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INITIAL AND BOUNDARY VALUE PROBLEMS FOR nth ORDER DIFERENCE EQUATIONS

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1. Introduction

In recent years there has been considerable interest in the theory and constructive methods of solutions of difference equations satisfying some boundary conditions, e.g., see [1—3, 8—12 and references therein]. In particular following the methods of Šeda [14] for continuous problems, Eloe [9] has discussed the existence and the uniqueness of solutions of the noth norder difference equations together with multi-point boundary conditions. In this paper we shall consider the nth order difference equation

$$\Delta(\varrho(t)\Delta^{n-1}u(t))=f(t, u(t), \Delta u(t), \ldots, \Delta^{n-1}u(t)),\ t\in I(0, N) \tag{1.1}$$

and use shooting type methods to prove the existence and the uniqueness of solutions satisfying two point boundary conditions. These methods have been analysed successfully for the continuous problem in [7, 13, 15].

In the following, for two nonnegative integers p and q $(p<q)$, $I(p, q)$ represents the discrete set $\{p, p+1, \ldots, q\}$, whereas $I(p)=\{p, p+1, \ldots\}$. For any function $g(t)$ the sum $\sum_{s=q}^{p} g(s)=0$. The $(r)^{(m)}$ is the usual factorial notation and stands for $r(r-1)\ldots(r-m+1)$ with $(0)^{(m)}=1$. Finally, $\Delta u(t)=u(t+1)-u(t)$.

In (1.1) the function f is assumed to be defined on $I(0, N)\times R^n$, the $\varrho(t)$ is defined and positive for all $t\in I(0, N+1)$. With these assumptions the solution of (1.1) can be constructed inductively once the initial conditions

$$\Delta^i u(0)=A_i, \qquad 0\leqslant i\leqslant n-1 \tag{1.2}$$

are prescribed. However, this is not the case if we seek the solution of (1.1) together with some boundary conditions, e.g., see [1].

2. Basic Lemmas

Lemma 2.1. [4, 5] *Let $u(t)$ be some function defined on $I(t_1) \subseteq I(0)$. Then, for $0 \leq k \leq p-1$, $t \in I(t_1)$ the following identity holds*

$$\Delta^k u(t) = \sum_{j=k}^{p-1} \frac{(t-t_1)^{(j-k)}}{(j-k)!} \Delta^j u(t_1) + \frac{1}{(p-k-1)!} \sum_{s=t_1}^{t-p+k} (t-s-1)^{(p-k-1)} \Delta^p u(s) .$$

Lemma 2.2. *Let $u(t)$ be some function defined on $I(0, n+q)$, and $\Delta^j u(0) = \varepsilon_j$, $0 \leq j \leq n-1$. Further,*

(i) *if $\varepsilon_j > 0$, $0 \leq j \leq n-1$ and $\Delta^{n-1} u(t) > 0$ for all $t \in I(0, q+1)$, then $\Delta^k u(t) > 0$ for all $t \in I(0, n+q-k)$, (and hence $\Delta^k u(t)$ is strictly increasing for all $t \in I(0, n+q-k)$) $0 \leq k \leq n-2$*

(ii) *if $\varepsilon_j = 0$, $0 \leq j \leq n-2$, then*

$$\begin{aligned} &\Delta^k u(t) = 0 \quad \text{for all } t \in I(0, n-k-2) \\ &\Delta^k u(n-k-1) = \varepsilon_{n-1}, 0 \leq k \leq n-1 \end{aligned} \tag{2.1}$$

also, if $\Delta^{n-1} u(t) > 0$, $t \in I(0, q+1)$, then $\Delta^k u(t) > 0$ for all $t \in I(n-k-1, n+q-k)$, $0 \leq k \leq n-2$ and for such t that

$$u(t) \leq \frac{1}{k!} (t-nk+1)^{(k)} \Delta^k u(t), 1 \leq k \leq n-2 . \tag{2.2}$$

