbf{X}\end{pmatrix}\begin{pmatrix}\varepsilon\\ \beta\end{pmatrix} \tag{13}$$

The minimum norm quadratic estimation which is a function of the matrix $\mathbf{S}$ (MINQE ($\mathbf{S}$)) is obtained by minimizing the Euclidean norm of the matrix $\mathbf{H}$ of the quadratic form (13) defined as follows

$$\mathbf{H}=\begin{pmatrix}\mathbf{S}^{1/2}\mathbf{A}\mathbf{S}^{1/2}-\mathbf{N} & \mathbf{S}^{1/2}\mathbf{A}\mathbf{X}\\ \mathbf{X}'\mathbf{A}\mathbf{S}^{1/2} & \mathbf{X}'\mathbf{A}\mathbf{X}\end{pmatrix} \tag{14}$$

The square of the Euclidean norm of $\mathbf{H}$ is

$$||\mathbf{H}||^2=tr\ (\mathbf{S}^{1/2}\mathbf{A}\mathbf{S}^{1/2}-\mathbf{N})^2+2tr\ \mathbf{X}'\mathbf{A}\mathbf{S}\mathbf{A}\mathbf{X}+tr\ (\mathbf{X}'\mathbf{A}\mathbf{X})^2. \tag{15}$$

We consider the class of invariant unbiased quadratic estimators of the function $\mathbf{f}'\theta$ in the linear model (1), i.e. the class of the statistics $\mathbf{Y}'\mathbf{A}\mathbf{Y}$, where the matrix $\mathbf{A}$ belongs to the class $\mathcal{A}_1$ defined in (8).

Definition 2.1. A minimum norm unbiased invariant quadratic estimator (MINQE(U, I, $\mathbf{S}$)) of the function $\mathbf{f}'\theta$ is a statistic $\hat{q}(\mathbf{Y}, \mathbf{V}, \mathbf{S})=\mathbf{Y}'\mathbf{A}\mathbf{Y}$, where the matrix $\mathbf{A}$ minimizes the expression (15) in the class $\mathcal{A}_1$.

Lemma 2.2. A quadratic estimator $\mathbf{Y}'\mathbf{A}\mathbf{Y}$ of the function $\mathbf{f}'\theta$ is the MINQE(U, I, $\mathbf{S}$) if the matrix $\mathbf{A}$ minimizes the expression

$$tr\ \mathbf{A}\mathbf{S}\mathbf{A}\mathbf{S}-2\sum_{i=1}^{m}\chi_i\ tr\ \mathbf{S}\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{S}\mathbf{A} \tag{16}$$

in the class $\mathcal{A}_1$.

Proof. As the matrix $\mathbf{A}$ satisfies the condition $\mathbf{A}\mathbf{X}=\mathbf{0}$ the expression (15) is of the following form

$$||\mathbf{H}||^2 = tr\,(\mathbf{S}^{1/2}\mathbf{A}\mathbf{S}^{1/2} - \mathbf{N})^2 = tr\,\mathbf{S}^{1/2}\mathbf{A}\mathbf{S}\mathbf{A}\mathbf{S}^{1/2} - 2tr\,\mathbf{S}^{1/2}\mathbf{A}\mathbf{S}^{1/2}\mathbf{N} + tr\,\mathbf{N}^2 =$$

$$= tr\,\mathbf{ASAS} - 2\sum_{i=1}^{m} \chi_i tr\,\mathbf{S}\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{S}\mathbf{A} + tr\,\mathbf{N}^2.$$

The expression $tr\,\mathbf{N}^2$ is indenpendent of the matrix $\mathbf{A}$ and hence we have the MINQE(U, I, $\mathbf{S}$) of $\mathbf{f}'\,\theta$ in the form $\mathbf{Y}'\mathbf{A}\mathbf{Y}$, where the matrix $\mathbf{A}$ minimizes the expression (16) in the class $\mathcal{A}_1$.

