Blahoslav Harman

Maximal ergodic theorem on a logic

Mathematica Slovaca, Vol. 35 (1985), No. 4, 381--386

Persistent URL: http://dml.cz/dmlcz/136405

Math. Slovaca 35, 1985, No. 4, 381—386

MAXIMAL ERGODIC THEOREM ON A LOGIC

BLAHOSLAV HARMAN

Introduction

The aim of the present paper is to prove and formulate the maximal ergodic theorem (MET) on a logic analogical to the classical one. The classical MET is studied in a space $(X, \mathcal{S}, \mu, T)$, where X is a nonempty set, $\mathcal{S}$ is σ-algebra on X, μ is a measure on $\mathcal{S}$ and $T: X \to X$ is a measure μ preserving transformation. For our purposes the most suitable formulation is the following:

Let $f: X \to R$ be an μ-integrable function. Let us denote

$$E_n = \{x \in X; \exists k \leqq n: f(x) + f(Tx) + \ldots + f(T^{k-1}x) \geqq 0\}.$$

Then $\int f \chi_{E_n} \, d\mu \geqq 0$.

This theorem plays the most important role in proving the classical individual ergodic theorem. In the case of logics the variants of the individual ergodic theorems have been studied (see [1], [2]), but no formulations of a MET have appeared.

1. Notations and preliminary results

Let $\mathcal{L}$ be a logic, that is a σ-latice with the first element 0 and the last element 1, with an orthocomplementation $\perp: \mathcal{L} \to \mathcal{L}$. The following conditions on $\mathcal{L}$ must be fulfilled:

i) if $a \in \mathcal{L}$ then $(a^\perp)^\perp = a$
ii) if $a < b$ then $b^\perp < a^\perp$
iii) if $a < b$ then $b = a \vee (b \wedge a^\perp)$
iv) $a \vee a^\perp = 1$ for all $a \in \mathcal{L}$.

Two elements a, $b \in \mathcal{L}$ are orthogonal ($a \perp b$) iff $a < b^\perp$, compatible ($a \leftrightarrow b$) iff there are three pairwise orthogonal elements a_1, b_1, c such that $a = a_1 \vee c$ and $b = b_1 \vee c$.

By the symbol $\mathcal{B}(R^1)$ there is denoted the set of all Borel sets on R^1. An observable x: $\mathcal{B}(R^1) \to \mathcal{L}$ is the map which satisfies the conditions:

i) $x(\emptyset) = 0$

ii) if $E, F \in \mathscr{B}(R^1)$, $E \cap F = \emptyset$ then $x(E) \perp x(F)$

iii) if $E_i \in \mathscr{B}(R^1)$ for $i \in N$, $E_i \cap E_j = \emptyset$ for $i \neq j$ then $x\left(\bigcup_{i=1}^{\infty} E_i\right) = \bigvee_{i=1}^{\infty} x(E_i)$.

Let f: $R^1 \to R^1$ be a Borel measurable function. It is easy to see that xf^{-1}: $\mathscr{B}(R^1) \to \mathscr{L}$, $E \mapsto x(f^{-1}(E))$ is an observable. Two observables x and y are compatible ($x \leftrightarrow y$) iff $x(E) \leftrightarrow y(F)$ for all $E, F \in \mathscr{B}(R^1)$.

If $x_1, x_2, \ldots, x_n$ are pairwise compatible observables, then it is possible to define the sum of them in the following way (see [3], theorem 6.17):

Let π_i: $R^n \to R^1$, $(u_1, u_2, \ldots, u_n) \mapsto u_i$ $(i = 1, 2, \ldots, n)$ be projections, h be the map h: $R^n \to R^1$, $(u_1, u_2, \ldots, u_n) \mapsto u_1 + u_2 + \ldots + u_n$.

Let $\varkappa$: $\mathscr{B}(R^n) \to \mathscr{L}$ be a σ-homomorphism such that $x_i = \varkappa\pi_i^{-1}$ for $i = 1, 2, \ldots, n$. Then we define

$$x_1 + x_2 + \ldots + x_n = \varkappa h^{-1}.$$

The state on $\mathscr{L}$ is the map m: $\mathscr{L} \to \langle 0, 1 \rangle$ which satisfied the following conditions:

i) $m(1) = 1$

ii) if $a_i \in \mathscr{L}$ for $i \in N$, $a_i \perp a_j$ for $i \neq j$, then $m\left(\bigvee_{i=1}^{\infty} a_i\right) = \sum_{i=1}^{\infty} m(a_i)$.