Proof. From lemma 2.1 ($p = n-1$, $t_1 = 0$), for all $0 \leq k \leq n-2$, we have

$$\Delta^k u(t) = \sum_{j=k}^{n-2} \frac{(t)^{(j-k)}}{(j-k)!} \varepsilon_j + \frac{1}{(n-k-2)!} \sum_{s=0}^{t-n+k+1} (t-s-1)^{(n-k-2)} \Delta^{n-1} u(s) \tag{2.3}$$

and hence if $\varepsilon_j > 0$, $0 \leq j \leq n-1$ and $\Delta^{n-1} u(t) > 0$ for all $t \in I(0, q+1)$, then $\Delta^k u(t) > 0$ as long as $t-n+k+1 \leq q+1$, i.e., $t \leq q-k$.

From part (ii), the equality (2.3) reduces to the following

$$\Delta^k u(t) = \frac{1}{(n-k-2)!} \sum_{s=0}^{t-n+k+1} (t-s-1)^{(n-k-2)} \Delta^{n-1} u(s) \tag{2.4}$$

and from this (2.2) immediately follows. Further, if $\Delta^{n-1} u(t) > 0$ for all $t \in I(0, q+1)$, then, from (2.4), $\Delta^k u(t) > 0$ as long as $0 \leq t-n+k+1 \leq q+1$, i.e., $t \in I(n-k-1, n+q-k)$.

Finally, to prove (2.2), once again from lemma 2.1 ($p = k$, $k = 0$, $t_1 = 0$), we have

$$u(t) = \frac{1}{(k-1)!} \sum_{s=n-k-1}^{t-k} (t-s-1)^{(k-1)} \Delta^k u(s), 1 \leq k \leq n-2 . \tag{2.5}$$

Since, $\Delta^k u(t) > 0$ and strictly increasing for all $t \in I(n-k-1, n+q-k)$, $1 \leq k \leq n-2$, from (2.5) we find

$$u(t)<\frac{1}{(k-1)!}\sum_{s=n-k-1}^{t-k}(t-s-1)^{(k-1)}\Delta^k u(t)=-\frac{1}{k!}\sum_{s=n-k-1}^{t-k}\Delta(t-s)^{(k)}\Delta^k u(t)=$$

$$=-\frac{1}{k!}[(t-s)^{(k)}]_{t-n-1}^{t-k+1}\Delta^k u(t)=\frac{1}{k!}(t-n+k+1)^{(k)}\Delta^k u(t)\,.$$

Remark 1. Throughout lemma 2.2 the strict inequalities can be replaced by less than or equal to inequalities.

3. Comparison results

Theorem 3.1. *Assume that*

(i) $f(t, u_0, u_1, \ldots, u_n)$ *is nondecreasing in* $u_0, u_1, \ldots, u_{n-1}$ *for a fixed* $t\in I(0, N)$

(ii) $v(t)$ *and* $w(t)$ *are defined for all* $t\in I(0, N+n)$ *and for all* $t\in I(0, N)$ *one of the inequalities*

$$\Delta(\varrho(t)\Delta^{n-1}v(t))\leqslant f(t, v(t), \Delta v(t), \ldots, \Delta^{n-1}v(t)) \tag{3.1}$$

$$\Delta(\varrho(t)\Delta^{n-1}w(t))\geqslant f(t, w(t), \Delta w(t), \ldots, \Delta^{n-1}w(t)) \tag{3.2}$$

is strict.