Theorem 2.3. a) The MINQE(U, I, $\mathbf{S}$) of the function $\mathbf{f}'\theta$ is the statistic $\mathbf{Y}'\mathbf{A}_1\mathbf{Y}$, where

$$\mathbf{A}_1 = \sum_{i=1}^{m} \chi_i \mathbf{Q}_S'\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{Q}_S - \sum_{i=1}^{m} \gamma_i \mathbf{Q}_S'\mathbf{S}^{-1}\mathbf{V}_i\mathbf{S}^{-1}\mathbf{Q}_S \tag{17}$$

and $\mathbf{Q}_S = \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-}\mathbf{X}'\mathbf{S}^{-1}$, the vector $\chi = (\chi_1, \ldots, \chi_m)'$ is any solution of the linear system (11) and the vector $\gamma = (\gamma_1, \ldots, \gamma_m)'$ is any solution of the linear system

$$\sum_{i=1}^{m} \gamma_i\, tr\,\mathbf{V}_j\mathbf{Q}_S'\mathbf{S}^{-1}\mathbf{V}_i\mathbf{S}^{-1}\mathbf{Q}_S = \sum_{i=1}^{m} \chi_i\, tr\,\mathbf{V}_j\mathbf{Q}_S'\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{Q}_S - f_j \tag{18}$$

for $j = 1, 2, \ldots, m$.

b) The MINQE(U, I, $\mathbf{S}$) of the function $\mathbf{f}'\theta$ exists if and only if the systems (11) and (18) are consistent.

Proof. It is evident that the matrix $\mathbf{A}_1$ is symmetric. The equation $\mathbf{A}_1\mathbf{X} = \mathbf{0}$ is satisfied because of $\mathbf{Q}_S\mathbf{X} = \mathbf{0}$. The equations $tr\,\mathbf{A}_1\mathbf{V}_j = \mathbf{f}_j$ for $j = 1, 2, \ldots, m$ are satisfied because the equation (18) holds. It suffices to prove that the matrix $\mathbf{A}_1$ minimizes the expression (16) in the class $\mathcal{A}_1$.

Let $\mathbf{D}$ be a matrix for which

$$\mathbf{D}' = \mathbf{D};\ \mathbf{DX} = \mathbf{0};\ tr\,\mathbf{DV}_i = \mathbf{0}, \quad i = 1, 2, \ldots, m \tag{19}$$

holds. The matrix $\mathbf{A}_1$ minimizes the expression (16) in the class $\mathcal{A}_1$ if for each matrix $\mathbf{D}$ which satisfies the conditions (19) $tr\,(\mathbf{A}_1 + \mathbf{D})\mathbf{S}(\mathbf{A}_1 + \mathbf{D})\mathbf{S} - 2\sum_{i=1}^{m} \chi_i\, tr\,\mathbf{S}\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{S}(\mathbf{A}_1 + \mathbf{D}) \geqq tr\,\mathbf{A}_1\mathbf{S}\mathbf{A}_1\mathbf{S} - 2\sum_{i=1}^{m} \chi_i\, tr\,\mathbf{S}\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{S}\mathbf{A}_1$ holds.

$$tr\,(\mathbf{A}_1 + \mathbf{D})\mathbf{S}(\mathbf{A}_1 + \mathbf{D})\mathbf{S} - 2\sum_{i=1}^{m} \chi_i\, tr\,\mathbf{S}\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{S}(\mathbf{A}_1 + \mathbf{D}) =$$

$$= tr\,\mathbf{A}_1\mathbf{S}\mathbf{A}_1\mathbf{S} - 2\sum_{i=1}^{m} \chi_i\, tr\,\mathbf{S}\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{S}\mathbf{A}_1 + tr\,\mathbf{DSDS} +$$

$$+ 2\, tr\,\mathbf{A}_1\mathbf{SDS} - 2\sum_{i=1}^{m} \chi_i\, tr\,\mathbf{S}\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{SD}.$$

With regard to the fact that the expression $tr\ \mathbf{DSDS}$ is nonnegative it suffices to prove that $\sum_{i=1}^{m}\chi_i\ tr\ \mathbf{SV}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{SD} = tr\ \mathbf{A}_1\mathbf{SDS}$.