If x is an observable associated with a logic $\mathscr{L}$, then the map m_x: $\mathscr{B}(R^1) \to \langle 0, 1 \rangle$, $E \mapsto m(x(E))$ is a probability measure on $\mathscr{B}(R^1)$. A σ-homomorphism τ of a logic is the map τ: $\mathscr{L} \to \mathscr{L}$ which has satisfied the following conditions:

i) $\tau(0) = 0$

ii) $\tau(a^\perp) = (\tau(a))^\perp$ for all $a \in \mathscr{L}$

iii) if $a_i \in \mathscr{L}$ for $i \in N$, then $\tau\left(\bigvee_{i=1}^{\infty} a_i\right) = \bigvee_{i=1}^{\infty} \tau(a_i)$.

Let x be an observable associated with a logic $\mathscr{L}$, m be a state on $\mathscr{L}$. τ is said to be an x-measurable σ-homomorphism iff $\tau(x(\mathscr{B}(R^1))) \subset x(\mathscr{B}(R^1))$. It is said to be an invariant σ-homomorphism iff $m(\tau(a)) = m(a)$ for all $a \in \mathscr{L}$. If moreover from $\tau(a) = a$ it follows that $a \in \{0, 1\}$, then τ is said to be an ergodic homomorphism.

If τ is a σ-homomorphism of a logic $\mathscr{L}$, x an observable associated with $\mathscr{L}$, it is evident that τx: $\mathscr{B}(R^1) \to \mathscr{L}$, $E \mapsto \tau(x(E))$ is an observable associated with $\mathscr{L}$.

If τ_i is a σ-homomorphism of a logic $\mathscr{L}$ for $i = 1, 2, \ldots, n$ and x is an observable, then if $\tau_i x$ are pairwise compatible observables, we shall write the sum of them in the shortened form as follows: $\tau_1 x + \tau_2 x + \ldots + \tau_n x = (\tau_1 + \tau_2 + \ldots + \tau_n)x$. By the symbol $\mathbf{1}$ we shall denote an identical σ-homomorphism on $\mathscr{L}$.

2. Maximal ergodic theorem on a logic

The two first assertions of this part are proved in [1]. The Theorem 8 and Theorem 9 are the main assertions.

Lemma 1. *Let x be an observable. A homomorphism $\tau: \mathscr{L} \to \mathscr{L}$ is x-measurable iff there is a Borel measurable transformation $T: R^1 \to R^1$ such that $\tau x = xT^{-1}$.*

Lemma 2. *Let x be an observable. If a homomorphism $\tau: \mathscr{L} \to \mathscr{L}$ is x-measurable, then for the above transformation T we have $\tau^n x = xT^{-n}$, $n \in N$. If τ is an ergodic homomorphism in a state m, then T is an m_x-measure preserving transformation from R^1 into itself.*

From the proof of Lemma 2 it follows that if τ is an invariant homomorphism, then T is a measure m_x-preserving transformation.

In order to prove certain assertions we need in addition a part of Lemma 6.7, from [3]. Let us present it as Lemma 3.

Lemma 3. *Let $a, b \in \mathscr{L}$, $\mathscr{L}$ being any logic. The following statement are equivalent:*

a) $a \leftrightarrow b$

b) *there exist an observable x and two Borel sets A and B of the real line such that $x(A) = a$ and $x(B) = b$.*

Lemma 4. *Let x be an observable associated with a logic $\mathscr{L}$, let m be a state on $\mathscr{L}$, $f \in L_1(m_x)$. Let E, $F \in \mathscr{B}(R^1)$ such that $x(E) = x(F)$. Then*

$$\int f \chi_E dm_x = \int f \chi_F dm_x.$$

Proof: Let $E, F \in \mathscr{B}(R^1)$. Since $x(F) \perp x(F^c)$ and $x(F) \vee x(F^c) = 1$ it follows that $x(E - F) = x(E \cap F^c) = x(E) \wedge x(F^c) = x(E) \wedge x(F)^{\perp} = x(E) \wedge x(E)^{\perp} = 0$ and then $m_x(E - F) = 0$. Analogically $m_x(F - E) = 0$, which implies $m_x(E \triangle F) = 0$. Functions $f\chi_E$ and $f\chi_F$ are equal almost evrywhere, which proves the lemma.

Lemma 5. *Let X be a nonempty set, $\mathscr{S}$ be a σ-algebra of subsets of the set X. Let $f_i: X \to R^1$, $i = 1, 2, \ldots, n$ be $\mathscr{S}$-measurable functions. Let $F: X \to R^n$, $u \mapsto (f_1(u), f_2(u), \ldots, f_n(u))$. Let $h: R^n \to R^1$, $(u_1, u_2, \ldots, u_n) \mapsto (u_1 + u_2 + \ldots + u_n)$. Then*

i) F^{-1}: $\mathscr{B}(R^n) \to \mathscr{S}$, $\mathscr{E} \to \{u \in X;\ (f_1(u), f_2(u), \ldots, f_n(u)) \in \mathscr{E}\}$ *is a σ-homomorphism*

ii) $f_i^{-1} = F^{-1}\pi_i^{-1}$ *for* $i = 1, 2, \ldots, n$

iii) $F^{-1}h^{-1} = (f_1 + f_2 + \ldots + f_n)^{-1}$.