(iii)
$$\Delta^k v(0)<\Delta^k w(0),\ 0\leqslant k\leqslant n-1\,. \tag{3.3}$$

Then, for all $t\in I(0, N+n-k)$

$$\Delta^k v(t)<\Delta^k w(t),\ 0\leqslant k\leqslant n-1\,. \tag{3.4}$$

Proof. From lemma 2.2 it suffices to show that for all $t\in I(0, N+1)$ the inequality $\Delta^{n-1}v(t)<\Delta^{n-1}w(t)$ holds. For this, let us assume that $r\in I(1, N+1)$ be the first point where $\Delta^{n-1}v(t)\geqslant\Delta^{n-1}w(t)$. Then, from lemma 2.2 for all $t\in I(0, n+r-k-2)$, $\Delta^k v(t)<\Delta^k w(t)$, $0\leqslant k\leqslant n-2$. Thus, in particular for all $0\leqslant k\leqslant n-1$, $\Delta^k v(r-1)<\Delta^k w(r-1)$. However, from the inequalities (3.1), (3.2) we have

$$\Delta(\varrho(r-1)\Delta^{n-1}w(r-1))-\Delta(\varrho(r-1)\Delta^{n-1}v(r-1))>$$

$$>f(r-1, w(r-1), \Delta w(r-1), \ldots, \Delta^{n-1}w(r-1))-$$

$$-f(r-1, v(r-1), \Delta v(r-1), \ldots, \Delta^{n-1}v(r-1))$$

and hence from the nondecreasing nature of f, we find

$$\Delta(\varrho(r-1)\Delta^{n-1}w(r-1))-\Delta(\varrho(r-1)\Delta^{n-1}v(r-1))>0\,,$$

which is the same as

$$\varrho(r)\Delta^{n-1}(w(r)-v(r))>\varrho(r-1)\Delta^{n-1}(w(r-1)-v(r-1))\,. \tag{3.5}$$

Since $\varrho(t)>0$ for all $t\in I(0, N+1)$, inequality (3.5) implies that $\Delta^{n-1}w(r)>\Delta^{n-1}v(r)$. This contradiction completes the proof.

Corollary 3.2. *Let the conditions of theorem* 3.1 *be satisfied with strict inequality in both* (3.1) *and* (3.2). *Further, let* $u(t)$ *be the solution of the initial value problem* (1.1), (1.2) *and* $\Delta^k v(0)<A_k<\Delta^k w(0)$, $0\leqslant k\leqslant n-1$. *Then, for all* $t\in I(0, N+n-k)$ *the inequality* $\Delta^k v(t)<\Delta^k u(t)<\Delta^k w(t)$ *holds.*

Corollary 3.3. *Assume that the conditions* (i) *and* (ii) *of theorem* 3.1 *are satisfied, and* $\Delta^k v(0)\leqslant\Delta^k w(0)$, $0\leqslant k\leqslant n-2$ *and* $\Delta^{n-1}v(0)\gtrless\Delta^{n-1}w(0)$. *Then,* $\Delta^k v(t)\leqslant\Delta^k w(t)$ *for all* $t\in I(0, N+n-k)$ *and in particular for all* $t\in I(n-k-1, N+n-k)$ *the strict inequality* $\Delta^k v(t)<\Delta^k w(t)$, $0\leqslant k\leqslant n-1$ *holds.*

Lemma 3.4. *Assume that* $q_i(t)$, $0\leqslant i\leqslant n-1$ *are defined and nonnegative on* $I(0, N)$. *Then, for all* $\alpha>0$ *the solutions of the initial value problems*

$$\Delta(\varrho(t)\Delta^{n-1}v(t))=\sum_{i=0}^{n-1}q_i(t)\Delta^i v(t) \tag{3.6}$$

$$\begin{aligned}&\Delta^i v(0)=0,\ 0\leqslant i\leqslant n-2\\ &\Delta^{n-1}v(0)=\alpha>0\end{aligned} \tag{3.7}$$

have the property that $\Delta^k v(t)\geqslant 0$ *for all* $t\in I(0, N+n-k)$ *and in particular for all* $t\in I(n-k-1, N+n-k)$ *the strict inequality* $\Delta^k v(t)>0$, $0\leqslant k\leqslant n-1$ *holds.*

Proof. Let $r\in I(1, N+1)$ be the first point where $\Delta^{n-1}v(t)\leqslant 0$, then from lemma 2.2, $\Delta^k v(t)\geqslant 0$ for all $t\in I(0, n+r-k-2)$ and in particular $\Delta^k v(r-1)\geqslant 0$, $0\leqslant k\leqslant n-2$. However, from the difference equation (3.6), we have

$$\varrho(r)\Delta^{n-1}v(r)=\varrho(r-1)\Delta^{n-1}v(r-1)+\sum_{i=0}^{n-1}q_i(r-1)\Delta^i v(r-1)>0\,.$$

This contradiction completes the proof.