$$tr\ \mathbf{A}_1\mathbf{SDS} = tr\ \Big(\sum_{i=1}^{m}\chi_i\mathbf{Q}_S'\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{Q}_S - \sum_{i=1}^{m}\gamma_i\mathbf{Q}_S'\mathbf{S}^{-1}\mathbf{V}_i\mathbf{S}^{-1}\mathbf{Q}_S)\mathbf{SDS} =$$

$$= \sum_{i=1}^{m}\chi_i\ tr\ \mathbf{SQ}_S'\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{Q}_S\mathbf{SD} - \sum_{i=1}^{m}\gamma_i\ tr\ \mathbf{SQ}_S'\mathbf{S}^{-1}\mathbf{V}_i\mathbf{S}^{-1}\mathbf{Q}_S\mathbf{SD} =$$

$$= \sum_{i=1}^{m}\chi_i\ tr\ (\mathbf{SV}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{SD} - \mathbf{X}(\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-}\mathbf{X}'\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{SD}) -$$

$$- \sum_{i=1}^{m}\gamma_i\ tr\ (\mathbf{SS}^{-1}\mathbf{V}_i\mathbf{S}^{-1}\mathbf{SD} - \mathbf{X}(\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-}\mathbf{X}'\mathbf{S}^{-1}\mathbf{V}_i\mathbf{S}^{-1}\mathbf{SD}) =$$

$$= \sum_{i=1}^{m}\chi_i\ tr\ (\mathbf{SV}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{SD} - \mathbf{DX}(\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-}\mathbf{X}'\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{S}) -$$

$$- \sum_{i=1}^{m}\gamma_i\ tr\ (\mathbf{V}_i\mathbf{D} - \mathbf{DX}(\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-}\mathbf{X}'\mathbf{S}^{-1}\mathbf{V}_i) = \sum_{i=1}^{m}\chi_i\ tr\ \mathbf{SV}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{SD}.$$

The proof of statement b) is clear.

In the next three corollaries we shall be using the notation: $(tr\ \mathbf{V}_j\mathbf{AV}_i\mathbf{B})$ is a matrix of the type $m \times m$, whose (j, i)-*th* element is $tr\ \mathbf{V}_j\mathbf{AV}_i\mathbf{B}$.

Corollary 2.4. One choice of the MINQE(U, I, **S**) of the function $\mathbf{f}'\theta$ is

$$\mathbf{f}'(\mathbf{M}^{-}\mathbf{u} - \mathbf{M}^{-}\mathbf{LK}^{-}\mathbf{v} + \mathbf{K}^{-}\mathbf{v}), \tag{20}$$

where the vectors $\mathbf{u} = (\mathbf{Y}'\mathbf{Q}_S'\mathbf{V}^{-1}\mathbf{V}_1\mathbf{V}^{-1}\mathbf{Q}_S\mathbf{Y}, \ldots, \mathbf{Y}'\mathbf{Q}_S'\mathbf{V}^{-1}\mathbf{V}_m\mathbf{V}^{-1}\mathbf{Q}_S\mathbf{Y})'$ $\mathbf{v} = (\mathbf{Y}'\mathbf{Q}_S'\mathbf{S}^{-1}\mathbf{V}_1\mathbf{S}^{-1}\mathbf{Q}_S\mathbf{Y}, \ldots, \mathbf{Y}'\mathbf{Q}_S'\mathbf{S}^{-1}\mathbf{V}_m\mathbf{S}^{-1}\mathbf{Q}_S\mathbf{Y})'$ and the matrices $\mathbf{M} = (tr\ \mathbf{V}_j\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1})$, $\mathbf{L} = (tr\ \mathbf{V}_j\mathbf{Q}_S'\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{Q}_S)$ and $\mathbf{K} = (tr\ \mathbf{V}_j\mathbf{Q}_S'\mathbf{S}^{-1}\mathbf{V}_i\mathbf{S}^{-1}\mathbf{Q}_S)$.