Proof: Straightforward

q.e.d.

Let $\mathscr{S}$ be a σ-algebra of subsets of a set X. Let $E \in \mathscr{S}$, $E_i \in \mathscr{S}$ for $i \in N$. If we put $E^{\perp} = E^c$, $\bigvee_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} E_i$, then $\mathscr{S}$ is a logic with the first element $\emptyset$ and the last element X. If $f: X \to R^1$ is a $\mathscr{S}$-measurable function, then f^{-1}: $\mathscr{B}(R^1) \to \mathscr{S}$ is an observable associated with a logic $\mathscr{S}$. For the sum of observables of this type the following assertion is valid.

Lemma 6. *Let $\mathcal{S}$ be a σ-algebra of subsets of a set X. Let f_i: $X \to R^1$, $i=1, 2, \ldots, n$ be $\mathcal{S}$-measurable functions. Then*

$$f_1^{-1}+f_2^{-1}+\ldots+f_n^{-1}=(f_1+f_2+\ldots+f_n)^{-1}.$$

Proof: The assertion of Lemma 6 is a straightforward consequence of the preceding lemma and of the definition of the sum of the compatible observables.

q.e.d.

Lemma 7. *Let x be an observable associated with a logic $\mathcal{L}$. Let f_i: $R^1 \to R^1$, $i=1, 2, \ldots, n$ be Borel measurable functions. Then*

$$xf_1^{-1}+xf_2^{-1}+\ldots+xf_n^{-1}=x(f_1^{-1}+f_2^{-1}+\ldots+f_n^{-1}).$$

Proof: Because of $xf_i^{-1}(E) \in x(\mathcal{B}(R^1))$ for $i=1, 2, \ldots, n$ and for any $E \in \mathcal{B}(R^1)$, the observables xf_1^{-1}, xf_2^{-1}, ..., xf_n^{-1} are mutually compatible (see Lemma 3). Let us denote $\varkappa = F^{-1}$, where F, F^{-1} are the maps from Lemma 5. Due to Lemma 5, $f_i^{-1}=\varkappa\pi_i^{-1}$ for $i=1, 2, \ldots, n$ and then $f_1^{-1}+f_2^{-1}+\ldots+f_n^{-1}=\varkappa h^{-1}$.

Let us denote $\varkappa^*=x\varkappa$. Evidently $\varkappa^*$: $\mathcal{B}(R^n) \to \mathcal{L}$ is a σ-homomorphism. Moreover the following is valid

$$\varkappa^*\pi_i^{-1}=x\varkappa\pi_i^{-1}=xf_i^{-1} \quad i=1, 2, \ldots, n.$$

From the definition of the sum of compatible observables we have

$$xf_1^{-1}+xf_2^{-1}+\ldots+xf_n^{-1}=\varkappa^*h^{-1}=x(f_1^{-1}+f_2^{-1}+\ldots+f_n^{-1}).$$

q.e.d.

Theorem 8. (Maximal ergodic theorem.)

Let $\mathcal{L}$ be a logic, x an observable associated with a logic $\mathcal{L}$. Let m be a state on $\mathcal{L}$, τ an x-measurable σ-homomorphism on $\mathcal{L}$ which is invariant in a state m. Let $f \in L_1(m_x)$. Let us denote

$$\Omega = \bigvee_{k=1}^{n} (\mathbf{1}+\tau+\ldots+\tau^{k-1})xf^{-1}\langle 0, +\infty).$$

Then there is $E \in \mathcal{B}(R^1)$ such that $x(E)=\Omega$, and $\int f \chi_E dm_x \geqq 0$.

Proof: Firstly we proof that xf^{-1}, τxf^{-1}, ..., $\tau^{k-1}xf^{-1}$ are pairwise compatible observables associated with a logic $\mathcal{L}$. Due to the x-measurability of the τ we get consequently $\tau^k(x(\mathcal{B}(R^1)))=\tau^{k-1}(\tau(x(\mathcal{B}(R^1)))) \subset \tau^{k-1}(x(\mathcal{B}(R^1))) \subset x(\mathcal{B}(R^1))$. Hence τ^k is an x-measurable σ-homomorphism on $\mathcal{L}$.