Theorem 3.5. *Assume that*

(i) $g(t, u_0, u_1, \ldots, u_{n-1})$ *is defined for all* $(t, u_0, u_1, \ldots, u_{n-1})\in I(0, N)\times R^n$ *and nondecreasing in* $u_0, u_1, \ldots, u_{n-1}$ *for a fixed* $t\in I(0, N)$, *also for* $\lambda>1$

$$\lambda g(t, u_0, u_1, \ldots, u_{n-1})\leqslant g(t, \lambda u_0, \lambda u_1, \ldots, \lambda u_{n-1})$$

(ii) *for a fixed* $t\in I(0, N)$ *and* $u_i\in R_+$, $0\leqslant i\leqslant n-1$

$$f(t, u_0, u_1, \ldots, u_{n-1})\geqslant g(t, u_0, u_1, \ldots, u_{n-1})+l(t)u_0+\sum_{i=1}^{n-2}q_i(t)u_i$$

where $q_i(t)\geqslant 0$, $1\leqslant i\leqslant n-2$ *and* $l(t)$ *are defined on* $I(0, N)$ *and*

$$l(t)+\sum_{i=1}^{n-2} q_i(t)\frac{(i)!}{(t-n+i+1)^{(i)}}\geqslant 0 \tag{3.8}$$

(iii) *$u(t, 0, \beta)$ is the solution of (1.1) satisfying the initial conditions*

$$\Delta^i u(0)=0,\ 0\leqslant i\leqslant n-2,\ \Delta^{n-1}u(0)=\beta$$

(iv) *there exists a solution $v(t, 0, \alpha)$ of the difference equation*

$$\Delta(\varrho(t)\Delta^{n-1}v(t))=g(t, v(t), \Delta v(t), \ldots, \Delta^{n-1}v(t))+l(t)v(t)+\sum_{i=1}^{n-2} q_i(t)\Delta^i v(t) \tag{3.9}$$

satisfying the initial conditions $\Delta^i v(0)=0$, $0\leqslant i\leqslant n-2$, $\Delta^{n-1}v(0)=\alpha>0$ such that $\Delta^{n-1}v(t, 0, \alpha)>0$ for all $t\in I(0, N+1)$.

Then, for all $t\in I(0, N+n-i)$

$$0\leqslant\frac{\beta-\varepsilon}{\alpha}\Delta^i v(t, 0, \alpha)\leqslant\Delta^i u(t, 0, \beta),\ 0\leqslant i\leqslant n-1 \tag{3.10}$$

where $\varepsilon\geqslant 0$ and $\beta-\varepsilon\geqslant\alpha$. In particular $\Delta^i u(t, 0, \beta)>0$ for all $t\in I(n-i-1, N+n-i)$, $0\leqslant i\leqslant n-1$.

Proof. Since $\Delta^{n-1}v(t, 0, \beta)>0$ for all $t\in I(0, N+1)$ and $\Delta^i v(0, 0, \beta)=0$, $0\leqslant i\leqslant n-2$, lemma 2.2 ensures that $\Delta^i v(t, 0, \beta)\geqslant 0$ for all $t\in I(0, N+n-i)$ and in particular the strict inequality holds for all $t\in I(n-i-1, N+n-i)$, $0\leqslant i\leqslant n-1$. Thus, it suffices to show that $\frac{\beta-\varepsilon}{\alpha}\Delta^i v(t, 0, \alpha)\leqslant\Delta^i u(t, 0, \beta)$, $0\leqslant i\leqslant n-1$ holds for all $t\in I(0, N+n-i)$. For this, we define a function $\Phi(t)$, $t\in I(0, N+n)$ as follows