Proof. The MINQE(U, I, **S**) of the function $\mathbf{f}'\theta$ is, according to (17),

$$\mathbf{Y}'\mathbf{A}_1\mathbf{Y} = \sum_{i=1}^{m}\chi_i\mathbf{Y}'\mathbf{Q}_S'\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{Q}_S\mathbf{Y} - \sum_{i=1}^{m}\gamma_i\mathbf{Y}'\mathbf{Q}_S'\mathbf{S}^{-1}\mathbf{V}_i\mathbf{S}^{-1}\mathbf{Q}_S\mathbf{Y} = \chi'\mathbf{u} - \gamma'\mathbf{v}.$$

For the vectors χ and γ of (11) and (18) we have $\chi' = \mathbf{f}'\mathbf{M}^{-}$ and $\gamma' = \mathbf{f}'(\mathbf{M}^{-}\mathbf{L} - \mathbf{I})\mathbf{K}^{-}$.

$$\mathbf{Y}'\mathbf{A}_1\mathbf{Y} = \chi'\mathbf{u} - \gamma'\mathbf{v} = \mathbf{f}'\mathbf{M}^{-}\mathbf{u} - \mathbf{f}'(\mathbf{M}^{-}\mathbf{L} - \mathbf{I})\mathbf{K}^{-}\mathbf{v} =$$

$$= \mathbf{f}'(\mathbf{M}^{-}\mathbf{u} - \mathbf{M}^{-}\mathbf{LK}^{-}\mathbf{v} + \mathbf{K}^{-}\mathbf{v}).$$

Corollary 2.5. One choice of the MINQE(U, I, **S**) of the vector of unknown variance components $\theta = (\theta_1, \ldots, \theta_m)'$ is

$$\hat{\theta} = \mathbf{M}^{-}\mathbf{u} - \mathbf{M}^{-}\mathbf{LK}^{-}\mathbf{v} + \mathbf{K}^{-}\mathbf{v}. \tag{21}$$

Corollary 2.6. The MINQE(U, I, $\mathbf{S}$) of the function $\mathbf{f}'\theta$ exists if and only if $\mathbf{f} \in \mathcal{R}(\mathbf{M})$ and $(\mathbf{LM}^- - \mathbf{I})\mathbf{f} \in \mathcal{R}(\mathbf{K}')$, where $\mathcal{R}(\mathbf{N})$ denotes the range of the matrix $\mathbf{N}$.

The estimations MINQE(U, I, $\mathbf{S}$) obtained in this paper ((17)) and MINQE(U, I) obtained by Rao, discused in *Lemma* 1.2, are unbiased invariant quadratic estimations of the function $\mathbf{f}'\theta$. We shall compare both these results from the standpoint of their variance. Here, the following two possibilities are interesting. The first, $\mathbf{S} = \mathbf{V}$, it means that the matrix $\mathbf{S}$ does not contribute to an estimated situation by a new information; then the MINQE(U, I, $\mathbf{S}$) and the MINQE(U, I) are the same (Theorem 2.7). The second, $\mathbf{S} = \mathbf{V}_\theta$, the information obtained from the matrix $\mathbf{S}$ is precise; then $D_\theta(\text{MINQE(U, I, }\mathbf{S}\text{)}) \leqq D_\theta(\text{MINQE(U, I)})$ (Theorem 2.8).

Theorem 2.7. If $\mathbf{S} = \mathbf{V}$, then the MINQE(U, I, $\mathbf{S}$) of the function $\mathbf{f}'\theta$ is equal to the MINQE(U, I).

Proof. If $\mathbf{S} = \mathbf{V}$, then MINQE(U, I, $\mathbf{S}$) of the function $\mathbf{f}'\theta$ (17) is

$$\hat{q} = \sum_{i=1}^{m} (\chi_i - \gamma_i)\mathbf{Y}'\mathbf{Q}_V'\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{Q}_V\mathbf{Y}, \tag{22}$$

where $\mathbf{Q}_V = \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^-\mathbf{X}'\mathbf{V}^{-1}$ and the vector $(\chi_1 - \gamma_1, \ldots, \chi_m - \gamma_m)'$ is any solution of the linear system

$$\sum_{i=1}^{m} (\chi_i - \gamma_i)\ tr\ \mathbf{V}_j\mathbf{Q}_V'\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{Q}_V = f_j. \tag{23}$$

Denoting $(\chi_i - \gamma_i) = \delta_i$ in (22), (23) we obtain the MINQE(U, I) defined in *Lemma* 1.2.