Evidently $\tau^k xf^{-1}(\mathcal{B}(R^1)) \subset \tau^k x(\mathcal{B}(R^1))$ and then due to the x-measurability of τ^k

$$\tau^k xf^{-1}(E) \in x(\mathcal{B}(R^1))$$

and

$$\tau^i xf^{-1}(F) \in x(\mathcal{B}(R^1))$$

for $0 \leqq i \leqq k \leqq n$ and for all E, $F \in \mathcal{B}(R^1)$. From Lemma 3 we have $\tau^k xf^{-1}(E) \leftrightarrow$

$\tau^i x f^{-1}(F)$, which implies $\tau^k x f^{-1} \leftrightarrow \tau^i x f^{-1}$. Let T: $R^1 \to R^1$ be a transformation from Lemma 1. Let us denote

$$E_k = \{x \in R^1;\, f(x) + f(Tx) + \ldots + f(T^{k-1}x) \geqq 0\}.$$

By application of Lemmas 1, 6 and 7 it follows consequently

$$\begin{aligned} x(E_k) &= x(f + fT + \ldots + fT^{k-1})^{-1}\langle 0, +\infty) = \\ &= x(f^{-1} + (fT)^{-1} + \ldots + (fT^{k-1})^{-1})\langle 0, +\infty) = \\ &= x(f^{-1} + T^{-1}f^{-1} + \ldots + T^{-k+1}f^{-1})\langle 0, +\infty) = \\ &= (xf^{-1} + xT^{-1}f^{-1} + \ldots + xT^{-(k-1)}f^{-1})\langle 0, +\infty) = \\ &= (xf^{-1} + \tau x f^{-1} + \ldots + \tau^{k-1} x f^{-1})\langle 0, +\infty) = \\ &= (\mathbf{1} + \tau + \ldots + \tau^{k-1}) x f^{-1}\langle 0, +\infty). \end{aligned}$$

Let $E = \{x \in R^1;\, \exists k \leqq n\colon f(x) + f(Tx) + \ldots + f(T^{k-1}x) \geqq 0\}$. It is easy to see that $E = \bigcup_{k=1}^{n} E_k$ and then

$$x(E) = x\left(\bigcup_{k=1}^{n} E_k\right) = \bigvee_{k=1}^{n} x(E_k) = \bigvee_{k=1}^{n} (\mathbf{1} + \tau + \ldots + \tau^{k-1}) x f^{-1}\langle 0, +\infty),$$

that is $x(E) = \Omega$.

Due to Lemma 2 the transformation T is measure m_x-preserving. By application of the classical maximal ergodic theorem we have

$$\int f\, \chi_E\, dm_x \geqq 0.$$

q.e.d.

The direct consequence of Theorem 8 is the following assertion:

Theorem 9. *Let $\mathscr{L}$, x, m, τ be as in the preceding theorem. Let $a \in R^1$. Let us denote*

$$\Omega^{(a)} = \bigvee_{k=1}^{n} (\mathbf{1} + \tau + \ldots + \tau^{k-1}) x f^{-1}\langle a, +\infty).$$

Let $E \in \mathscr{B}(R^1)$ be such that $x(E) = \Omega^{(a)}$. Then $\int f\, \chi_E\, dm_x \geqq am(\Omega^{(a)})$.

Proof: It is easy to see that $f^{-1}\langle a, +\infty) = (f - a)^{-1}\langle 0, \infty)$. Hence

$$\Omega^{(a)} = \bigvee_{k=1}^{n} (\mathbf{1} + \tau + \ldots + \tau^{k-1}) x (f - a)^{-1}\langle 0, +\infty).$$

Due to theorem 8 it follows $\int (f - a)\, \chi_E\, dm_x \geqq 0$. After a short arrangement we have

$$\int f\, \chi_E\, dm_x \geqq a \int \chi_E\, dm_x = am(x(E)) = am(\Omega^{(a)}).$$

q.e.d.

REFERENCES

[1] DVUREČENSKIJ, A.—RIEČAN, B.: On the individual ergodic theorem on a logic. CMUC 21, 2, 1980.
[2] PULMANOVÁ, S.: On the individual ergodic theorem on a logic. Mathematica Slovaca, to appear.
[3] VARADARAJAN, V. S.: Geometry of Quantum Theory, Van Nostrand New York, 1968.

Received July 28, 1983

Vysoká vojenská technická škola
Katedra matematiky
031 19 Liptovský Mikuláš

МАКСИМАЛЬНАЯ ЭРГОДИЧЕСКАЯ ТЕОРЕМА НА ЛОГИКАХ

Blahoslav Harman

Резюме

В работе рассматриваются вопросы, связанные с доказательством максимальной эргодической теоремы на логиках и её формулировкой.