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Distributive associative near lattices

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DISTRIBUTIVE ASSOCIATIVE NEAR LATTICES

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The aim of the note presented is to classify partially ordered sets by endowing them, in a natural wav, with two binary operations and thereby producing algebras. These algebras are formally similar to lattices and this parallel can be used to classify partially ordered sets in the usual algebraic way by way of homomorphisms, subalgebras, and direct products. A syntactical theory is also presented. In the note we study the lower part of the lattice of varieties of near lattices. The atoms of this lattice are the variety of distributive lattices generated by the two-element-lattice and a subvariety of distributive near lattices generated by a two-element-near-lattice with incomparable elements.

1. Fundamental concepts

Def. 1.1. *The algebra* $(L; \wedge, \vee)$ *is called a near lattice if the following equations are fulfilled:*

1) $x\wedge(y\wedge z)=(x\wedge y)\wedge(y\wedge z)$	1′) $(x\vee y)\vee z=(x\vee y)\vee(y\vee z)$
2) $x\wedge(x\wedge y)=x\wedge y$	2′) $(x\vee y)\vee y=x\vee y$
3) $x\wedge x=x$	3′) $x\vee x=x$
4) $x\wedge y=x\wedge(y\wedge x)$	4′) $x\vee y=(y\vee x)\vee y$
5) $x\wedge(x\vee y)=x$	5′) $(y\wedge x)\vee x=x$
6) $(y\vee x)\wedge x=x$	6′) $x\vee(x\wedge y)=x$

Prop. 1.2 [5]. *To every near lattice* $[L; \wedge, \vee)$ *there corresponds a poset* $(L; \leqq)$ *defined by* $a\leqq b$ *if and only if* $b\wedge a=a$.

We notice that $b\wedge a=a$ if and only if $b\vee a=b$.

Furthermore we have $b\leqq a\vee b$ and $a\wedge b\leqq a$.

Prop. 1.3 [5]. *If* $a\wedge b\leqq b$, *then the element* $a\wedge b$ *is the infimum of* a, b *and has the property* $a\wedge b=b\wedge a$.

If $a\leqq a\vee b$, *then the element* $a\vee b$ *is the supremum of* a, b *and has the property* $a\vee b=b\vee a$.

Prop. 1.4 [5]. *A near lattice* $(L; \wedge, \vee)$ *is a lattice if and only if at least one of the commutative laws holds.*

Remark. As the operations $\wedge$, $\vee$ are not commutative one has to notice that a duality σ has the properties

$$\sigma(x \wedge y) = \sigma(y) \vee \sigma(x) \quad \text{and} \quad \sigma(x \vee y) = \sigma(y) \wedge \sigma(x).$$

If one applies the principle of duality to the category of near lattices, one has to observe this fact.

Theorem 1.5 [5]. *To every poset* $(L; \leqq)$ *there corresponds a near lattice* $(L; \wedge, \vee)$ *which has this poset as an order relation.*

Proof. One defines the operations on an arbitrary poset by

$$x \wedge y = \begin{cases} y & \text{if } \; y \leqq x \\ x & \text{else} \end{cases}$$

and

$$x \vee y = \begin{cases} x & \text{if } \; x \geqq y \\ y & \text{else} \end{cases}$$

and verifies the equations.

Obviously in most cases there are more than one near lattice corresponding to a given poset.

2. Distributive and associative near lattices

Def. 2.1. *A near lattice* $(L; \wedge, \vee)$ *is called associative if the associative laws hold:*

$$x \wedge (y \wedge z) = (x \wedge y) \wedge z \quad \textit{and} \quad x \vee (y \vee z) = (x \vee y) \vee z \quad \textit{hold.}$$

$(L; \wedge, \vee)$ *is called distributive if the following laws hold:*

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \qquad (x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$$
$$(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z) \qquad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

By direct calculation one can prove the following

Proposition 2.1. *Let* $(L; \wedge, \vee)$ *be a distributive and associative near lattice and* $b \in L$. *Then* θ *defined by* $(x, y) \in \theta$ *iff* $b \wedge x = b \wedge y$ *is a congruence relation.*
Of course the dual form of proposition 2.1 also holds.

Theorem 2.2. *Every distributive associative near lattice L is the disjoint union of lattices* V_i, $i \in I$, *such that every element of* V_i *is incomparable with any of the elements of* V_j *for* $i \neq j$.