Theorem 2.8. Let the realization vector $\mathbf{Y}$ have a normal distribution $N_n(\mathbf{X}\beta, \mathbf{V}_\theta)$, let $\mathbf{S} = \mathbf{V}_\theta$ and let the expressions $tr\ \mathbf{V}_\theta\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{V}_\theta\mathbf{A}$ be indenpendent of the matrix $\mathbf{A}$ ($i = 1, \ldots, m$). Then the MINQE(U, I, $\mathbf{S}$) of the function $\mathbf{f}'\theta$ has a minimum variance in the class $\mathcal{A}_1$ for such $\theta \in \mathcal{O}$ that $\mathbf{V}_\theta = \mathbf{S}$ is valid.

Proof. If $\mathbf{S} = \mathbf{V}_\theta$, then according to *Lemma* 2.2 we have that the MINQE(U, I, $\mathbf{S}$) is a statistic $\mathbf{Y}'\mathbf{A}_1\mathbf{Y}$, where $\mathbf{A}_1$ minimizes the expression $tr\ \mathbf{A}\mathbf{V}_\theta\mathbf{A}\mathbf{V}_\theta - 2\sum_{i=1}^{m}\chi_i\ tr\ \mathbf{V}_\theta\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{V}_\theta\mathbf{A}$. With respect to the assumptions $tr\ \mathbf{V}_\theta\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{V}_\theta\mathbf{A}$ are indenpendent of the matrix $\mathbf{A}$, the matrix $\mathbf{A}_1$ in the estimator $\mathbf{Y}'\mathbf{A}_1\mathbf{Y}$ minimizes the expression $tr\ \mathbf{A}\mathbf{V}_\theta\mathbf{A}\mathbf{V}_\theta$ in the class $\mathcal{A}_1$. Since $D_\theta(\mathbf{Y}'\mathbf{A}\mathbf{Y}) = 2\ tr\ \mathbf{A}\mathbf{V}_\theta\mathbf{A}\mathbf{V}_\theta$ (*Lemma* 1.3), the theorem is proved.

Corollary 2.9. If the assumptions of the theorem 2.8 are satisfied, then

$$D_\theta(\text{MINQE(U, I, }\mathbf{S}\text{)}) \leqq D_\theta(\text{MINQE(U, I)}). \tag{24}$$

Proof is obvious.

Example.

Let $\mathbf{Y}=\mathbf{X}\beta+\mathbf{e}$, where $\mathbf{X}=(1,1,1)'$ and

$$D(\mathbf{Y})=D(\mathbf{e})=\begin{pmatrix} \theta_1 & \theta_2-\theta_1 & 0 \\ \theta_2-\theta_1 & \theta_1 & 0 \\ 0 & 0 & \theta_2 \end{pmatrix}=$$

$$=\theta_1\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}+\theta_2\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}=\theta_1\mathbf{V}_1+\theta_2\mathbf{V}_2=\mathbf{V}_\theta .$$

Let $\alpha_1=\alpha_2=1$, i.e. $\mathbf{V}=\mathbf{V}_1+\mathbf{V}_2=\mathbf{I}$ and let the matrix $\mathbf{S}$ contains prior values of elements of the matrix $\mathbf{V}_\theta$.

We wish to find the MINQE(U, I) and the MINQE(U, I, $\mathbf{S}$) of the function $\mathbf{f}'\theta=f_1\theta_1+f_2\theta_2$.

The matrix $\mathbf{A}$ of the the estimator $\mathbf{Y}'\mathbf{A}\mathbf{Y}$ is an element of the class $\mathcal{A}_1$ and therefore

$$\mathbf{A}=\frac{1}{6}\begin{pmatrix} f_1-f_2-6a & -f_1+f_2 & 6a \\ -f_1+f_2 & 3f_1+3f_2+6a & -2f_1-4f_2-6a \\ 6a & -2f_1-4f_2-6a & 2f_1+4f_2 \end{pmatrix}$$

where $a\in(-\infty,\infty)$.