$$\Phi(t)=u(t, 0, \beta)-\frac{\beta-\varepsilon}{\alpha}v(t, 0, \alpha)\,.$$

Then, $\Delta^i\Phi(0)=0$, $0\leqslant i\leqslant n-2$ and $\Delta^{n-1}\Phi(0)=\varepsilon>0$, and from lemma 2.2 and remark 1 note that we need to prove $\Delta^{n-1}\Phi(t)\geqslant 0$ for all $t\in I(0, N+1)$. Let $r\in I(1, N+1)$ be the first point where $\Delta^{n-1}\Phi(t)<0$. Then, from lemma 2.2, $\Delta^k\Phi(t)\geqslant 0$ for all $t\in I(0, n+r-k-2)$, $0\leqslant k\leqslant n-1$. Hence, in particular $\Delta^k u(r-1)\geqslant 0$, $0\leqslant k\leqslant n-1$. Since, $\varrho(t)>0$ for all $t\in I(0, N+1)$, we have

$$\Delta(\varrho(r-1)\Delta^{n-1}\Phi(r-1))=\varrho(r)\Delta^{n-1}\Phi(r)-\varrho(r-1)\Delta^{n-1}\Phi(r-1)<0\,. \tag{3.11}$$

Next, using the conditions on the functions and the inequality (3.11), we successively obtain

$$f(r-1, u(r-1), \Delta u(r-1), \ldots, \Delta^{n-1}u(r-1))=\Delta(\varrho(r-1)\Delta^{n-1}u(r-1))=$$

$$=\Delta(\varrho(r-1)\Delta^{n-1}\Phi(r-1))+\frac{\beta-\varepsilon}{\alpha}\Delta(\varrho(r-1)\Delta^{n-1}v(r-1))<$$

$$<\frac{\beta-\varepsilon}{\alpha}\Delta(\varrho(r-1)\Delta^{n-1}v(r-1))=$$

$$=\frac{\beta-\varepsilon}{\alpha}\Big[g(r-1,\, v(r-1),\, \Delta v(r-1),\, \dots,\, \Delta^{n-1}v(r-1))+l(r-1)v(r-1)+$$

$$+\sum_{i=1}^{n-2}q_i(r-1)\Delta^i v(r-1)\Big]\leqslant$$

$$\leqslant g(r-1,\, \frac{\beta-\varepsilon}{\alpha}v(r-1),\, \frac{\beta-\varepsilon}{\alpha}\Delta v(r-1),\, \dots,\, \frac{\beta-\varepsilon}{\alpha}\Delta^{n-1}v(r-1))+$$

$$+\frac{\beta-\varepsilon}{\alpha}\Big[l(r-1)v(r-1)+\sum_{i=1}^{n-2}q_i(r-1)\Delta^i v(r-1)\Big]\leqslant$$

$$\leqslant f(r-1,\, u(r-1),\, \Delta u(r-1),\, \dots,\, \Delta^{n-1}u(r-1))-l(r-1)u(r-1)-$$

$$\sum_{i=1}^{n-2}q_i(r-1)\Delta^i u(r-1)+$$

$$+\frac{\beta-\varepsilon}{\alpha}\Big[l(r-1)v(r-1)+\sum_{i=1}^{n-2}q_i(r-1)\Delta^i v(r-1)\Big]=$$

$$=f(r-1,\, u(r-1),\, \Delta u(r-1),\, \dots,\, \Delta^{n-1}u(r-1))-l(r-1)\Phi(r-1)-$$

$$-\sum_{i=1}^{n-2}q_i(r-1)\Delta^i\Phi(r-1)\leqslant$$

$$\leqslant f(r-1,\, u(r-1),\, \Delta u(r-1),\, \dots,\, \Delta^{n-1}u(r-1))-$$

$$-\Big[l(r-1)+\sum_{i=1}^{n-2}\frac{q_i(r-1)(i)!}{(r-n+i)^{(i)}}\Big]\Phi(r-1)\,,$$

which is not true from (3.8) and the fact that $\Phi(r-1)\geqslant 0$. This contradiction completes the proof.