Proof. Let $a \in L$ and consider the maximal sublattice V_1 of L which contains the element a. Assume there exists an element $b \in L \setminus V_1$ which is comparable with an element $d \in V_1$. If we have $b \vee x = x \vee b$ and $b \wedge x = x \wedge b$ for every $x \in V_1$, then V_1 was not maximal. Therefore there is an element $a \in V_1$ with $b \vee a \neq a \vee b$ or $b \wedge a \neq a \wedge b$. We assume $d < b$ and put $a \wedge d = c$. We have $a \vee (b \wedge c) = a \vee c = a$ and also $(a \vee b) \vee (a \vee c) = (a \vee b) \wedge a \leqq a \vee b$. Therefore we have $a \leqq a \vee b$ by distributivity and $a \vee b = b \vee a$ by proposition 1.3. Now we have $b \vee (a \wedge b) = (b \vee a) \wedge b = (a \vee b) \wedge b = b$. Therefore $a \wedge b \leqq b$ and by proposition 1.3 there is $a \wedge b = b \wedge a$. Contradiction. The remaining case $b < d$ leads in a similar way to a contradiction.

Lemma 2.3. *Let L be a distributive associative near lattice which is the disjoint union of the lattices V_i, $i \in I$ with $|V_i| > 2$ for every $i \in I$. Let η be a congruence relation of L so that for some $a, b \in V_i$, $a \neq b$ we have $(a, b) \in \eta$. Then η is not the smallest non-identical congruence.*

Proof. We consider the following congruences (proposition 2.1) $(x, y) \in \theta_1$ iff $a \wedge x = a \wedge y$ and $(x, y) \in \theta_2$ iff $x \vee a = y \vee a$. If we assume $n \subseteq \theta_1 \cap \theta_2$ we get a contradiction.

Lemma 2.4. *Let L fulfill the same assumption as in Lemma 2.3. Let η be a congruence of L such that for some $a, b \in L$ with $a \in V_i$, $b \in V_j$, $i \neq j$, we have $(a, b) \in \eta$. Then η is not the smallest nonidentical congruence.*

Proof. We define a relation θ by $(c, d) \in \theta$ iff there is an index i, $i \in I$, such that $c, d \in V_i$. One can show that θ is a congruence and we have $\eta \nsubseteq \theta$.

Notation. By D_2 we denote the distributive lattice consisting of two elements and by D^2 we denote the only (up to isomorphism) distributive associative near lattice consisting of two incomparable elements.

Theorem 2.5. *The near lattices D_2 and D^2 are the only subdirectly irreducible distributive associative near lattices (besides the one-element-lattice).*

Proof: By the theorem and the lemmas above a subdirectly irreducible near lattice L can only be the disjoint union of one- or two-element lattices. But L cannot contain a subnear-lattices consisting of three elements. For $|L| > 2$ one can show that L is not subdirectly irreducible using the considerations of lemma 2.4 and lemma 2.3.

Theorem 2.6. *Let L be a distributive associative near lattice such that L is the disjoint union of lattices which have a least and a greatest element. Then L is the direct product of a distributive lattice with a near lattice of the variety HSP (D^2).*

Proof. Let L be the disjoint union of $\{L_i | i \in I\}$, a family of lattices with least and greatest elements.

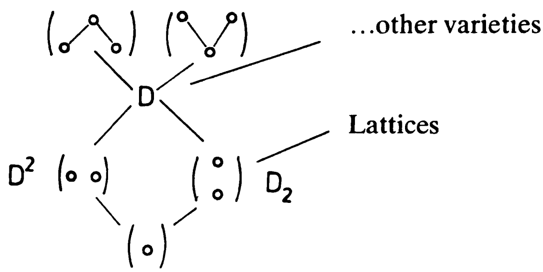
It is sufficient to show $L_i \simeq L_j$ where c_i is the least element of L_i, c_j respectively that of L_j and d_i is the greatest element of L_i, d_j respectively that of L_j. We consider the following mappings:

$$\alpha\colon\ L_i \to L_j,\ \alpha(x) = x \vee c_j$$
$$\beta\colon\ L_j \to L_i,\ \beta(y) = y \vee c_i.$$

α and β are near lattice homomorphisms. We have to show $\beta \circ \alpha = \mathrm{id}_{L_i}$.

We consider $d_j \wedge (c_i \vee c_j) = (d_j \wedge c_i) \vee (d_j \wedge c_j) = (d_j \wedge c_i) \vee c_j = d_j \wedge c_i$ and on the other hand $d_j \wedge (c_i \vee c_j) = c_i \vee c_j$.

From $(d_j \wedge c_i) \vee c_i = (c_i \vee c_j) \vee c_i$ we derive $c_j \vee c_i = c_i$. The following diagrams describe the lower part of the lattice of the varieties of near lattices:



$D = \mathrm{HSP}(D_2, D^2)$ the variety of the distributive and associative near lattices. By theorem 2.1 in [5] it follows that D^2 and D_2 are the only atoms in the lattice of the varieties of near lattices.

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ДИСТРИБУТИВНЫЕ АССОЦИАТИВНЫЕ ПОЧТИ РЕШЕТКИ

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Резюме

В работе изучается решетка подмногообразий почти решеток. Приведено описание нижней части этой решетки. В частности показано, что эта решетка подмногообразий имеет только два атома: дистрибутивные решетки и подмногообразие прождено двухэлементной антицепью.