If we estimate $\theta_1+\theta_2$ $(f_1=f_2=1)$, then

$$\mathbf{A}=\begin{pmatrix} -a & 0 & a \\ 0 & 1+a & -1-a \\ a & -1-a & 1 \end{pmatrix},$$

where $a\in(-\infty,\infty)$.

The MINQE(U, I) of the function $\theta_1+\theta_2$ is the estimator $\mathbf{Y}'\mathbf{A}\mathbf{Y}$, where we assing the parameter a of the matrix $\mathbf{A}$ additionally in such a way that

$$tr\ \mathbf{AVAV}=6a^2+6a+4=min .$$

The expressions $tr\ \mathbf{V}_\theta\mathbf{V}^{-1}\mathbf{V}_i\mathbf{V}^{-1}\mathbf{V}_\theta\mathbf{A}$ $(i=1,2)$ are indenpendent of the matrix $\mathbf{A}$ (They are indenpendent of the parameter a) because $tr\ \mathbf{V}_\theta\mathbf{V}^{-1}\mathbf{V}_1\mathbf{V}^{-1}\mathbf{V}_\theta\mathbf{A}=2(\theta_1-\theta_2)^2f_1$ and $tr\ \mathbf{V}_\theta\mathbf{V}^{-1}\mathbf{V}_2\mathbf{V}^{-1}\mathbf{V}_\theta\mathbf{A}=2(\theta_1\theta_2-\theta_1^2)f_1+\theta_2^2f_2$. If $\mathbf{S}=\mathbf{V}_\theta$ holds too, then the MINQE(U, I, $\mathbf{S}$) of the function $\theta_1+\theta_2$ is the estimator $\mathbf{Y}'\mathbf{A}\mathbf{Y}$, where we assign the parameter a of the matrix $\mathbf{A}$ additionally in such a way that

$$tr\ \mathbf{AV}_\theta\mathbf{AV}_\theta=(\theta_1+\theta_2)^2-6(\theta_2^2-2\theta_1\theta_2)a-6(\theta_2^2-2\theta_1\theta_2)a^2=min .$$

Since $D_\theta(\mathbf{Y}'\mathbf{A}\mathbf{Y})=2\ tr\ \mathbf{AV}_\theta\mathbf{AV}_\theta$, it is obvious that the MINQE(U, I, $\mathbf{S}$) of the function $\theta_1+\theta_2$ is an estimator with minimal variance in the class $\mathcal{A}_1$.

REFERENCES

[1] KLEFFE, J.: Best quadratic unbiased estimators for variance components in mixed linear models. Sankhya 38, 1976, 179—186.
[2] RAO, C. R.: Estimation of variance and covariance components-MINQUE theory. Journ. Multivariat. Analysis 1, 1971, 257—275.
[3] RAO, C. R.—MITRA, K. S.: Generalized Inverse of Matrices and its Application. J. Wiley, N. York 1971.
[4] RAO, C. R.—KLEFFE, J.: Estimation of Variance Components. In: Krisnaiah, P. R., ed. Handbook of Statistics, Vol. I. 1—40, North Holland, N. York 1980.
[5] SEELY, J.: Linear spaces and unbiased estimation—application to the mixed linear model. Ann. Math. Statistics, 41, 1970, 1735—1748.

Received July 5, 1984

Chemickotechnologická fakulta SVŠT
Jánska 1 a
812 37 Bratislava

НЕСМЕЩЕННАЯ КВАДРАТИЧЕСКАЯ ОЦЕНКА ВАРИАЦИОННЫХ КОМПОНЕНТОВ С МИНИМАЛЬНОЙ ДИСПЕРСИЕЙ

Štefan Varga

Резюме

На основе реализации матрицы **S**, которая является оценкой ковариационной матрицы $\mathbf{V}_\Theta = \Theta_1\mathbf{V}_1 + \ldots + \Theta_m\mathbf{V}_m$ случайного вектора **Y**, получается оценка MINQE(U, I, **S**) линейной функции $f\Theta_1 + \ldots + f_m\Theta_m$ в форме $\mathbf{Y}'\mathbf{A}(\mathbf{S})\mathbf{Y}$. Показана ситуация, когда MINQE(U, I, **S**) имеет минимальную дисперсию.