Corollary 3.6. *Assume that $u(t, 0, \beta)$ be the solution of (1.1) satisfying the initial conditions $\Delta^i u(0)=0$, $0\leqslant i\leqslant n-2$, $\Delta^{n-1}u(0)=\beta$, and let for a fixed $t\in I(0, N)$ and $u_i\in R_+$, $0\leqslant i\leqslant n-1$*

$$f(t,\, u_0,\, u_1,\, \dots,\, u_{n-1})\geqslant\sum_{i=0}^{n-1}q_i(t)u_i$$

where $q_i(t)\geqslant 0$, $0\leqslant i\leqslant n-1$ and defined on $I(0, N)$. Then, the conclusion of theorem 3.5 holds.

Proof. In view of lemma 3.4 we see that all the conditions of theorem 3.5 are satisfied.

4. Boundary value problems

Theorem 4.1. *In addition to the assumption* (i) *of theorem* 3.5, *we assume that*

(1) *for a fixed* $t \in I(0, N)$ *and* $u_i \geq \bar{u}_i$, $0 \leq i \leq n-1$

$$f(t, u_0, u_1, \ldots, u_{n-1}) - f(t, \bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{n-1} \geq$$

$$\geq g(t, u_0 - \bar{u}_0, u_1 - \bar{u}_1, \ldots, u_{n-1} - \bar{u}_{n-1}) + l(t)(u_0 - \bar{u}_0) + \sum_{i=1}^{n-2} q_i(t)(u_i - \bar{u}_i)$$

where $l(t)$ and $q_i(t)$, $1 \leq i \leq n-2$ *are the same as in condition* (ii) *of theorem* 3.5.

(2) *for each* $\alpha > 0$ *the condition* (iv) *of theorem* 3.5 *holds.*

Then, the difference equation (1.1) *satisfying the boundary conditions*

$$\Delta^i u(0) = A_i,\ 0 \leq i \leq n-2 \tag{4.2}$$

$$\Delta^q u(N+n-q) = B_q,\ 0 \leq q \leq n-1 \text{ and fixed} \tag{4.2}$$

has a unique solution.

Proof. Let A denote the vector $(A_1, A_2, \ldots, A_{n-2})$ and $u(t, A, \gamma_i)$, $i = 1, 2$ be the solution of (1.1), (4.1) and $\Delta^{n-1}u(0, A, \gamma_i) = \gamma_i$. For $\gamma_1 > \gamma_2$, we define $w(t, A, \gamma_1, \gamma_2) = u(t, A, \gamma_1) - u(t, A, \gamma_2)$, then $w(t, A, \gamma, \gamma_2)$ is the solution of the initial value problem

$$\Delta(\varrho(t)\Delta^{n-1}w(t, A, \gamma_1, \gamma_2)) = F(t, w(t, A, \gamma_1, \gamma_2), \Delta w(t, A, \gamma_1, \gamma_2), \ldots,$$
$$\Delta^{n-1}w(t, A, \gamma_1, \gamma_2)) \tag{4.3}$$
$$\Delta^i w(0, A, \gamma_1, \gamma_2) = 0,\ 0 \leq i \leq n-2$$
$$\Delta^{n-1}w(0, A, \gamma_1, \gamma_2) = \gamma_1 - \gamma_2 > 0$$

where

$$F(t, w(t, A, \gamma_1, \gamma_2), \Delta w(t, A, \gamma_1, \gamma_2), \ldots, \Delta^{n-1} w(t, A, \gamma_1, \gamma_2)) =$$
$$= f(t, w(t, A, \gamma_1, \gamma_2) + u(t, A, \gamma_2), \Delta w(t, A, \gamma_1, \gamma_2) +$$
$$+ \Delta u(t, A, \gamma_2), \ldots, \Delta^{n-1}w(t, A, \gamma_1, \gamma_2) +$$
$$+ \Delta^{n-1}u(t, A, \gamma_2)) - f(t, u(t, A, \gamma_2), \Delta u(t, A, \gamma_2), \ldots, \Delta^{n-1}u(t, A, \gamma_2)).$$

By (1) the function F satisfies the conditions of theorem 3.5. Thus, for the solution $w(t, A, \gamma_1, \gamma_2)$ of (4.3) and $v(t, 0, \alpha)$ of (3.9) with $\gamma_1 - \gamma_2 > \alpha > 0$, we find

$$0 \leq \frac{\gamma_1 - \gamma_2}{\alpha} \Delta^i v(t, 0, \alpha) \leq \Delta^i w(t, A, \gamma_1, \gamma_2),\ 0 \leq i \leq n-1,\ t \in I(0, N+n-i)$$

and $\Delta^i w(t, A, \gamma_1, \gamma_2) > 0$ for all $t \in I(n-i-1, N+n-i)$.

Thus, in particular $\Delta^q w(N+n-q, A, \gamma_1, \gamma_2) = \Delta^q u(N+n-q, A, \gamma_1) - \Delta^q u(N+n-q, A, \gamma_2) > 0$. Hence, for a fixed $\gamma_2 \in R$, we get $\lim_{\gamma_1 \to \infty} \Delta^q u(N+n-q, A, \gamma_1) = \infty$ and for a fixed $\gamma_1 \in R$, $\lim_{\gamma_2 \to -\infty} \Delta^q u(N+n-q, A, \gamma_2) = -\infty$. This implies that $\Delta^q u(N+n-q, A, \gamma) - B_q$ is a continuous function of γ and its range must be the whole real line R. Hence, there exists a $\gamma_q^* \in R$ such that $\Delta^q u(N+n-q, A, \gamma_q^*) = B_q$. This $u(t, A, \gamma_q^*)$ is a solution of the boundary value problem (1.1), (4.1), (4.2).

Next, we shall prove the uniqueness of the solution. For this, let $u_1(t)$ and $u_2(t)$ be two solutions of (1.1), (4.1), (4.2). Since the solutions of the initial value problems (1.1), (1.2) and are unique, it is necessary that $\Delta^{n-1}u_1(0) \neq \Delta^{n-1}u_2(0)$. Without loss of generality we can assume that $\Delta^{n-1}u_1(0) = \alpha_1 > \alpha_2 = \Delta^{n-1}u_2(0)$. Then, as in the existence proof we easily arrive at the inequality $\Delta^q u_1(t) - \Delta^q u_2(t) > 0$ for all $t \in I(n-q-1, N+n-q)$ and in particular $\Delta^q u_1(N+n-q) > \Delta^q u_2(N+n-q)$. This contradiction completes the proof of the theorem.

Corollary 4.2. *Let us assume that for a fixed $t \in I(0, N)$ and $u_i \geq \bar{u}_i$, $0 \leq i \leq n-1$*

$$f(t, u_0, u_1, \ldots, u_{n-1}) - f(t, \bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{n-1}) \geq \sum_{i=0}^{n-1} q_i(t)(u_i - \bar{u}_i)$$

where $q_i(t) \geq 0$, $0 \leq i \leq n-1$ and are defined for all $t \in I(0, N)$, (in particular f is non-decreasing in all u_i, $0 \leq i \leq n-1$). Then, the boundary value problem (1.1), (4.1), (4.2) *has a unique solution.*

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НАЧАЛЬНАЯ И КРАЕВАЯ ЗАДАЧА ДЛЯ РАЗНОСТНЫХ УРАВНЕНИЙ n-ТОГО ПОРЯДКА

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Резюме

В этой работе методом стрельбы доказано существование и единственность решений нелинейных разностных уравнений, удовлетворяющих двухточечным краевым условиям. Результаты получены с помощью теорем сравнения для начальных задач.